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CHARACTERIZATIONS OF SOME BIVARIATE MODELS USING RECIPROCAL COORDINATE SUBTANGENTS

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1. INTRODUCTION

The concept of reciprocal coordinate subtangent (RCST) has been used in the statistical literature as a useful tool to study the monotone behaviors of a continuous density function and for characterizing distributions through its functional forms. It is considered as a measure for strongly unimodal property (see Hajek and Sidak, 1967). Let X be an absolutely continuous univariate random variable (rv) having a probability density function (PDF) $f(t)$ such that $f'(t)$ exists. Then RCST to a curve $y = f(t)$ of the rv X is given by

$$\eta(t) = -\frac{f'(t)}{f(t)} = -\frac{\partial}{\partial t}\log f(t) \tag{1}$$

RCST also plays a very important role in reliability analysis, however, used it rather unknowingly. For example, failure rate or hazard rate is the coordinate subtangent measured on the curve $y = \bar{F}(t)$, where $\bar{F}(t)$ is the survival function (SF). Since many of the failure rate functions have complex expressions, Glaser (1980) identified $\eta(t)$ (but not called as 'RCST') as an easy statistical tool to determine the monotone behaviour or shape of a failure rate function. $\eta(t)$ can also be expressed in terms of the failure rate $h(t) = -\frac{\bar{F}'(t)}{\bar{F}(t)}$ by $\eta(t) = h(t) - \frac{h'(t)}{h(t)}$, provided $h'(t)$ exist. However, Mukherjee and Roy (1989) identified RCST as a measure to characterize various models by a unique determination of $f(t)$ using (1) is given by

$$f(t) = k \exp\left[-\int_0^t \eta(x)dx\right]$$

where k is a normalizing constant. Mukherjee and Roy (1989) also studied some properties and applications of $\eta(t)$ and proved characterization results to certain important life distributions viz.

exponential, Lomax and finite range models. For more details about $\eta(t)$, we refer to Gupta and Warren (2001); Navarro and Hernandez (2004); Mi (2004); Lai and Xie (2006); Navarro (2008); Sunoj and Sreejith (2012) and the references therein.

Recently, Roy and Roy (2010) extended the concept of RCST to the bivariate and multivariate setup. In the bivariate case, for a non negative vector variable (X,Y) with PDF $f_{(X,Y)}(t,s)$, the vector valued RCST is defined by

$$\eta_X(t,s) = -\frac{\partial}{\partial t}\log f_{(X,Y)}(t,s) \tag{2}$$

and

$$\eta_Y(t,s) = -\frac{\partial}{\partial s}\log f_{(X,Y)}(t,s). \tag{3}$$

If the bivariate RCST $(\eta_X(t,s),\eta_Y(t,s))$ is continuous and $f_{(X,Y)}(0,0) > 0$, then Roy and Roy (2010) proved that the density curve can be uniquely determined from the following two alternative forms:

$$f_{(X,Y)}(t,s) = C\exp\left[-\int_0^t \eta_X(x,0)dx - \int_0^s \eta_Y(t,y)dy\right] \tag{4}$$

and

$$f_{(X,Y)}(t,s) = C\exp\left[-\int_0^s \eta_Y(0,y)dy - \int_0^t \eta_X(x,s)dx\right], \tag{5}$$

where C is a normalizing constant.

In modeling statistical data, the standard practice is either to derive the appropriate model based on the physical properties of the system or to choose a flexible family of distributions and then find a member of the family that is appropriate to the data. In both the situations it would be helpful if we find characterization theorems that explain the distribution using important measures of indices (see for example, Nair and Sankaran, 1991; Ruiz and Navarro, 1994; Asadi, 1998; Sankaran and Nair, 2000; Sankaran and Sunoj, 2004).

Characterizations of the bivariate distribution or joint density through their conditional densities or survival functions are an important problem studied by many researchers. Arnold (1995, 1996) and Arnold and Kim (1996) have studied several classes of conditional survival models. The identification of the joint distribution of (X,Y) when conditional distributions of $(X \mid Y = s)$ and $(Y \mid X = t)$ are known is one important problem studied in the past. This approach of identifying a bivariate density using the conditionals is called the conditional specification of the joint distribution (see Arnold et al., 1999). These conditional models are often useful in many two component reliability systems where the operational status of one component is known. Another important problem closely associated to this is the identification of the joint distribution of (X,Y) when the conditional distribution or corresponding reliability measures of the rv's $(X \mid Y > s)$ and $(Y \mid X > t)$ are known. That is, instead of conditioning on a component failing (down) at a specified time, we study the system when the survival time of one component is known. For a recent study of these models, we refer to Navarro and Sarabia (2011);

Navarro et al. (2011) and the references therein.

In the present paper, we derive some new characterizations to certain important families of distributions using the bivariate RCST function given in (2) and (3). The paper is organized as follows. In Section 2, we proved characterization results for a general bivariate model whose conditional distributions are proportional hazard rate models, Sarmanov family and Ali-Mikhail-Haq family of bivariate distributions and establish a relationship between local dependence function and RCST. In Section 3 and 4, we define RCST for conditionally specified distributions and some characterization results are proved.

2. BIVARIATE RCST

In this section, we consider the bivariate RCST given in (2) and (3) and study its relationship with some important families of distributions.

