STATISTICA, anno LXVI, n. 2, 2006

EXPONENTIATED WEIBULL DISTRIBUTION

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1. INTRODUCTION

A new family of distributions, namely the exponentiated exponential distribution was introduced by Gupta *et al.* (1998). The family has two parameters (scale and shape) similar to the Weibull or gamma family. Properties of the distribution were studied by Gupta and Kundu (2001). They observed that many properties of the new family are similar to those of the Weibull or gamma family. Hence the distribution can be used as an alternative to a Weibull or gamma distribution. The two-parameter Weibull and Gamma distributions are the most popular distributions used for analyzing lifetime data. The gamma distribution has wide application other than that in survival analysis. However, its major drawback is that its survival function cannot be obtained in a closed form unless the shape parameter is an integer. This makes the Gamma distribution a little less popular than the Weibull distribution, whose survival function and failure rate have very simple and easy-to-study forms. In recent years the Weibull distribution has become rather popular in analyzing lifetime data because in the presence of censoring it is very easy to handle.

In this paper we consider the exponentiated Weibull family that was introduced by Mudholkar and Srivastava (1993). It has a scale parameter and two shape parameters. The Weibull family and the exponentiated exponential family are found to be particular cases of this family. The distribution has been compared with the two-parameter Weibull and gamma distributions with respect to failure rate. The maximum likelihood estimators of the parameters and their asymptotics have been discussed. Finally the distribution has been fitted to a real life data and the fit has been found to be good.

2. EXPONENTIATED WEIBULL DISTRIBUTION

The exponentiated Weibull (EW) distribution is defined in the following way.

It has distribution function given by

$$G_\alpha(z) = [1-\exp\{-(\lambda z)^\gamma\}]^\alpha\,,\ z>0, \quad \alpha,\lambda,\gamma>0\,, \tag{1}$$

and therefore its probability density function (pdf) is of the form

$$g_\alpha(z) = \alpha\gamma\lambda^\gamma z^{\gamma-1}[1-\exp\{-(\lambda z)^\gamma\}]^{\alpha-1}\exp\{-(\lambda z)^\gamma\},\ \ z>0\,.$$

The corresponding survival function is

$$S_\alpha(z) = 1-[1-\exp\{-(\lambda z)^\gamma\}]^\alpha$$

and the failure rate is

$$r_\alpha(z) = \frac{\alpha\gamma\ \lambda^\gamma z^{\gamma-1}\exp\{-(\lambda z)^\gamma\}[1-\exp\{-(\lambda z)^\gamma\}]^{\alpha-1}}{(1-[1-\exp\{-(\lambda z)^\gamma\}]^\alpha)}\,.$$

Here (α,γ) denote the shape parameters and λ is the scale parameter. For $\gamma=1$, it represents the exponentiated exponential (EE) family, and for $\alpha=1$, it represents the Weibull family. Thus, EW is a generalization of the exponentiated exponential family as well as the Weibull family. EW distribution also has a very nice physical interpretation. If there are *n* components in a parallel system and the lifetimes of the components are independently and identically distributed as EW, then the system lifetime is also EW.

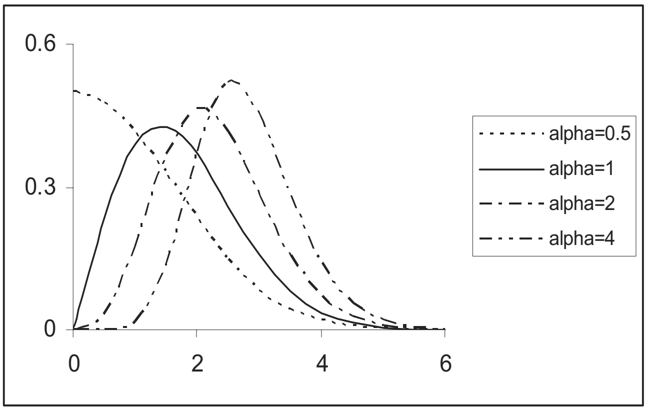

Figure 1 – Showing the EW pdf for λ =0.5, γ =2, when α = 0.5, 1, 2, 4.

