STATISTICA, anno LXXIV, n. 1, 2014

MULTIVARIATE NORMAL-LAPLACE DISTRIBUTION AND PROCESSES

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1. INTRODUCTION

The normal-Laplace distribution, which results from the convolution of independent normal and Laplace random variables is introduced by Reed and Jorgensen (2004). Normal-Laplace distribution is a new distribution which (in its symmetric form) behaves somewhat like the normal distribution in the middle of its range, and like the Laplace distribution in its tails. In general, the normal-asymmetric Laplace distribution can be used to model positively skewed, negatively skewed as well as symmetric data (see Lishamol (2008)). Reed and Jorgensen (2004) also introduced a generalized normal-Laplace distribution, which is useful in financial applications for obtaining an alternative stochastic process model to Brownian motion for logarithmic prices, in which the increments exhibit fatter tails than the normal distribution. Reed (2007) developed Brownian-Laplace motion for modelling financial asset price returns.

Multivariate normal distribution, a generalization of univariate normal distribution is studied by various authors. The multivariate normal distribution has applications in statistical inference, image processing etc. Ernst (1998) introduced multivariate extension of symmetric Laplace distributions via an elliptic contouring. Many properties in the univariate laws can be extended to this class of distributions. With an appropriate limit of the parameters of multivariate hyperbolic distributions, one can obtain a multivariate and asymmetric extension of the Laplace laws, see Blaesid (1981).

The analysis of time series is usually based on the assumption that an observed time series is a Gaussian process. But in many practical situations, the data show a tendency of asymmetry or follow heavy tailed distributions. The use of autoregressive representation of a stationary time series or the innovation approach in the analysis of time series has recently been attracting attention of many researchers and this time domain approach provides answers to many problems. Linear AR(1) structure is simple, useful and interpretable in a wide range of contexts. Theoretical results concerning stationarity, moments and correlation structure have been proven for many particular AR(1) models. Sim (1994) discussed a general theory of model-building approach that consists of model identification, estimation, diagnostic checking and forecasting for a model with a given marginal distribution. Damsleth and El-Shaaravi (1989) developed a time series model with Laplace noise as an alternative to the normal distribution. Jose et al (2008) and Lishamol and Jose (2009,2010) developed a unified theory for Gaussian and non-Gaussian autoregressive processes through normal-Laplace and generalized normal-Laplace distributions. Jose et al (2010), Lishamol and Jose (2011) discussed time series models with Skew Laplace III Marginals and geometric normal-Laplace marginals. Jose and Manu (2011) developed autoregressive processes with generalized Laplacian distribution marginals. Krishna and Jose (2011) developed autoregressive processes with Marshall-Olkin generalized asymmetric Laplace marginal distributions. George and George (2013) developed processes with Marshall-Olkin Esscher transformed Laplace distribution. Jose and Mariyamma (2012) developed integer valued autoregressive processes with generalized discrete Mittag-Leffler marginals. Jose and Manu (2012) developed a product autorgressive model with log-Laplace marginal distribution.

The present article is organized as follows. In Section 2, we consider the univariate normal-Laplace distribution. In Section 3, we introduce the multivariate normal-Laplace distribution and study its properties. Section 4 deals with first order autoregressive processes with multivariate normal-Laplace marginals. In section 5, multivariate generalized normal-Laplace distribution is introduced. In Section 6, we introduce the geometric generalized normal-Laplace distribution and study its properties. Section 7 deals with the estimation of parameters. Some applications are discussed in Section 8.

2. NORMAL-LAPLACE DISTRIBUTION AND ITS PROPERTIES

The normal-Laplace distribution introduced by Reed and Jorgensen (2004), arises as the convolution of an independent normal and an asymmetric Laplace densities. A normal-Laplace random variable X with parameters μ, σ^2, α and β can be represented as

$$X \stackrel{d}{=} Z + W \tag{1}$$

where Z and W are independent random variables with Z following normal distribution with mean μ and variance σ^2 and W following an asymmetric Laplace distribution with parameters α, β. The corresponding normal-Laplace distribution shall be denoted by $\mathrm{NL}(\mu, \sigma^2, \alpha, \beta)$. Various results on normal-Laplace distribution are available in Reed (2006). A normal-Laplace random variable X can also be expressed as