2.1. Bivariate model with conditional distributions are proportional hazard models

Recently, Navarro and Sarabia (2011) studied the reliability properties in two classes of bivariate continuous distributions based on the specification of conditional hazard functions. These classes were constructed by conditioning on two types of events viz. events of the type $\{X = t\}$ and type $\{Y = s\}$ and events of the $\{X > t\}$ and type $\{Y > s\}$ respectively, that has been used in Arnold and Kim (1996). In survival studies the most widely used semi parametric regression model is the proportional hazard rate (PHR) model. The univariate Cox PHR model is a class of modelling distributions with PDF and SF given by

$$f(t;\alpha) = \alpha\lambda(t)\exp\{-\alpha\Lambda(t)\}, \quad t \geq 0 \tag{6}$$

and

$$\bar{F}(t;\alpha) = \exp\{-\alpha\Lambda(t)\}, \quad t \geq 0,$$

where $\alpha > 0$, $\lambda(t)$ is the baseline hazard rate function and $\Lambda(t) = \int_0^t \lambda(x)dx$ is the baseline cumulative hazard function. The hazard (or failure) rate function of $f(t;\alpha)$ is $h(t;\alpha) = f(t;\alpha)/\bar{F}(t;\alpha) = \alpha\lambda(t)$. A rv with the PDF (6) can be denoted by $X \sim PHR(\alpha;\Lambda(t))$. Special cases of $f(t;\alpha)$ and $\bar{F}(t;\alpha)$ include exponential, Burr, Pareto and Weibull. Navarro and Sarabia (2011) obtained a general form of a bivariate PDF with conditional distributions satisfying $(X \mid Y = s) \sim PHR(\alpha_1(s);\Lambda_1(t))$ and $(Y \mid X = t) \sim PHR(\alpha_2(t);\Lambda_2(s))$, given by

$$f_{(X,Y)}(t,s) = c(\phi)a_1a_2\lambda_1(t)\lambda_2(s)\exp\left[-a_1\Lambda_1(t) - a_2\Lambda_2(s) - \phi a_1a_2\Lambda_1(t)\Lambda_2(s)\right] \tag{7}$$

for $t, s \geq 0$, where $a_1, a_2 > 0$ and $\phi \geq 0$. The model given in (7) is a reparametrization of the bivariate conditional proportional hazard model due to Arnold and Kim (1996). The case when $\phi = 0$ corresponds to the case of independence. In particular, if $\Lambda_1(t) = t$ and $\Lambda_2(s) = s$, we obtain the class of bivariate distributions with exponential conditionals considered by Arnold and Strauss (1988). Navarro and Sarabia (2011) also obtained a bivariate PDF with conditional distributions satisfying $(X \mid Y > s) \sim PHR(\alpha_1(s); \Lambda_1(t))$ and $(Y \mid X > t) \sim PHR(\alpha_2(t); \Lambda_2(s))$, with joint pdf is given by

$$f_{(X,Y)}(t,s) = a_1 a_2 \lambda_1(t) \lambda_2(s) \left(\frac{\alpha_1(s)\alpha_2(t)}{a_1 a_2} - \phi \right), \quad \exp\left[-a_1\Lambda_1(t) - a_2\Lambda_2(s) - \phi a_1 a_2 \Lambda_1(t)\Lambda_2(s)\right] \tag{8}$$

where in both cases $\alpha_1(s) = a_1[1+\phi a_2\Lambda_2(s)]$ and $\alpha_2(t) = a_2[1+\phi a_1\Lambda_1(t)]$, Λ_1 and Λ_2 are two cumulative hazard functions and λ_1 and λ_2 are their respective hazard rate functions.

Now we have the following characterization theorem.

THEOREM 2.1. *For a non negative random vector (X,Y), the relationships*

$$\eta_X(t,s) = a_1[1+\phi a_2\Lambda_2(s)]\lambda_1(t) - \frac{\lambda_1'(t)}{\lambda_1(t)} \tag{9}$$

and

$$\eta_Y(t,s) = a_2[1+\phi a_1\Lambda_1(t)]\lambda_2(s) - \frac{\lambda_2'(s)}{\lambda_2(s)}, \tag{10}$$

hold if, and only if, $f_{(X,Y)}(t,s)$ is of the form (7).

PROOF. Assume that equations (9) and (10) hold, then using (4) we obtain

$$f_{(X,Y)}(t,s) = C\exp\left[-a_1\int_0^t \lambda_1(x)dx + \int_0^t \frac{\lambda_1'(x)}{\lambda_1(x)}dx - a_2(1+\phi a_1\Lambda_1(t))\int_0^s \lambda_2(y)dy + \int_0^s \frac{\lambda_2'(y)}{\lambda_2(y)}dy\right],$$

and thus

$$f_{(X,Y)}(t,s) = C\exp\left[-a_1\Lambda_1(t) + \log\lambda_1(t) - a_2(1+\phi a_1\Lambda_1(t))\Lambda_2(s) + \log\lambda_2(s)\right],$$

and we have the model (7). The other part is quite straightforward.

EXAMPLE 2. Bivariate exponential (i.e., $\Lambda_1(t) = t$ and $\Lambda_2(s) = s$) with joint PDF

$$f_{(X,Y)}(t,s) = c(\phi)a_1a_2\exp\left[-a_1t - a_2s - \phi a_1a_2ts\right],$$

obtains characterizing relationships $\eta_X(t,s) = a_1 + \phi a_1 a_2 s$ and $\eta_Y(t,s) = a_2 + \phi a_1 a_2 t$.