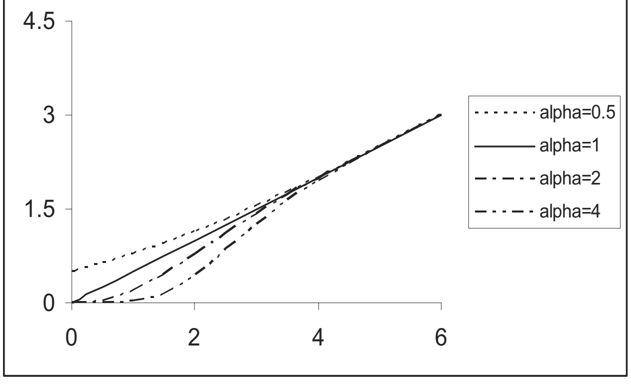

Figure 2 – Showing the failure rate curves for λ =0.5, γ =2, when α = 0.5, 1, 2, 4.

Figure 1 shows that the density function of EW is unimodal and, for fixed λ and γ, it becomes more and more symmetric as α increases.

Figure 2 shows that the failure rate is a non-decreasing function of α for fixed λ and γ.

We observe that for the EW distribution,

(i) if $\alpha = \gamma = 1$, the failure rate is constant;

(ii) if $\alpha = 1$, the failure rate is increasing for $\gamma > 1$ and decreasing for $\gamma < 1$;

(iii) if $\gamma = 1$, the failure rate is increasing for $\alpha > 1$ and decreasing for $\alpha < 1$.

Let us consider the Gamma and Weibull distributions with scale parameter λ and shape parameter $\delta = \alpha\gamma$:

$$f_G(x) = \frac{\lambda^\delta}{\Gamma(\delta)} x^{\delta-1} \exp(-\lambda x), \quad x > 0$$

$$f_W(x) = \delta\lambda^\delta x^{\delta-1} \exp\{-(\lambda x)^\delta\}, \quad x > 0 .$$

A comparison of the failure rates of the three distributions are given in the table below.

TABLE 1

Failure rate of the distributions

Parameter	Gamma	Weibull	EW
$\alpha = \gamma = 1$	Constant	Constant	Constant
$\alpha = 1$	Increasing from 0 to λ for $\gamma > 1$ and decreasing from ∞ to λ for $\gamma < 1$	Increasing from 0 to ∞ for $\gamma > 1$ and decreasing from ∞ to 0 for $\gamma < 1$	Increasing from 0 to ∞ for $\gamma > 1$ and decreasing from ∞ to 0 for $\gamma < 1$
$\gamma = 1$	Increasing from 0 to λ for $\alpha > 1$ and decreasing from ∞ to λ for $\alpha < 1$	Increasing from 0 to ∞ for $\alpha > 1$ and decreasing from ∞ to 0 for $\alpha < 1$	Increasing from 0 to λ for $\alpha > 1$ and decreasing from ∞ to λ for $\alpha < 1$
$\alpha, \gamma > 1$	Increasing from 0 to λ	Increasing from 0 to ∞	Increasing from 0 to ∞
$\alpha, \gamma < 1$	Decreasing from ∞ to λ	Decreasing from ∞ to 0	Decreasing from ∞ to 0

Thus, the failure rate of EW behaves more like the failure rate of the Weibull distribution than the Gamma distribution.

Now we consider the moments of the EW distribution. From the formulas 3.381(4) in Gradshteyn and Ryzhik (1965) and the binomial expansion, the k-th moment of exponentiated Weibull variable Z with distribution function (1) is obtained as

$$E(Z^k) = \begin{cases} \alpha\lambda^{-k}\Gamma\left(\frac{k}{\gamma}+1\right)\sum_{i=0}^{\alpha-1}\binom{\alpha-1}{i}(-1)^i(i+1)^{-\frac{k}{\gamma}-1}, & \text{if } \alpha \in N \\ \alpha\lambda^{-k}\Gamma\left(\frac{k}{\gamma}+1\right)\sum_{i=0}^{\infty}\frac{{}_{\alpha-1}P_i}{i!}(-1)^i(i+1)^{-\frac{k}{\gamma}-1}, & \text{if } \alpha \notin N, \text{ for } k = 0,1,2,..., \end{cases} \tag{2}$$

where ${}_aP_i = a(a-1)(a-2)....(a-i+1)$, and N is the set of natural numbers.