$$X \stackrel{d}{=} Z + E_1 - E_2 \tag{2}$$

where E_1 and E_2 are independent exponential variables with parameters α and β respectively and $Z \sim N(\mu, \sigma^2)$ independent of E_1 and E_2. Hence the characteristic function (c.f.) of $NL(\mu, \sigma^2, \alpha, \beta)$ can be obtained as the product of the c.f.'s of its normal and Laplace components and is given by Reed and Jorgensen (2004) as,

$$\varphi_X(t) = \left[\exp(i\mu t - \frac{\sigma^2}{2}t^2)\right]\left[\frac{\alpha\beta}{(\alpha - it)(\beta + it)}\right]$$
$$= \left[\exp(i\mu t - \frac{\sigma^2}{2}t^2)\right]\left[\frac{1}{1 + t^2\gamma^2 - i\eta t}\right],$$

$$\textit{where } \gamma^2 = \frac{1}{\alpha\beta},\ \eta = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

The normal-Laplace distribution is infinitely divisible and is closed under linear transformation. The mean, variance and cumulants exist for the distribution. Fiure 1 shows the probability density functions of the normal-Laplace, Laplace and normal distributions. Figure 2(a), (b) represent the density functions of the normal Lapalce distribution with changes in values of α and β respectively.

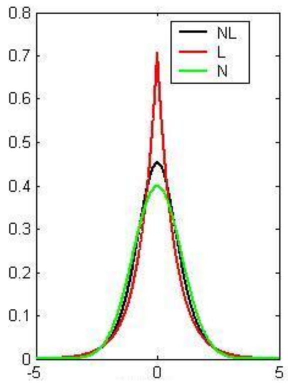

Figure 1 - Probability density functions of the normal-Laplace, Laplace and normal distributions

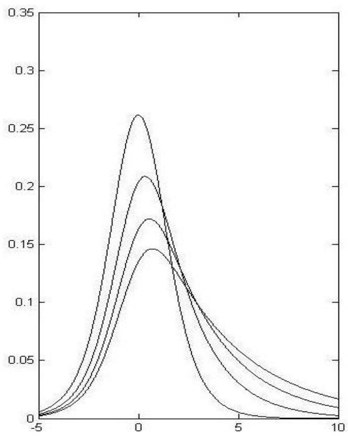

Figure 2 - (a) changes in values of α

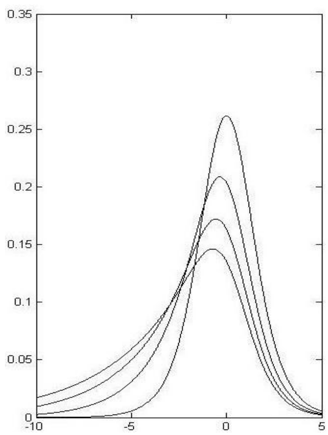

Figure 2 - (b) changes in values of β

3. MULTIVARIATE NORMAL-LAPLACE DISTRIBUTION

A multivariate extension of the normal-Laplace distribution of Reed and Jorgensen (2004), namely the multivariate normal-Laplace distribution can be obtained as the convolution of multivariate normal (with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$) and multivariate asymmetric Laplace (with parameters $\mathbf{m}$ and V) random vectors. The c.f. of multivariate normal-Laplace distribution is given by

$$\phi_X(t)=\left(\exp\left(it'\boldsymbol{\mu}-\frac{1}{2}t'\boldsymbol{\Sigma}t\right)\right)\left(\frac{1}{1+\frac{1}{2}t'Vt-im't}\right),\ t,m\in\mathbb{R}^p,\boldsymbol{\Sigma}>0,V>0. \tag{3}$$

When $m=0$, we get the symmetric case. Then the c.f. is given by

$$\phi_X(t)=\left(\exp\left(it'\boldsymbol{\mu}-\frac{1}{2}t'\boldsymbol{\Sigma}t\right)\right)\left(\frac{1}{1+\frac{1}{2}t'Vt}\right),t\in\mathbb{R}^p,\boldsymbol{\Sigma}>0,V>0. \tag{4}$$