EXAMPLE 3. Bivariate Weibull (i.e., $\Lambda_1(t) = t^{\gamma_1}$ and $\Lambda_2(s) = s^{\gamma_2}$) with joint PDF

$$f_{(X,Y)}(t,s)=c(\phi)a_1a_2\gamma_1t^{\gamma_1-1}\gamma_2s^{\gamma_2-1}\exp\left[-a_1t^{\gamma_1}-a_2s^{\gamma_2}-\phi a_1a_2t^{\gamma_1}s^{\gamma_2}\right],$$

the relationships are

$$\eta_X(t,s)=\gamma_1t^{\gamma_1-1}\left(a_1+\phi a_1a_2s^{\gamma_2}\right)-\frac{\gamma_1-1}{t}$$

and

$$\eta_Y(t,s)=\gamma_2s^{\gamma_2-1}\left(a_2+\phi a_1a_2t^{\gamma_1}\right)-\frac{\gamma_2-1}{s}.$$

EXAMPLE 4. Bivariate Pareto (i.e., $\Lambda_1(t)=\log\frac{\beta_1+t}{\beta_1}$ and $\Lambda_2(s)=\log\frac{\beta_2+s}{\beta_2}$) with joint PDF

$$f_{(X,Y)}(t,s)=c(\phi)a_1a_2\beta_1^{a_1}\beta_2^{a_2}\left(\frac{1}{\beta_1+t}\right)^{a_1+1}\left(\frac{1}{\beta_2+s}\right)^{a_2+1}\exp\left[-\phi a_1a_2\log\frac{\beta_1+t}{\beta_1}\log\frac{\beta_2+s}{\beta_2}\right],$$

characterizes

$$\eta_X(t,s)=\frac{1}{\beta_1+t}\left(1+a_1+\phi a_1a_2\log\frac{\beta_2+s}{\beta_2}\right)$$

and

$$\eta_Y(t,s)=\frac{1}{\beta_2+s}\left(1+a_2+\phi a_1a_2\log\frac{\beta_1+t}{\beta_1}\right).$$

EXAMPLE 5. Bivariate Burr (i.e., $\Lambda_1(t)=\log\frac{\beta_1+t^{\gamma_1}}{\beta_1}$ and $\Lambda_2(s)=\log\frac{\beta_2+s^{\gamma_2}}{\beta_2}$) with joint PDF

$$f_{(X,Y)}(t,s)=c(\phi)a_1a_2\gamma_1\gamma_2\beta_1^{a_1}\beta_2^{a_2}t^{\gamma_1-1}s^{\gamma_2-1}\left(\frac{1}{\beta_1+t^{\gamma_1}}\right)^{a_1+1}\left(\frac{1}{\beta_2+s^{\gamma_2}}\right)^{a_2+1}$$
$$\exp\left[-\phi a_1a_2\log\frac{\beta_1+t^{\gamma_1}}{\beta_1}\log\frac{\beta_2+s^{\gamma_2}}{\beta_2}\right],$$

we have

$$\eta_X(t,s)=\frac{\gamma_1t^{\gamma_1-1}}{\beta_1+t^{\gamma_1}}\left(1+a_1+\phi a_1a_2\log\frac{\beta_2+s^{\gamma_2}}{\beta_2}\right)-\frac{\gamma_1-1}{t}$$

and

$$\eta_Y(t,s)=\frac{\gamma_2 s^{\gamma 2-1}}{\beta_2+s^{\gamma_2}}\left(1+a_2+\phi a_1 a_2\log\frac{\beta_1+t^{\gamma_1}}{\beta_1}\right)-\frac{\gamma_2-1}{s}.$$

THEOREM 6. For a non negative random vector, the relationships

$$\eta_X(t,s)=\alpha_1(s)\lambda_1(t)\left(1-\frac{\phi a_1 a_2}{\alpha_1(s)\alpha_2(t)-\phi a_1 a_2}\right)-\frac{\lambda_1'(t)}{\lambda_1(t)} \tag{11}$$

and

$$\eta_Y(t,s)=\alpha_2(t)\lambda_2(s)\left(1-\frac{\phi a_1 a_2}{\alpha_1(s)\alpha_2(t)-\phi a_1 a_2}\right)-\frac{\lambda_2'(s)}{\lambda_2(s)} \tag{12}$$

hold if and only if $f_{(X,Y)}(t,s)$ *is* (8).

PROOF. Assume that equations (11) and (12) holds, then using (4) we obtain

$$f_{(X,Y)}(t,s)=C\exp\left[-\int_0^t\left(a_1\lambda_1(x)\left(1-\frac{\phi a_1 a_2}{a_1\alpha_2(x)-\phi a_1 a_2}\right)-\frac{\lambda_1'(x)}{\lambda_1(x)}\right)dx\right]$$
$$\exp\left[-\int_0^s\left(\alpha_2(t)\lambda_2(y)\left(1-\frac{\phi a_1 a_2}{\alpha_1(y)\alpha_2(t)-\phi a_1 a_2}\right)-\frac{\lambda_2'(y)}{\lambda_2(y)}\right)dy\right],$$

Equivalently, we have

$$f_{(X,Y)}(t,s)=C\exp\left[-\int_0^t\left(a_1\lambda_1(x)-\frac{\phi a_1\lambda_1(x)}{[1+\phi a_1\Lambda_1(x)]-\phi}-\frac{\lambda_1'(x)}{\lambda_1(x)}\right)dx\right]$$
$$\exp\left[-\int_0^s\left(\alpha_2(t)\lambda_2(y)-\frac{\phi a_2\alpha_2(t)\lambda_2(y)}{[1+\phi a_2\Lambda_2(y)]\alpha_2(t)-\phi a_2}-\frac{\lambda_2'(y)}{\lambda_2(y)}\right)dy\right]$$

which on further simplification yield

$$f_{(X,Y)}(t,s)=C^*\lambda_1(t)\lambda_2(s)\left([1+\phi a_1\Lambda_1(t)][1+\phi a_2\Lambda_2(s)]-\phi\right)$$
$$\exp\left[-a_1\Lambda_1(t)-a_2\Lambda_2(s)-\phi a_1 a_2\Lambda_1(t)\Lambda_2(s)\right],$$

reduces to the model (8). The first part is direct.