Since (2) is a convergent series for all $k \geq 0$, all moments exist.
In particular,

$$\mathrm{E}(Z) = \begin{cases} \alpha\lambda^{-1}\Gamma\left(\dfrac{1}{\gamma}+1\right)\displaystyle\sum_{i=0}^{\alpha-1}\binom{\alpha-1}{i}(-1)^i(i+1)^{-\frac{1}{\gamma}-1}, & \text{if } \alpha \in N \\ \alpha\lambda^{-1}\Gamma\left(\dfrac{1}{\gamma}+1\right)\displaystyle\sum_{i=0}^{\infty}\frac{{}_{\alpha-1}P_i}{i!}(-1)^i(i+1)^{-\frac{1}{\gamma}-1}, & \text{if } \alpha \notin N \end{cases}$$

and

$$\mathrm{E}(Z^2) = \begin{cases} \alpha\lambda^{-2}\Gamma\left(\dfrac{2}{\gamma}+1\right)\displaystyle\sum_{i=0}^{\alpha-1}\binom{\alpha-1}{i}(-1)^i(i+1)^{-\frac{2}{\gamma}-1}, & \text{if } \alpha \in N \\ \alpha\lambda^{-2}\Gamma\left(\dfrac{2}{\gamma}+1\right)\displaystyle\sum_{i=0}^{\infty}\frac{{}_{\alpha-1}P_i}{i!}(-1)^i(i+1)^{-\frac{2}{\gamma}-1}, & \text{if } \alpha \notin N \end{cases}$$

The variance of Z can be easily obtained from the above.

For $\gamma = 2$, we get the exponentiated Rayleigh distribution. From formula 3.462 in Gradshteyn and Ryzhik (1965), the moment generating function of the exponentiated Rayleigh distribution is obtained as

$$\begin{aligned} \mathrm{E}[\exp(tZ)] &= \alpha\sum_{i=0}^{\alpha-1}\binom{\alpha-1}{i}\frac{(-1)^i}{i+1}\exp\left(\frac{\lambda^{-2}t^2}{8(i+1)}\right)D_{-2}\left(-\frac{\lambda^{-1}t}{\sqrt{2(i+1)}}\right), \text{ if } \alpha \in N \\ &= \alpha\sum_{i=0}^{\infty}\frac{{}_{\alpha-1}P_i}{i!}\frac{(-1)^i}{i+1}\exp\left(\frac{\lambda^{-2}t^2}{8(i+1)}\right)D_{-2}\left(-\frac{\lambda^{-1}t}{\sqrt{2(i+1)}}\right), \text{ if } \alpha \notin N, \end{aligned}$$

where $D_{-2}(a) = -(\sqrt{\pi/2})\exp\left(\dfrac{a^2}{4}\right)[(\sqrt{\pi/2})\exp\left(-\dfrac{a^2}{2}\right) - a(1-\Phi(a/\sqrt{2}))]$ (See formula 9.254 in Gradshteyn and Ryzhik (1965), and Φ (.) is the standardized normal distribution function.

3. MAXIMUM LIKELIHOOD ESTIMATORS AND FISHER'S INFORMATION MATRIX

In this section we discuss the maximum likelihood estimators of the parameters of the EW distribution and their asymptotic properties.

Let $Z_1, Z_2, ..., Z_n$ be a random sample of size n from the EW distribution given by (1). Then the log likelihood function (LL) comes out to be

$$\begin{aligned} L(\alpha,\lambda,\gamma) = {} & n\ln\alpha + n\ln\gamma + n\gamma\ln\lambda + (\gamma-1)\sum\ln z_i + \\ & + (\alpha-1)\sum\ln[1-\exp\{-(\lambda z_i)^\gamma\}] - \sum(\lambda z_i)^\gamma \end{aligned} \tag{3}$$

Therefore the MLEs of α, λ, γ which maximize (3) must satisfy the normal equations given by

$$\frac{\partial}{\partial \alpha} L(\alpha,\lambda,\gamma) = \frac{n}{\alpha} + \sum \ln[1-\exp(-(\lambda z_i)^\gamma)] = 0 \tag{4}$$

$$\frac{\partial}{\partial \lambda} L(\alpha,\lambda,\gamma) = \frac{n\gamma}{\lambda} + (\alpha-1)\gamma\lambda^{\gamma-1}\sum \frac{e^{-(\lambda z_i)^\gamma}}{1-e^{-(\lambda z_i)^\gamma}} z_i^\gamma - \gamma\lambda^{\gamma-1}\sum z_i^\gamma = 0 \tag{5}$$

$$\begin{aligned}\frac{\partial}{\partial \gamma} L(\alpha,\lambda,\gamma) &= \frac{n}{\gamma} + n\ln\lambda + \sum \ln z_i \\ &\quad + (\alpha-1)\lambda^\gamma \sum \frac{\exp\{-(\lambda z_i)^\gamma\}}{1-\exp\{-(\lambda z_i)^\gamma\}} z_i^\gamma \ln(\lambda z_i) - \lambda^{\gamma 1}\sum z_i^\gamma \ln(\lambda z_i) = 0\end{aligned} \tag{6}$$