3.1 Some properties

A p-variate normal-Laplace distribution with parameters $\boldsymbol{\mu},\boldsymbol{\Sigma}$ and V can be denoted by $NL_p(\boldsymbol{\mu},\boldsymbol{\Sigma},V)$. Let $X\sim NL_p(\boldsymbol{\mu},\boldsymbol{\Sigma},V)$, then X can be expressed as

$$X\overset{d}{=}Z+Y \tag{5}$$

where Z and Y are independent random vectors with Z following a p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and dispersion matrix $\boldsymbol{\Sigma}\left(\mathrm{N}_p\left(\boldsymbol{\mu},\boldsymbol{\Sigma}\right)\right)$ and Y following a p-variate symmetric Laplace distribution with parameter $V\left(L_p\left(V\right)\right)$). Another representation is

$$X\overset{d}{=}Z+\sqrt{W}Y, \tag{6}$$

where Z follows a p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and dispersion matrix $\boldsymbol{\Sigma}$, W is a standard exponential variable and Y follows p-variate normal distribution with mean vector 0 and dispersion matrix $\boldsymbol{\Sigma}$.

Moments.

The k$^{\text{th}}$ moment $M_k(x,\boldsymbol{\mu})$ of a random vector x and $\boldsymbol{\mu}$ is defined by

$$M_k(x,\boldsymbol{\mu})=E[(x-\boldsymbol{\mu})\otimes(x-\boldsymbol{\mu})'\otimes\cdots\otimes(x-\boldsymbol{\mu})'\otimes(x-\boldsymbol{\mu})],$$

if $k=2m-1$ and

$$M_k(x,\mu) = E[(x-\mu)\otimes(x-\mu)'\otimes\cdots\otimes(x-\mu)\otimes(x-\mu)'],$$

if $k = 2m$, where the Kronecker product of matrices $A = (a_{ij}) : m\times n$ and $B : p\times q$ is defined as a $mp\times nq$ matrix $A\otimes B = [a_{ij}B]$. When $\mu = 0$, we write only $M_k(x)$. By vector differentiation of the c.f. of a random vector, we can obtain the moment vector,

$$M_k = \frac{1}{i^k}\frac{\partial^k \phi_X(t)}{\partial t\partial t'\ldots}\bigg|_{t=0}.$$

The mean vector and variance-covariance matrix of $NL_p(\mu,\Sigma,V)$ can be obtained as

$$E(X) = \mu \qquad Cov(X) = \Sigma + V. \tag{7}$$

Cumulants.

The k^{th} cumulant of random vector x is denoted by $C_k(x)$. The cumulants are obtained as the matrix derivatives of the function

$$\varphi_X(t) = \ln\phi_X(t)$$

by

$$C_k = \frac{1}{i^k}\frac{\partial^k \varphi_X(t)}{\partial t\partial t'\ldots}\bigg|_{t=0}.$$

For multivariate normal-Laplace distribution,

$$C_1 = \mu \qquad C_2 = \Sigma + V.$$

The class of elliptical distributions.

The class of elliptical distributions, introduced by Kelker (1970), is a generalization of multivariate normal distributions. These distributions are symmetric and may not adequately represent the data when some asymmetry is present.

DEFINITION 1. *The random vector X has a multivariate elliptical distribution if its c.f. can be expressed as*

$$\phi_X(t) = \exp(it'V)\psi(\frac{1}{2}t'\Sigma t) \tag{8}$$

for some column vector μ, $n\times n$ *positive matrix* Σ *and for some function* $\psi(t)\in\psi_n$, *which is called the characteristic generator.*

The multivariate normal and symmetric Laplace distributions belong to elliptical family, since the c.f. of multivariate normal and symmetric Laplace distributions can be factorized as (8). As a consequence the multivariate normal-Laplace distribution belongs to the class of elliptical distributions because the sum of elliptical distributions is elliptical, see Fang et al. (1987). This property is very important when we deal with portfolio of assets represented by sum.

Marginal distributions of elliptical distributions are also elliptical distributions. So the marginal distributions of multivariate normal-Laplace distributions are also elliptical. Also all odd order moments of an elliptical distribution are zero and hence the result holds for multivariate normal-Laplace distribution.

Infinite divisibility.

The multivariate normal-Laplace distribution is infinitely divisible. Since the c.f. of $NL_p(\boldsymbol{\mu}, \Sigma, V)$ can be written as

$$\phi_X(t) = \left[e^{\left(it'\frac{\boldsymbol{\mu}}{n} - \frac{1}{2}t'\frac{\Sigma}{n}t\right)} \left(\frac{1}{1+\frac{1}{2}t'Vt} \right)^{\frac{1}{n}} \right]^n,$$

for any integer $n > 0$. The term in brackets is the c.f. of a random vector expressed as $Z+Y$, where $Z \sim N_p(\frac{\boldsymbol{\mu}}{n}, \frac{\Sigma}{n})$ and Y following a multivariate generalized symmetric Laplace distribution with parameters V, $\frac{1}{n}\left(\mathrm{L}_p(V, \frac{1}{n})\right)$.