2.2. *Sarmanov family of bivariate distributions*

Assume that $f_X(t)$ and $f_Y(s)$ are univariate PDF's with supports defined on $A_X\subseteq R$ and

$A_Y \subseteq R$. Let $\phi_X(t)$ and $\phi_Y(t)$ be bounded nonconstant functions such that $\int_{-\infty}^{\infty} \phi_X(x) f_X(x) dx = 0$ and $\int_{-\infty}^{\infty} \phi_Y(y) f_Y(y) dy = 0$. Then the function defined by

$$f_{(X,Y)}(t,s) = f_X(t) f_Y(s) [1 + \omega \phi_X(t) \phi_Y(s)] \tag{13}$$

is a bivariate joint density with specified marginals $f_X(t)$ and $f_Y(s)$, provided ω is a real number which satisfies the condition $1 + \omega \phi_X(t) \phi_Y(s) \geq 0$ for all t and s. This is called the Sarmanov family of bivariate distributions. For various applications of this family, we refer to Willett and Thomas (1985, 1987) and Lee (1996). When $\phi_X(t) = 1 - 2F_X(t)$ and $\phi_Y(s) = 1 - 2F_Y(s)$ the Sarmanov family becomes the Farlie-Gumbel-Morgenstern (FGM) family (see Bairamov *et al.*, 2001).

THEOREM 7. *For a non negative random vector* (X, Y)

$$\eta_X(t,s) = \eta_X(t) - \frac{\omega \phi'_X(t) \phi_Y(s)}{1 + \omega \phi_X(t) \phi_Y(s)} \tag{14}$$

and

$$\eta_Y(t,s) = \eta_Y(s) - \frac{\omega \phi_X(t) \phi'_Y(s)}{1 + \omega \phi_X(t) \phi_Y(s)} \tag{15}$$

if and only if $f_{(X,Y)}(t,s)$ *is* (13).

The proof is similar to Theorem 1.

EXAMPLE 8. Bivariate distributions with the beta marginals. In this case, we have $\phi_X(t) = t - \frac{a_1}{a_1 + b_1}$ and $\phi_Y(s) = s - \frac{a_2}{a_2 + b_2}$. Then (14) and (15) becomes

$$\eta_X(t,s) = \frac{t(a_1 + b_1 - 2) - a_1 + 1}{t(1-t)} - \frac{\omega\left(s - \frac{a_2}{a_2 + b_2}\right)}{1 + \omega\left(t - \frac{a_1}{a_1 + b_1}\right)\left(s - \frac{a_2}{a_2 + b_2}\right)}$$

and

$$\eta_Y(t,s) = \frac{s(a_2 + b_2 - 2) - a_2 + 1}{s(1-s)} - \frac{\omega\left(t - \frac{a_1}{a_1 + b_1}\right)}{1 + \omega\left(t - \frac{a_1}{a_1 + b_1}\right)\left(s - \frac{a_2}{a_2 + b_2}\right)}.$$

EXAMPLE 9. Bivariate distributions with the gamma marginals. In this case, we have

$$\phi_X(t) = e^{-t} - \left(1+\frac{1}{\lambda_1}\right)^{-\alpha_1} \text{ and } \phi_Y(s) = e^{-s} - \left(1+\frac{1}{\lambda_2}\right)^{-\alpha_2}.$$

Then

$$\eta_X(t,s) = \lambda_1 - \frac{\alpha_1 - 1}{t} + \frac{\omega e^{-t}\left(e^{-s} - \left(\frac{\lambda_2}{\lambda_2+1}\right)^{\alpha_2}\right)}{1+\omega\left(e^{-t} - \left(\frac{\lambda_1}{\lambda_1+1}\right)^{\alpha_1}\right)\left(e^{-s} - \left(\frac{\lambda_2}{\lambda_2+1}\right)^{\alpha_2}\right)}$$

and

$$\eta_Y(t,s) = \lambda_2 - \frac{\alpha_2 - 1}{s} + \frac{\omega e^{-s}\left(e^{-t} - \left(\frac{\lambda_1}{\lambda_1+1}\right)^{\alpha_1}\right)}{1+\omega\left(e^{-t} - \left(\frac{\lambda_1}{\lambda_1+1}\right)^{\alpha_1}\right)\left(e^{-s} - \left(\frac{\lambda_2}{\lambda_2+1}\right)^{\alpha_2}\right)}$$

EXAMPLE 10. FGM family. In this case, we have $\phi_X(t) = 1 - 2F_X(t)$ and $\phi_Y(s) = 1 - 2F_Y(s)$. Then

$$\eta_X(t,s) = \eta_X(t) + \frac{2\omega f_X(t)\left(1-2F_Y(s)\right)}{1+\omega\left(1-2F_X(t)\right)\left(1-2F_Y(s)\right)}$$

and

$$\eta_Y(t,s) = \eta_Y(s) + \frac{2\omega f_Y(s)\left(1-2F_X(t)\right)}{1+\omega\left(1-2F_X(t)\right)\left(1-2F_Y(s)\right)}.$$