From (4) we obtain the MLE of α as a function of (λ,γ), say $\hat{\alpha}(\lambda,\gamma)$ given by

$$\hat{\alpha} = \hat{\alpha}(\lambda,\gamma) = -\frac{n}{\sum \ln[1-\exp(-(\lambda z_i)^\gamma)]} \tag{7}$$

Multiplying (5) by λ/γ we get

$$n + \lambda^\gamma[(\alpha-1)\sum \frac{\exp\{-(\lambda z_i)^\gamma\}}{1-\exp\{-(\lambda z_i)^\gamma\}} z_i^\gamma - \sum z_i^\gamma] = 0 \tag{8}$$

Subtracting $\ln\lambda$ times (8) from (6) we have

$$\frac{n}{\gamma} + \sum \ln z_i + \lambda^\gamma[(\alpha-1)\sum \frac{\exp\{-(\lambda z_i)^\gamma\}}{1-\exp\{-(\lambda z_i)^\gamma\}} z_i^\gamma \ln z_i - \sum z_i^\gamma \ln z_i] = 0 \tag{9}$$

Using (7) in (8) and (9) we get two equations, which are satisfied by the MLEs $\hat{\lambda}$ and $\hat{\gamma}$ of λ and γ, respectively.

Because of the complicated form of the likelihood equations, algebraically it is very difficult to prove that the solution to the normal equations give a global maximum or at least a local maximum, though numerical computation during data analysis showed the presence of at least local maximum. However, the following properties of the log-likelihood function have been algebraically noted:

(a) for given (λ,γ), L is a strictly concave function of α. Further, the optimal value of α, given by (7), is a concave increasing function of λ for given γ;

(b) for given (α,γ), and $\alpha \geq 1$, L is a strictly concave function of λ.

To obtain confidence interval we use the asymptotic normality results. We have that, if $\hat{\theta} = (\hat{\alpha},\hat{\lambda},\hat{\gamma})'$ denotes the MLE of $\theta = (\alpha,\lambda,\gamma)'$, then

$$\sqrt{n}(\hat{\theta}-\theta) \to N_3(0, I^{-1}(\theta)) \tag{10}$$

where $I^{-1}(\theta)$ is Fisher's information matrix given by

$$I(\theta)=-\frac{1}{n}\begin{pmatrix} \mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha^2}\right) & \mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right) & \mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha\partial\gamma}\right) \\ \mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right) & \mathrm{E}\left(\frac{\partial^2 L}{\partial\lambda^2}\right) & \mathrm{E}\left(\frac{\partial^2 L}{\partial\lambda\partial\gamma}\right) \\ \mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha\partial\gamma}\right) & \mathrm{E}\left(\frac{\partial^2 L}{\partial\lambda\partial\gamma}\right) & \mathrm{E}\left(\frac{\partial^2 L}{\partial\gamma^2}\right) \end{pmatrix}$$

We give below expressions for some elements of the Fisher's Information matrix, which may be useful in practice.

For $\alpha > 2$,

$$\mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha^2}\right) = -\frac{n}{\alpha^2}$$

$$\mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right) = \frac{n\gamma\alpha}{\lambda}\left[\frac{\psi(\alpha)-\psi(1)}{\alpha-1}-\frac{\psi(\alpha+1)-\psi(1)}{\alpha}\right]$$
$$= \frac{n\gamma}{\lambda}\left[\frac{\alpha}{\alpha-1}\{\psi(\alpha)-\psi(1)\}-\{\psi(\alpha+1)-\psi(1)\}\right],$$

using formula 4.293(8) in Gradshteyn and Ryzhik, 1965,

$$\mathrm{E}\left(\frac{\partial^2 L}{\partial\lambda^2}\right) = -\frac{n\gamma^2}{\lambda^2}\left[\frac{1}{\gamma}+\frac{\alpha(\alpha-1)}{(\alpha-2)}\{\psi'(1)-\psi'(\alpha-1)\}+\{\psi(\alpha-1)-\psi'(1)\}^2\right]$$
$$-\frac{n\gamma^2}{\lambda^2}[\psi'(1)-\psi(\alpha)+\{\psi(\alpha)-\psi(1)\}^2]-$$
$$\frac{n\alpha(\alpha-1)\gamma(\gamma-1)}{\lambda^2}\left[\frac{\psi(\alpha)-\psi(1)}{\alpha-1}-\frac{\psi(\alpha+1)-\psi(1)}{\alpha}\right]$$
$$-\frac{n\gamma(\gamma-1)}{\lambda^2}[\psi(\alpha+1)-\psi(1)],$$

where Ψ (.) is the digamma function and Ψ' (.) is its first order derivative.