Property 1. If $X \sim \mathrm{NL}_p(\boldsymbol{\mu}, \Sigma, V)$ and $Y = AX + b$, where A is a $p \times p$ matrix and $b \in \mathbb{R}^p$, then $Y \sim \mathrm{NL}_p(A\boldsymbol{\mu} + b, A\Sigma A', AVA')$.

PROOF. The c.f. of Y . is

$$\begin{aligned} \phi\, Y(t) &= E\left(e^{it'(AX+b)}\right) \\ &= e^{it'b} \phi\, X(A't) \\ &= \frac{e^{it'(A\boldsymbol{\mu}+b) - \frac{1}{2}t'A\Sigma A'}}{1+\frac{1}{2}t'AVA't}. \end{aligned}$$

Hence $Y \sim NL_p(A\boldsymbol{\mu} + b, A\Sigma A', AVA')$.

Property 2. If $X \sim NL_p(\boldsymbol{\mu}, \Sigma, V)$, *then* $a'Xa \in \mathbb{R}^p$ follows a univariate normal-Laplace distribution.

PROOF. This theorem can also be easily be proved using the c.f.

Property 3. Let $X \sim \mathrm{NL}_p(\boldsymbol{\mu}, \Sigma, V)$ and partition $X, \boldsymbol{\mu}, \Sigma$ and V as

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \end{bmatrix} \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \end{bmatrix} \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \\ \end{bmatrix} \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \\ \end{bmatrix},$$

where X_1 and μ_1 are $k \times 1$ vectors and Σ_{11} and V_{11} are $k \times k$ matrices. Then $X_1 \sim NL_p(\mu_1, \Sigma_{11}, V_{11})$ and $X_1 \sim NL_p(\mu_2, \Sigma_{22}, V_{22})$. When $\Sigma_{12} = 0$, X_1 and X_2 are independently distributed.

PROOF. By using the joint c.f., we can easily prove the result.

DEFINITION 2. A random variable X with c.f. φ is said to be semi-self decomposable, if for some $0 < a < 1$, there exists a c.f. φ_a such that $\varphi(t) = \varphi_X(at)\,\varphi_a(t), \forall\, t \in \mathbb{R}$. If this relation holds for every $0 < a < 1$, then φ is self-decomposable or the corresponding distribution belongs to L class.

Property 4. Multivariate normal-Laplace c.f. φ is self decomposable or the corresponding distribution belongs to the class L.

PROOF. The c.f. of multivariate normal-Laplace distribution is

$$\begin{aligned}
\phi(t) &= \left(\exp\left(it'\boldsymbol{\mu} - \frac{1}{2}t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}t'Vt}\right) \\
&= \left(\exp\left(iat'\boldsymbol{\mu} - \frac{1}{2}a^2 t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}a^2 t'Vt}\right) \\
&\quad \times \left(\exp\left(it'(1-a)\boldsymbol{\mu} - \frac{1}{2}t'(1-a^2)\Sigma t\right)\right)\left[a^2 + (1-a^2)\frac{1}{1+\frac{1}{2}t'Vt}\right] \\
&= \phi_X(at)\phi_a(t).
\end{aligned}$$

The second term in the expression is also a c.f., which can be seen in (10).

4. MULTIVARIATE NORMAL-LAPLACE PROCESSES

Consider the usual linear, additive first order autoregressive model given by

$$X_n = aX_{n-1} + \varepsilon_n, \quad 0 < a \le 1, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{9}$$

where X_n and innovations ε_n are independent p - variate random vectors.

In terms of c.f., we have

$$\phi_X(t) = \phi_X(at)\phi_\varepsilon(t)$$

The c.f. of $\{\varepsilon_n\}$ can be obtained as

$$\phi_\varepsilon(t) = \left(\exp\left(it'(1-a)\mu - \frac{1}{2}t'(1-a^2)\Sigma t\right)\right)\left[a^2 + (1-a^2)\frac{1}{1+\frac{1}{2}t'Vt}\right]. \tag{10}$$

From this, we can obtain the distribution of the innovation sequence as

$$\varepsilon \overset{d}{=} Z_1 + L$$

where $Z_1 \sim N_p((1-a)\boldsymbol{\mu}, (1-a^2)\Sigma)$ and L can be treated as a sequence of random vectors of the form

$$L = \begin{cases} 0, & \text{with probability} \quad a^2 \\ L_p, & \text{with probability} \quad (1-a^2) \end{cases}$$

where L_p's are independently and identically distributed symmetric multivariate Laplace random vectors.