2.3. *Ali-Mikhail-Haq family of bivariate distributions*

The family of bivariate distributions proposed by (Ali *et al.*, 1978) is given by

$$F_{(X,Y)}(t,s) = \frac{F_X(t)F_Y(s)}{1-\alpha\overline{F}_X(t)\overline{F}_Y(s)}, \quad -1 \le \alpha \le 1,$$

where $F_X(t)$ and $F_Y(s)$ are the marginal distribution functions of X and Y, $\overline{F}_X(t) = 1 - F_X(t)$ and $\overline{F}_Y(s) = 1 - F_Y(s)$. The above family of bivariate distributions is indexed by a single parameter and contains Gumbel Type I distributions as well as the case of independent rv's. The parameter α is essentially a parameter of association between X and Y. A special case of the above model is Gumbel bivariate logistic distribution given by $F_{(X,Y)}(t,s)=\dfrac{1}{1+e^{-t}+e^{-s}}$. A simple way of describing the model would be through the joint distribution $F_{(X,Y)}(u,v)$ for the rv's (U,V), where $U=F_X(t)$ and $V=F_Y(s)$, and we obtain the copula

$$F_{(U,V)}(u,v)=\frac{u\,v}{1-\alpha\,\bar{u}\,\bar{v}} \tag{16}$$

where $\bar{u}=1-u$ and $\bar{v}=1-v$. It can be verified that, for the model (16), the joint density is given by

$$f_{(U,V)}(u,v)=\frac{(1-\alpha)(1-\alpha\,\bar{u}\,\bar{v})+2\alpha\,u\,v}{(1-\alpha\,\bar{u}\,\bar{v})^3} \tag{17}$$

for $0<u<1$ and $0<v<1$.

THEOREM 2.4. For a non negative random vector (X,Y), the relationships

$$\eta_U(u,v)=\frac{3\alpha\,\bar{v}}{1-\alpha\,\bar{u}\,\bar{v}}-\frac{(1-\alpha)\alpha\,\bar{v}+2\alpha\,v}{(1-\alpha)(1-\alpha\,\bar{u}\,\bar{v})+2\alpha\,u\,v}$$

and

$$\eta_V(u,v)=\frac{3\alpha\,\bar{u}}{1-\alpha\,\bar{u}\,\bar{v}}-\frac{(1-\alpha)\alpha\,\bar{u}+2\alpha\,u}{(1-\alpha)(1-\alpha\,\bar{u}\,\bar{v})+2\alpha\,u\,v}$$

are satisfied if and only if $f_{(U,V)}(u,v)$ is the model (17).

The proof is similar to Theorem 2.1.

2.4. Local dependence function and RCST

Let (X,Y) be a bivariate random vector in the support of $(a_1,b_1)\times(a_2,b_2), b_i>a_i,\ i=1,2$, where (a_i,b_i) is an interval on the real line with an absolutely continuous distribution function $F_{(X,Y)}(t,s)$, and PDF $f_{(X,Y)}(t,s)$. Assume that mixed partial derivative of $f_{(X,Y)}(t,s)$ exists. The local dependence function (see Holland and Wang, 1987) of (X,Y) is given by,

$$\gamma_f(t,s)=\frac{\partial^2}{\partial t\partial s}\log f_{(X,Y)}(t,s).$$

The relation between local dependence function and RCST is

$$\gamma_f(t,s)=-\frac{\partial}{\partial s}\eta_X(t,s),\quad \text{where}\quad \eta_X(t,s)=-\frac{\partial}{\partial t}\log f_{(X,Y)}(t,s)$$

or

$$\gamma_f(t,s) = -\frac{\partial}{\partial t}\eta_Y(t,s), \text{ where } \eta_Y(t,s) = -\frac{\partial}{\partial s}\log f_{(X,Y)}(t,s).$$

THEOREM 11. For a non negative random vector (X,Y) with continuous RCST functions, the following conditions are equivalent:

(i). (X,Y) *follows a bivariate distribution with joint PDF* $f_{(X,Y)}(t,s) = a(t;\theta)b(s;\theta)e^{\theta ts}$ *, for some appropriate functions* $a(t;\theta)$ *and* $b(s;\theta)$*;*

(ii). $\eta_X(t,s) = -\theta s - \frac{a'(t;\theta)}{a(t;\theta)}$ *and* $\eta_Y(t,s) = -\theta t - \frac{b'(s;\theta)}{b(s;\theta)}$ *; and*

(iii). $\gamma_f(t,s)$ *is a constant.*

PROOF. The sequence of relationships from (i) to (ii) and (iii) is direct. The proof of (i) from (iii) can be obtained from (Jones, 1998).

3. CONDITIONALLY SPECIFIED RCST FOR X GIVEN $Y = s$ AND FOR Y GIVEN $X = t$

In this specification we consider conditioning on events of the forms $\{X = t\}$ and $\{Y = s\}$. Then, let (X,Y) be a bivariate random variable with support $S = (0,\infty)\times(0,\infty)$. Suppose $f_{(X|Y=s)}(t\,|\,s)$ and $f_{(Y|X=t)}(s\,|\,t)$ be the conditional PDF of $(X\,|\,Y = s)$ and $(Y\,|\,X = t)$ respectively, then a direct extension of RCST (2) and (3) to the these conditional rv's are given by

$$\eta_{(X|Y=s)}(t\,|\,s) = -\frac{\partial}{\partial t}\log f_{(X|Y=s)}(t\,|\,s) \tag{18}$$

and

$$\eta_{(Y|X=t)}(s\,|\,t) = -\frac{\partial}{\partial s}\log f_{(Y|X=t)}(s\,|\,t) \tag{19}$$

Integrating both sides of (18) with respect to t over the integral 0 to t , we get

$$f_{(X|Y=s)}(t\,|\,s) = C_1(s)\exp\left[-\int_0^t \eta_{(X|Y=s)}(x\,|\,s)\,dx\right], \tag{20}$$

where $C_1(s)$ is constant of integration determined by $\int_X f_{(X|Y=s)}(x\,|\,s)dx = 1$. This implies that the conditionally specified RCST $\eta_{(X|Y=s)}(t\,|\,s)$ uniquely determines the conditional PDF $f_{(X|Y=s)}(t\,|\,s)$. Similarly by using (19), the conditional PDF $f_{(Y|X=t)}(s\,|\,t)$ is determined by

$$f_{(Y|X=t)}(s\,|\,t) = C_2(t)\,\exp\left[-\int_0^s \eta_{(Y|X=t)}(y\,|\,t)\,dy\right], \tag{21}$$

where $C_2(t)$ is constant of integration determined by $\int_Y f_{(Y|X=t)}(y\,|\,t)dy = 1$.