For $0<\alpha \leq 2$,

$$\mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha^2}\right) = -\frac{n}{\alpha^2}$$

$$\mathrm{E}\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right) = \frac{n\alpha\gamma}{\lambda}\int_0^\infty x\exp(-2x)(1-\exp(-x))^{\alpha-2}\,dx < \infty$$

$$\mathrm{E}\left(\frac{\partial^2 L}{\partial\lambda^2}\right) = -\frac{n\gamma}{\lambda^2} - \frac{n\alpha(\alpha-1)\gamma}{\lambda^2}\int_0^\infty x^2\exp(-2x)(1-\exp(-x))^{\alpha-3}\,dx$$
$$-\frac{n\alpha\gamma(\gamma-1)}{\lambda^2}\int_0^\infty x\exp(-x)(1-\exp(-x))^{\alpha-2}\,dx < \infty.$$

θ being unknown, we estimate $I^{-1}(\theta)$ by $I^{-1}(\hat{\theta})$ and can use this to obtain asymptotic confidence intervals for α, λ and γ.

4. DATA ANALYSIS

In this section we use uncensored data set from Nichols and Padgett, 2006. The data gives 100 observations on breaking stress of carbon fibres (in Gba):

3.7	2.74	2.73	2.5	3.6
3.11	3.27	2.87	1.47	3.11
4.42	2.41	3.19	3.22	1.69
3.28	3.09	1.87	3.15	4.9
3.75	2.43	2.95	2.97	3.39
2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.2	3.33	2.55
3.31	3.31	2.85	2.56	3.56
3.15	2.35	2.55	2.59	2.38
2.81	2.77	2.17	2.83	1.92
1.41	3.68	2.97	1.36	0.98
2.76	4.91	3.68	1.84	1.59
3.19	1.57	0.81	5.56	1.73
1.59	2	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18
3.51	2.17	1.69	1.25	4.38
1.84	0.39	3.68	2.48	0.85
1.61	2.79	4.7	2.03	1.8
1.57	1.08	2.03	1.61	2.12
1.89	2.88	2.82	2.05	3.65

Both exponentiated exponential distribution and exponentiated Weibull distribution were fitted by the method of maximum likelihood. A quasi-Newton algorithm in R (Ihaka and Gentleman, 1996) was used to solve the likelihood equations, and the following estimates were obtained:

For EE distribution with scale parameter λ and shape parameter α,

$\hat{\lambda} = 0.50345, \hat{\alpha} = 2.5808$, with $-L = 167.044$.

For EW distribution given by (1),

$$\hat{\alpha} = 1.17262, \hat{\lambda} = 0.35756, \hat{\gamma} = 2.57902, \text{ with } -L = 141.369,$$

where $-L$ denotes the negative logarithm of the maximized likelihood.

Thus, it follows by the standard likelihood ratio test that the exponentiated Weibull distribution is a much better model than the exponentiated exponential model for the given data. This observation is confirmed by the probability plots corresponding to the two fits, shown in Figure 3.

(a)