THEOREM 1. The process is stationary with $NL_p(\mu, \Sigma, V)$ marginals.

PROOF. We can prove this by the method of induction. We assume that

$$X_n \sim NL_p(\boldsymbol{\mu}, \Sigma, V).$$

Then

$$\phi_{X_n}(t) = \phi_{X_{n-1}}(at)\phi_{\varepsilon_n}(t)$$

$$= \left(\exp\left(iat'\boldsymbol{\mu} - \frac{1}{2}a^2 t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}a^2 t' Vt}\right)$$

$$\times\left(\exp\left(it'(1-a)\boldsymbol{\mu} - \frac{1}{2}t'(1-a^2)\Sigma t\right)\right)\left[a^2 + (1-a^2)\frac{1}{1+\frac{1}{2}t' Vt}\right]$$

$$= \left(\exp\left(it'\boldsymbol{\mu} - \frac{1}{2}t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}t' Vt}\right)$$

which is the c.f. of $NL_p(\boldsymbol{\mu},\Sigma,V)$. Therefore $\{X_n\}$ is strictly stationary with $NL_p(\boldsymbol{\mu},\Sigma,V)$ marginals.

THEOREM 2. *If X_0 is distributed arbitrary, then also the process is asymptotically Markovian with multivariate normal-Laplace marginal distribution.*

PROOF. We have $X_n = aX_{n-1} + \boldsymbol{\varepsilon}_n = a^n X_0 + \sum_{k=0}^{n-1} a^k \boldsymbol{\varepsilon}_{n-k}$.

In terms of c.f., we get

$$\phi_{X_n}(t) = \phi_{X_0}(a^n t)\prod_{k=0}^{n-1}\phi_{\varepsilon}(a^k t)$$

$$= \phi_{X_0}(a^n t)\prod_{k=0}^{n-1}\frac{\exp\left(it'a^k - \frac{1}{2}a^{2k}t'\Sigma t\right)}{1+\frac{1}{2}a^{2k}t' Vt}\frac{1+\frac{1}{2}a^{2(k+1)}t' Vt}{\exp\left(it'a^{k+1} - \frac{1}{2}a^{2(k+1)}t'\Sigma t\right)}$$

$$\to \left(\exp\left(it'\boldsymbol{\mu} - \frac{1}{2}t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}t' Vt}\right) \quad as \quad n \to \infty.$$

Hence even if X_0 is arbitrarily distributed, the process is asymptotically stationary Markovian with multivariate normal-Laplace marginals.

4.1. Distribution of sums and joint distribution of (X_n, X_{n+1})

Consider a stationary sequence $\{X_n\}$ satisfying (9). Then we have

$$X_{n+j} = a^j X_n + a^{j-1}\boldsymbol{\varepsilon}_{n+1} + a^{j-2}\boldsymbol{\varepsilon}_{n+2} + \cdots + \boldsymbol{\varepsilon}_{n+j}.$$

Hence

$$\begin{aligned} T_r &= X_n + X_{n+1} + \cdots + X_{n+r-1} \\ &= \sum_{j=0}^{r-1} [a^j X_n + a^{j-1}\boldsymbol{\varepsilon}_{n+1} + \cdots + \boldsymbol{\varepsilon}_{n+j}] \\ &= X_n \left(\frac{1-a^r}{1-a} \right) + \sum_{j=1}^{r-1} \boldsymbol{\varepsilon}_{n+j} \left(\frac{1-a^{r-j}}{1-a} \right). \end{aligned}$$