REMARK 12. Based on the definitions of conditional distributions, $f_{(X|Y=s)}(t\,|\,s) = f_{(X,Y)}(t,s)\,/\,f_Y(s)$ and $f_{(Y|X=t)}(s\,|\,t) = f_{(X,Y)}(t,s)\,/\,f_X(t)$, it can be easily seen that $\eta_{(X|Y=s)}(t\,|\,s) = \eta_X(t,s)$ and $\eta_{(Y|X=t)}(s\,|\,t) = \eta_Y(t,s)$, and therefore the characterization result in Theorem 2.1 can be obtained from Theorem 2.1 in (Navarro and Sarabia, 2011).

THEOREM 13. A necessary condition for the existence for a random vector (X,Y) with support $S_X \times S_Y$ satisfying (20) and (21) is that

$$\int_0^s \eta_{(Y|X=t)}(y\,|\,t)\,dy - \int_0^t \eta_{(X|Y=s)}(x\,|\,s)\,dx = u^*(t) + v^*(s) \tag{22}$$

holds for all (s,t) in $S_X \times S_Y$. Moreover, in this case, the PDF of (X,Y) can be obtained as

$$f_{(X,Y)}(t,s) = C\exp\left[u^*(t) - \int_0^s \eta_{(Y|X=t)}(y\,|\,t)dy\right], \tag{23}$$

or

$$f_{(X,Y)}(t,s) = C^*\exp\left[v^*(s) - \int_0^t \eta_{(X|Y=s)}(x\,|\,s)dx\right], \tag{24}$$

where C and C^* are normalizing constants.

PROOF. If (X,Y) exists and satisfies (20) and (21), then the densities $f_{(X|Y=s)}(t\,|\,s)$ and $f_{(Y|X=t)}(t\,|\,s)$ satisfy the compatibility condition (1.20) in Theorem 1.2 of (Arnold *et al.*, 1999, p. 8), that is,

$$\frac{f_{(X|Y=s)}(t\,|\,s)}{f_{(Y|X=t)}(s\,|\,t)} = u(t)v(s),$$

using (20) and (21), it becomes

$$\frac{C_1(s)\,\exp\left[-\int_0^t \eta_{(X|Y=s)}(x\,|\,s)\,dx\right]}{C_2(t)\,\exp\left[-\int_0^s \eta_{(Y|X=t)}(y\,|\,t)\,dy\right]} = u(t)v(s).$$

Equivalently,

$$\exp\left[-\left(\int_0^t \eta_{(X|Y=s)}(x\mid s)\,dx-\int_0^s \eta_{(Y|X=t)}(y\mid t)\,dy\right)\right]=u(t)C_2(t)\frac{v(s)}{C_1(s)}.$$

Taking logarithm on both sides, we get

$$-\left(\int_0^t \eta_{(X|Y=s)}(x\mid s)\,dx-\int_0^s \eta_{(Y|X=t)}(y\mid t)\,dy\right)=\log\left(u(t)C_2(t)\frac{v(s)}{C_1(s)}\right)=u^*(t)+v^*(s)$$

which gives (22), where

$$u^*(t)=\log\left(u(t)C_2(t)\right) \tag{25}$$

and

$$v^*(s)=\log\left(\frac{v(s)}{C_1(s)}\right).$$

From (21) and (25), if it exists, the joint PDF of (X,Y) becomes

$$f_{(X,Y)}(t,s)=f_X(t)f_{(Y|X=t)}(s\mid t)=f_X(t)C_2(t)\exp\left[-\int_0^s \eta_{(Y|X=t)}(y\mid t)\,dy\right]$$

$$=f_X(t)\frac{\exp(u^*(t))}{u(t)}\exp\left[-\int_0^s \eta_{(Y|X=t)}(y\mid t)\,dy\right]$$

Then, using that $u(t)$ is proportional to $f_X(t)$ (see (Arnold et al., 1999, p.8), we have

$$f_{(X,Y)}(t,s)=K\exp(u^*(t))\exp\left[-\int_0^s \eta_{(Y|X=t)}(y\mid t)\,dy\right],$$

thus obtains the form (23). In a similar fashion, we can obtain (24).

EXAMPLE 14. Suppose that $\eta_{(X|Y=s)}(t\mid s)$ and $\eta_{(Y|X=t)}(s\mid t)$ satisfies the relationships $\eta_{(X|Y=s)}(t\mid s)=\alpha_1(s)\lambda_1(t)-\frac{\lambda_1'(t)}{\lambda_1(t)}$ and $\eta_{(Y|X=t)}(s\mid t)=\alpha_2(t)\lambda_2(s)-\frac{\lambda_2'(s)}{\lambda_2(s)}$, where $\alpha_1(s)=a_1[1+\phi a_2\Lambda_2(s)]$ and $\alpha_2(t)=a_2[1+\phi a_1\Lambda_1(t)]$. Then we can easily show that it satisfies relationship (22) with $u^*(t)=\log\lambda_1(t)-a_1\Lambda_1(t)$ and $v^*(s)=a_2\Lambda_2(s)-\log\lambda_2(s)$. Now using (23) or (24), we have the model (7).