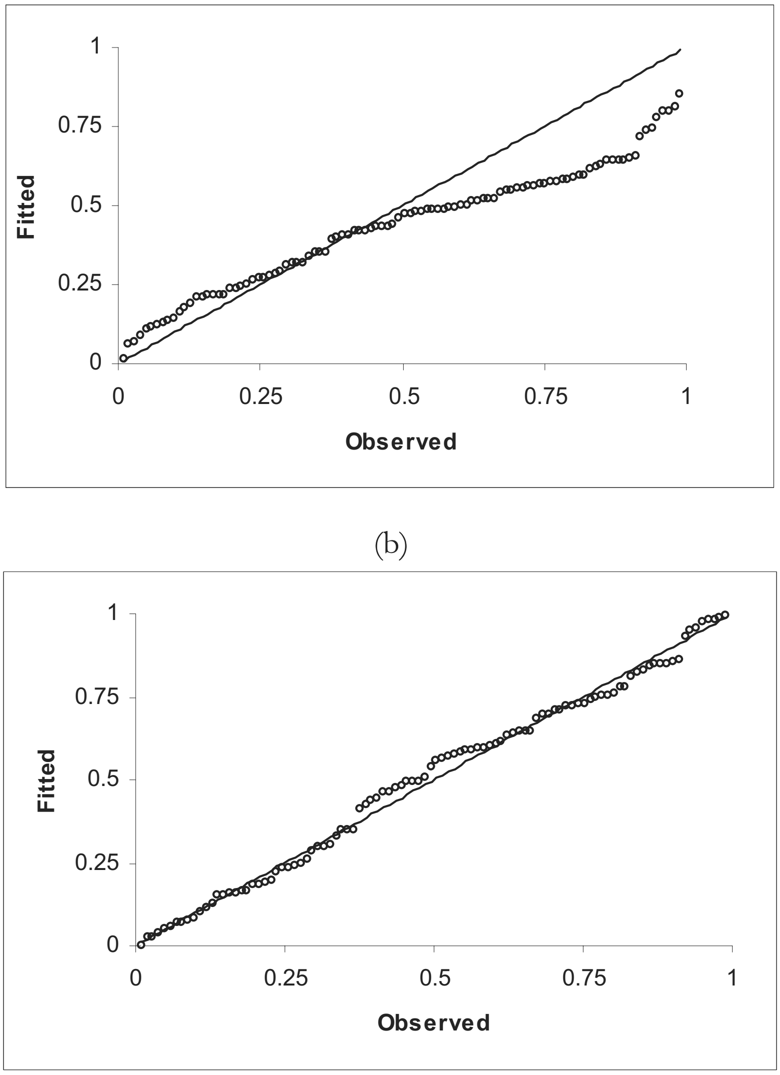

Figure 3 – Probability plots for the models based on the exponentiated exponential distribution (top) and the exponentiated Weibull distribution (bottom).

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ACKNOWLEDGEMENTS

The authors gratefully acknowledge the suggestions given by the anonymous referees, which have immensely helped to improve the presentation of the paper.

REFERENCES

L.J. BAIN (1976), *Statistical analysis of reliability and life testing model*, Marcel and Dekker Inc., New York.

I.S. GRADSHTEYN, I.M. RYHZIK (1965), *Tables of integrals, series and products*, Academic Press, New York.

R.D. GUPTA, D. KUNDU (2001), *Exponentiated exponential family: an alternative to gamma and Weibull*, "Biometrical Journal", 43(1), pp. 117-130.

R. IHAKA, R. GENTLEMAN (1996), *R: A language for data analysis and graphics*, "Journal of Computational and Graphical Statistics", 5, pp. 299-314.

J.F. LAWLESS (1982), *Statistical models and methods for lifetime data*, Wiley, New York.

H. LINHART, W. ZUCCHINI (1986), *Model selection*, Wiley, New York.

N.L. JOHNSON, S. KOTZ, N. BALAKRISHNAN (1995), *Continuous univariate distributions,* Vol. 1, Wiley, New York.

G.S. MUDHOLKAR, D.K. SRIVASTAVA (1993), *Exponentiated Weibull family for analyzing bathtub failure rate data*, "IEEE Transactions on Reliability", 42, pp. 299-302.

M.D. NICHOLS, W.J. PADGETT (2006), A *bootstrap control chart for Weibull percentiles*, "Quality and Reliability Engineering International", 22, pp. 141-151.

RIASSUNTO

La distribuzione di Weibull esponenziata

In questo lavoro studiamo la famiglia di distribuzioni chiamata "famiglia di Weibull esponenziata". La distribuzione ha tre parametri (uno di scala e due di forma), e la distribuzione di Weibull e la distribuzione esponenziale esponenziata, discusse da Gupta *et al.* (1998), sono casi particolari di tale famiglia. La funzione di sopravvivenza, il tasso di fallimento e i momenti delle distribuzioni sono stati ottenuti utilizzando alcune particolari formule. Il comportamento del tasso di fallimento è stato studiato e confrontato con quelli delle distribuzioni Weibull e Gamma. La distribuzione è stata adattata a dati reali e l'adattamento si è dimostrato buono.

SUMMARY

Exponentiated Weibull distribution

In this paper we study the family of distributions termed as exponentiated Weibull distribution. The distribution has three parameters (one scale and two shape) and the Weibull distribution and the exponentiated exponential distribution, discussed by Gupta, *et al.* (1998), are particular cases of it. The survival function, failure rate and moments of the distributions have been derived using certain special formulas. The behavior of the failure rate has been studied and compared with those of the Weibull and Gamma distributions. The distribution has been fitted to a real life data set and the fit has been found to be very good.