The c.f. of T_r is given by

$$\begin{aligned} \phi_{T_r}(t) &= \phi_{X_n}\left(t \frac{1-a^r}{1-a} \right) \prod_{j=1}^{r-1} \phi_{\varepsilon}\left(t \frac{1-a^{r-j}}{1-a} \right) \\ &= \left[\exp\left(\left(\frac{1-a^r}{1-a} \right) it'\boldsymbol{\mu} - \frac{1}{2}\left(\frac{1-a^r}{1-a} \right)^2 t'\Sigma t \right) \right] \left(\frac{1}{1+\frac{1}{2}\left(\frac{1-a^r}{1-a} \right)^2 t'Vt} \right) \\ &\times \prod_{j=1}^{r-1} \left\{ \left(\exp\left(it'(1-a^{r-j})\boldsymbol{\mu} - \frac{1}{2}(1-a^{r-j})(1+a)t'\Sigma t \right) \right) \right. \\ &\left. \times \left[a^2 + (1-a^2) \frac{1}{1+\frac{1}{2}\left(\frac{1-a^{r-j}}{1-a} \right)^2 t'Vt} \right] \right\}. \end{aligned}$$

The distribution of T_r can be obtained by inverting the above expression. The joint distribution of contiguous observation vectors (X_n, X_{n+1}) can be given in terms of c.f. as

$$
\begin{aligned}
&\phi_{X_n,X_{n+1}}(t_1,t_2)\\
&= E[exp(it_{1'}X_n + it_{2'}X_{n+1})]\\
&=E[exp(it_{1'}X_n + it_{2'}(aX_n + e_{n+1}))]\\
&= E[exp(i(t_{1'} + at_{2'})X_n + it_{2'}\boldsymbol{\varepsilon}_{n+1}))]\\
&= \phi_{X_n}(t_1 + at_2)\phi_{\varepsilon_{n+1}}(t_2)\\
&= \left(exp\left(i(t_1 + at_2)'\boldsymbol{\mu} - \frac{1}{2}(t_1 + at_2)'\Sigma(t_1 + at_2)\right)\right)\left(\frac{1}{1+\frac{1}{2}(t_1 + at_2)'V(t_1 + at_2)}\right)\\
&\quad\times\left(exp\left(it_{2'}(1-a)\boldsymbol{\mu} - \frac{1}{2}t_{2'}(1-a^2)\Sigma t_2\right)\right)\left[a^2 + (1-a^2)\frac{1}{1+\frac{1}{2}t_{2'}Vt_2}\right].
\end{aligned}
$$

Here $\phi_{X_n,X_{n+1}}(t_1,t_2) \neq \phi_{X_n,X_{n+1}}(t_2,t_1)$. Therefore the process is not time reversible.

5. MULTIVARIATE GENERALIZED NORMAL-LAPLACE DISTRIBUTION

A multivariate generalized normal-Laplace distribution can be defined by introducing an additional parameter. Multivariate generalized normal-Laplace distribution can be obtained as the convolution of multivariate normal and multivariate generalized symmetric Laplace random vectors. The c.f. of multivariate generalized normal-Laplace distribution is given by

$$
\phi_X(t) = \left[\left(\exp\left(it'\boldsymbol{\mu} - \frac{1}{2}t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}t'Vt}\right)\right]^{\nu}, t \in \mathbb{R}^p, \Sigma > 0, V > 0, \nu > 0. \tag{11}
$$

A p-variate generalized normal-Laplace distribution with parameters $\boldsymbol{\mu}, \Sigma, V$ and ν can be denoted by $NL_p(\boldsymbol{\mu}, \Sigma, V, \nu)$. Let $X \sim NL_p(\boldsymbol{\mu}, \Sigma, V, \nu)$, then X can be expressed as

$$
X \overset{d}{=} Z + Y, \tag{12}
$$

where Z and Y are independent random vectors with Z following a p-variate normal distribution with mean vector $\nu\boldsymbol{\mu}$ and dispersion matrix $\nu\Sigma\left(N_p(\nu\boldsymbol{\mu},\nu\Sigma)\right)$ and Y following a p-variate generalized symmetric Laplace distribution with parameters V,ν $\left(\mathrm{L}_p(V,\nu)\right)$. $NL_p(\boldsymbol{\mu}, \Sigma, V, \nu)$ is infinitely divisible and self-decomposable.

6. MULTIVARIATE GEOMETRIC GENERALIZED NORMAL-LAPLACE DISTRIBUTION

Now we introduce a new distribution namely, the multivariate geometric generalized normal-Laplace distribution denoted by $GGNL_p(\boldsymbol{\mu}, \Sigma, V, \nu)$. The c.f. is given by

$$\psi(t) = \frac{1}{1 - it'\nu\boldsymbol{\mu} + \frac{1}{2}\nu t'\Sigma t' + \nu\log(1 + \frac{1}{2}t'Vt)}$$

the above c.f. can be written in the form $\phi(t) = \exp\left(1 - \frac{1}{\psi(t)}\right)$, where $\phi(t)$ is the c.f. of $NL_p(\boldsymbol{\mu}, \Sigma, V)$. Hence the multivariate geometric generalized normal-laplace distribution is geometrically infinitely divisible.