Obviously, from Remark 3.1, Theorem 3.1 can be used to obtain a compatibility condition for the bivariate RCST.

4. CONDITIONALLY SPECIFIED RCST FOR $(X \mid Y > s)$ AND FOR $(Y \mid X > t)$

In the case of bivariate survival models, instead of conditioning on a component failing at a specified time, it is sometimes more natural to condition on the component's having survived beyond a specified time (see Navarro and Sarabia, 2011). Then the conditional RCST for $(X \mid Y > s)$ and $(Y \mid X > t)$ are defined as

$$\eta_{(X|Y>s)}(t \mid s) = -\frac{\partial}{\partial t} \log f_{(X|Y>s)}(t \mid s) \tag{26}$$

and

$$\eta_{(Y|X>t)}(s \mid t) = -\frac{\partial}{\partial s} \log f_{(Y|X>t)}(s \mid t) \tag{27}$$

where $P(X > t \mid Y > s) = \int_t^\infty f_{(X|Y>s)}(x \mid s) dx$ and $P(Y > s \mid X > t) = \int_s^\infty f_{(Y|X>t)}(y \mid t) dy$ are the conditional SF's of $(X \mid Y > s)$ and $(Y \mid X > t)$ respectively and assume that $\frac{P(X > t \mid Y > s)}{P(Y > s \mid X > t)} = u(t)v(s)$ with $u(t)$ and $1/v(s)$ are two SF's (see Navarro and Sarabia, 2011). The conditional RCST functions given in (26) and (27) where used in (Navarro, 2008) to study ordering properties between series systems. Integrating both sides of (26) with respect to t over the limit 0 to t, we get

$$f_{(X|Y>s)}(t \mid s) = D_1(s) \exp\left[-\int_0^t \eta_{(X|Y>s)}(x \mid s)\, dx \right] \tag{28}$$

where $D_1(s)$ is constant of integration determined by $\int_{S_X} f_{(X|Y>s)}(x \mid s) dx = 1$. Similarly from (27), we have

$$f_{(Y|X>t)}(s \mid t) = D_2(t) \exp\left[-\int_0^s \eta_{(Y|X>t)}(y \mid t)\, dy \right], \tag{29}$$

where $D_2(t)$ is constant of integration determined by $\int_{S_Y} f_{(Y|X>t)}(y \mid t) dy = 1$. Therefore, like the conditional RCST for $(X \mid Y = s)$ and $(Y \mid X = t)$, the conditional RCST for $(X \mid Y > s)$ and $(Y \mid X > t)$ uniquely determines the conditional PDF's $f_{(X|Y>s)}(t \mid s)$ and $f_{(Y|X>t)}(s \mid t)$ through the relationships (28) and (29).

THEOREM 15. The RCST functions $\eta_{(X|Y>s)}(t \mid s)$ and $\eta_{(Y|X>t)}(s \mid t)$ are the conditional RCST functions of a non negative random vector (X, Y) with support $S_X \times S_Y$ if and only if

$$\frac{\int_t^{\infty} \exp\left[-\int_0^x \eta_{(X|Y>s)}(z \mid s)\, dz \right] dx}{\int_s^{\infty} \exp\left[-\int_0^y \eta_{(Y|X>t)}(z \mid t)\, dz \right] dy} = \frac{u(t)}{v(s)}$$

holds for $S_X \times S_Y$. *Moreover, in this case, the SF of* (X,Y) *can be obtained as*

$$\overline{F}_{(X,Y)}(t,s) = cv(s) \int_t^{\infty} \exp\left[-\int_0^x \eta_{(X|Y>s)}(z \mid s)\, dz \right] dx$$

or as

$$\overline{F}_{(X,Y)}(t,s) = c^* u(t) \int_s^{\infty} \exp\left[-\int_0^y \eta_{(Y|X>t)}(z \mid t)\, dz \right] dy,$$

where c *and* c^* *are constants of integration.*

PROOF. The proof is a consequence of Theorem 11.1 in (Arnold et al., 1999) and (28) and (29).

EXAMPLE 16. The model in (8) is characterized by

$$\eta_{(X|Y>s)}(t \mid s) = \alpha_1(s)\lambda_1(t) - \frac{\lambda_1'(t)}{\lambda_1(t)}$$

and

$$\eta_{(Y|X>t)}(s \mid t) = \alpha_2(t)\lambda_2(s) - \frac{\lambda_2'(s)}{\lambda_2(s)}.$$

EXAMPLE 17. The functions $\eta_{(X|Y>s)}(t \mid s) = \theta(s)$ and $\eta_{(Y|X>t)}(s \mid t) = \tau(t)$ are the conditional RCST functions of a random vector (X,Y) with support $(0,\infty)\times(0,\infty)$ if and only if $\theta(s) = \alpha + \gamma s$ and $\tau(t) = \beta + \gamma t$ where $\alpha, \beta > 0$ and $\gamma \geq 0$. In this case they characterize the Gumbel's type I bivariate exponential distribution with SF $\overline{F}_{(X,Y)}(t,s) = \exp(-\alpha t - \beta s - \gamma ts)$ for $t, s \geq 0$.

Other examples can be obtained from that included in (Arnold et al., 1999).