6.1. Multivariate geometric normal-Laplace disribution

When $\nu = 1$, we get the multivariate geometric normal-Laplace distribution. The c.f. is obtained as denoted by $GNL_p(\boldsymbol{\mu}, \Sigma, V)$, is given by

$$\psi(t) = \frac{1}{1 - it'\boldsymbol{\mu} + \frac{1}{2}t'\Sigma t' + \log(1 + \frac{1}{2}t'Vt)}.$$

THEOREM 3. *Let* $X_1, X_2, \cdots$ *are independently and identically distributed* $GGNL_p(\boldsymbol{\mu}, \Sigma, V, \nu)$ *random vectors and* N *be geometric with mean* $\frac{1}{p}$, *such that* $P[N = k] = p(1-p)^{k-1},\ k = 1, 2, \ldots,\ 0 < p < 1$. *This establishes that* $Y \sim GGNL_p(\boldsymbol{\mu}, \Sigma, V, \nu / p)$.

PROOF. The c.f. of Y is

$$\psi_Y(t) = \sum_{k=1}^{\infty}[\psi_X(t)]^k p(1-p)^{k-1}$$
$$= \frac{p\,/\,(1 - it'\,\nu\boldsymbol{\mu} + \frac{1}{2}\nu t'\Sigma t' + \nu\log(1+\frac{1}{2}t'\,Vt))}{1-((1-p)\,/\,(1 - it'\,\nu\boldsymbol{\mu} + \frac{1}{2}\nu t'\Sigma t' + \nu\log(1+\frac{1}{2}t'\,Vt))}$$
$$= \frac{1}{1 - it'\frac{\nu}{p}\boldsymbol{\mu} + \frac{1}{2}\frac{\nu}{p}t'\Sigma t' + \frac{\nu}{p}\log(1+\frac{1}{2}t'\,Vt)}.$$

Hence Y is distributed as GGNL $_p(\boldsymbol{\mu},\Sigma,V,\nu/p)$.

THEOREM 4. Multivariate geometric generalized normal-Laplace distribution $\mathrm{GGNL}_p(\boldsymbol{\mu},\Sigma,V,\nu)$ *is the limiting distribution of multivariate generalized normal-Laplace distribution.*

PROOF. We have

$$\left[\left(\exp\left(it'\boldsymbol{\mu}-\frac{1}{2}t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}t'\,Vt}\right)\right]^{\nu/n} = \left\{1+\left[\left(\exp\left(it'\boldsymbol{\mu}-\frac{1}{2}t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}t'\,Vt}\right)\right]^{\nu/n} - 1\right\}^{-1}$$

is the c.f. of a probability distribution since Multivariate generalized normal-Laplace distribution is infinitely divisible. Hence by Lemma 3.2 of Pillai (1990).

$$\psi_n(t) = \left\{1+n\left[\left(\exp\left(it'\boldsymbol{\mu}-\frac{1}{2}t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}t'\,Vt}\right)\right]^{\nu/n} - 1\right\}^{-1}$$

is the c.f. of geometric sum of independently and identically distributed multivariate generalized normal-Laplace random vectors . Taking limit as $n \to \infty$.

$$\psi(t) = \lim_{n\to\infty}\psi_n(t)$$

$$= \left\{1 + \lim_{n\to\infty}\left\{ n\left[\left[\left(\exp\left(it'\boldsymbol{\mu} - \frac{1}{2}t'\Sigma t\right)\right)\left(\frac{1}{1+\frac{1}{2}t'Vt}\right)\right]^{\nu/n}\right\} - 1\right\}^{-1}$$

$$= \frac{1}{1 - it'\nu\boldsymbol{\mu} + \frac{1}{2}\nu t'\Sigma t' + \nu\log(1+\frac{1}{2}t'Vt)}.$$

Hence GGNL $GGNL_p(\boldsymbol{\mu},\Sigma,V,\nu)$ is the limiting distribution of multivariate generalized normal-Laplace distribution.

7. ESTIMATION OF PARAMETERS

Kollo and Srivastava (2004) discussed the estimation of Multivariate Laplace distribution. We can use the method of moments to estimate the mean, covariance matrix and the skewness and kurtosis measures of multivariate normal-Laplace distribution.