The FGM family specified by the joint SF of a two dimensional random vector (X,Y),

$$\overline{F}_{(X,Y)}(t,s) = \overline{F}_X(t)\overline{F}_Y(s)\left[1 + \omega\left(1 - \overline{F}_X(t)\right)\left(1 - \overline{F}_Y(s)\right)\right], -1 \leq \omega \leq 1 \tag{30}$$

with specified marginal distributions through $\overline{F}_X(t)$ and $\overline{F}_Y(s)$.

THEOREM 18. The relationships

$$\eta_{(X|Y>s)}(t\mid s)=\eta_X(t)-\frac{2\omega f_X(t)F_Y(s)}{1+\omega\big(2F_X(t)-1\big)F_Y(s)} \tag{31}$$

and

$$\eta_{(Y|X>t)}(s\mid t)=\eta_Y(s)-\frac{2\omega f_Y(s)F_X(t)}{1+\omega\big(2F_Y(s)-1\big)F_X(t)} \tag{32}$$

hold if and only if (X,Y) *follows the FGM family with joint PDF (30).*

PROOF. Assume that (31) holds, then using (28) we have the conditional PDF

$$f_{(X|Y>s)}(t\mid s)=\frac{D_1(s)f_X(t)\big[1+\omega\big(2F_X(t)-1\big)F_Y(s)\big]}{[1-\omega F_Y(s)]}$$

Now applying the boundary condition $\int_0^\infty f_{(X|Y>s)}(x\mid s)dx=1$, we obtain

$$\frac{D_1(s)}{[1-\omega F_Y(s)]}\left[\big(1-\omega F_Y(s)\big)\int_0^\infty f_X(x)dx+2\omega F_Y(s)\int_0^\infty F_X(x)f_X(x)dx\right]=1,$$

thus obtains $D_1(s)=1-\omega F_Y(s)$, and therefore

$$f_{(X|Y>s)}(t\mid s)=f_X(t)\big[1+\omega(2F_X(t)-1)F_Y(s)\big], \tag{33}$$

the conditional PDF of $(X\mid Y>s)$ for FGM model given in (30). Integrating (33) between the limits t to ∞, we get

$$\overline{F}_{(X|Y>s)}(t\mid s)=\big(1-\omega F_Y(s)\big)\int_t^\infty f_X(x)dx+2\omega F_Y(s)\int_t^\infty F_X(x)f_X(x)dx,$$

where $\overline{F}_{(X|Y>s)}(t\mid s)$ denote the SF of $(X\mid Y>s)$. On simplification, we further obtain

$$\overline{F}_{(X|Y>s)}(t\mid s)=\overline{F}_X(t)[1+\omega F_X(t)F_Y(s)],$$

is the conditional SF of FGM with model (30). In a similar manner, using (32) and (29), we can obtain the conditional SF of $(Y\mid X>t)$ for FGM in (30). The other part is straightforward.

In Theorem 4.2, if we consider identical marginals, i.e., when $f_X(t)=f_Y(s)=f(t)$, we have $F_X(t)=F_Y(s)=F(t)$ and $\eta_X(t)=\eta_Y(s)=\eta(t)$. In this case, (31) and (32) are reduced to a single relationship in either t or s, which are illustrated in the following examples.

EXAMPLE 19. Uniform$[0,1]$ marginals. In this case $f(t)=1, F(t)=t$ and $\eta(t)=0$, then

$$\eta_{(X|Y>s)}(t \mid s) = \eta_{(Y|X>t)}(s \mid t) = -\frac{2\omega t}{1+\omega t(2t-1)}.$$

EXAMPLE 20. Exponential marginals, with $f(t) = \lambda e^{-\lambda t}$ we have $\eta(t) = \lambda$, then

$$\eta_{(X|Y>s)}(t \mid s) = \eta_{(Y|X>t)}(s \mid t) = \lambda - \frac{2\omega\lambda e^{-\lambda t}(1-e^{-\lambda t})}{1+\omega(1-2e^{-\lambda t})(1-e^{-\lambda t})}.$$

EXAMPLE 21. Pareto marginals, with $f(t) = (1+t)^{-2}$ and $\eta(t) = 2(1+t)^{-1}$, then

$$\eta_{(X|Y>s)}(t \mid s) = \eta_{(Y|X>t)}(s \mid t) = \frac{2}{1+t} - \frac{2\omega t(1+t)^{-3}}{1+\omega\left(2t^2(1+t)^{-2} - t(1+t)^{-1}\right)}.$$

EXAMPLE 22. Weibull marginals, with $f(t) = ct^{c-1}e^{-t^c}$ and $\eta(t) = t^{-1}\left(1-c(1-t^c)\right)$, then

$$\eta_{(X|Y>s)}(t \mid s) = \eta_{(Y|X>t)}(s \mid t) = \frac{1-c(1-t^c)}{t} - \frac{2\omega c t^{c-1}e^{-t^c}(1-e^{-t^c})}{1+\omega(1-2e^{-t^c})(1-e^{-t^c})}.$$

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SUMMARY

Characterizations of some bivariate models using reciprocal coordinate subtangents

In the present paper, we consider the bivariate version of reciprocal coordinate subtangent (RCST) and study its usefulness in characterizing some important bivariate models. In particular, characterization results are proved for a general bivariate model whose conditional distributions are proportional hazard rate models (see Navarro and Sarabia, 2011), Sarmanov family and Ali-Mikhail-Haq family of bivariate distributions. We also study the relationship between local dependence function and reciprocal subtangent and a characterization result is proved for a bivariate model proposed by Jones (1998). Further, the concept of reciprocal coordinate subtangent is extended to conditionally specified models.

Keywords: Reciprocal coordinate subtangent; Reliability measures; Characterizations; Conditionally specified models.