Skewness and Kurtosis.

Let x be a random vector with mean vector $\boldsymbol{\theta}$ and covariance matrix Σ and $y = \Sigma^{-1/2}(x - \boldsymbol{\theta})$ with mean vector 0, covariance matrix I_p and with third and fourth moments $M_3(y)$, $M_4(y)$ and cumulants $C_3(y)$, $C_4(y)$. Then skewness measure $\beta_{1p}(x)$ and kurtosis characteristic $\beta_{2p}(x)$ are defined by

$$\beta_{1p}(x) = tr[C_{3'}(y)C_3(y)] = tr[M_{3'}(y)M_3(y)]$$

and

$$\beta_{2p}(x) = tr[M_4(y)] = tr[C_4(y)] + p^2 + 2p.$$

Let $x_1, x_2, \cdots, x_n$ are independently identically distributed as $\mathrm{NL}_p(\boldsymbol{\mu},\Sigma,V)$. Then the estimates of mean vector and covariance matrix are obtained as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and

$$S = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})'$$

estimated using the random vector defined by

$$y_i = S^{-1/2}(x_i - \bar{x}), \quad i = 1,\ldots,n,$$

where $S^{-1/2}$ is any square root of S^{-1} such that $S^{-1} = S^{-1/2}(S^{-1/2})'$. Then the estimates of third and fourth moments are given by

$$\hat{M}_3 = \frac{1}{n}\sum_{i=1}^{n}(y_i \otimes y_{i'} \otimes y_i)$$

and

$$\hat{M}_4 = \frac{n+1}{n-1}\frac{1}{n}\sum_{i=1}^{n}(y_i \otimes y_{i'} \otimes y_i \otimes y_{i'}).$$

These estimates are unbiased estimators, see Mardia (1970). The estimates of skewness and kurtosis measures are given by

$$\hat{\beta}_{1p} = tr[\hat{M}_3' \hat{M}_3]$$

and

$$\hat{\beta}_{2p} = tr[\hat{M}_4].$$

8. APPLICATIONS

In insurance and financial markets, there is significant need for the development of a standard framework for the risk measurement. Landsman and Valdez (2003) derives explicit formulas for computing tail conditional expectations for elliptical distributions and extends them to multivariate case. Multivariate elliptical distributions are useful to model combinations of correlated risks. Since multivariate normal-Laplace distribution comes in the class of elliptical distributions and it also convolutes both Gaussian and non-Gaussian distributions, it also has a very important role in risk analysis. It seems to provide an attractive tool for actuarial and financial risk management because it allows a multivariate portfolio of risks to have the property of regular variation in the marginal tails.

The importance of normal-Laplace model lies in the fact that it is the first attempt to combine Gaussian and non-Gaussian marginals to model time series data, see Jose et al (2008) and Lishamol and Jose (2009). Applications of normal-Laplace distribution are wide spread in areas like financial modelling, Levy process, Brownian motion etc. Reed (2006) showed that it is the distribution of the stopped state of a Brownian motion with normally distributed starting value if the stopping hazard rate is constant. In financial modeling, a normal-Laplace model is a more realistic alternative for Gaussian models as logarithmic price returns do not follow exactly a normal distribution. But it is more realistic to consider multivariate data where several interdependent variables are discussed. The models developed in this paper can be used for modeling multivariate time series data.

ACKNOWLEDGEMENTS

The authors are grateful to the reviewer for the valuable suggestions which helped in improving the paper. The authors are also grateful to the Department of Science and Technology (DST), Govt. of India for the financial support under the INSPIRE Fellowship.The first author gratefully acknowledges the financial assistance from UGC, New Delhi in the form of a major research project grant.

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SUMMARY

Multivariate normal-Laplace distribution and processes

The normal-Laplace distribution is considered and its properties are discussed. A multivariate normal-Laplace distribution is introduced and its properties are studied. First order autoregressive processes with these stationary marginal distributions are developed and studied. A generalized multivariate normal-Laplace distribution is introduced. Multivariate geometric normal-Laplace distribution and multivariate geometric generalized normal-Laplace distributions are introduced and their properties are studied. Estimation of parameters and some applications are also discussed.

Keywords: multivariate normal-Laplace distribution; autoregressive processes; multivariate geometric normal-Laplace distribution; multivariate geometric generalized normal-Laplace distribution