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AN IMPROVED ESTIMATION OF PARAMETERS OF MORGENSTERN TYPE BIVARIATE LOGISTIC DISTRIBUTION USING RANKED SET SAMPLING

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1. INTRODUCTION

The concept of ranked set sampling (RSS) was first introduce by McIntyre (1952) as a process of improving the precision of the sample mean as an estimator of the population mean. Ranked set sampling as described in McIntyre (1952) is applicable whenever ranking of a set of sampling units can be done easily by a judgement method (see, Chan *et al.* , 2004). Ranking by judgement method is not recommendable if the judgement method is too crude and is not powerful for ranking by discriminating the units of a moderately large sample. In certain situations, one may prefer exact measurement of some easily measurable variable associated with the study variable rather than ranking the units by a crude judgement method. Suppose the variable of interest say Y , is difficult or much expensive to measure, but an auxiliary variable X , correlated with Y , is readily measurable and can be ordered exactly. In this case as an alternative to McIntyre (1952) method of ranked set sampling, Stokes (1977) used an auxiliary variable for the ranking of the sampling units. The procedure of ranked set sampling described by Stokes (1977) using auxiliary variate is as follows: Choose n^2 independent units, arrange them randomly into n sets each with n units and observe the value of the auxiliary variable X on each of these units. In the first set, that unit for which the measurement on the auxiliary variable is the smallest is chosen. In the second set, that unit for which the measurement on the auxiliary variable is the second smallest is chosen. The procedure is repeated until in the last set, that unit for which the measurement on the auxiliary variable is the largest is chosen. The resulting new set of n units chosen by one from each set as described above is called the RSS defined by Stokes (1977).

In biological studies, such as in the root zone analysis of bamboo plants (Bambusa arundinacea), the shoot height of the plant is a correlated character with root weight. Clearly shoot height can be measured very easily whereas root weight measurement requires uprooting of the sampled plants. Hence, in such situations, we can choose the most desired plants with respect to their shoot length value and on which we measure the root weight for further analysis as presented in any ranked set sampling (RSS).

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Stokes (1977) suggested the ranked set sample mean as an estimator for the mean of the study variate Y , when an auxiliary variable X is used for ranking the sample units, under the assumption that (X,Y) follows a bivariate normal distribution. Barnett and Moore (1977) improved it by deriving the best linear unbiased estimator (BLUE) of the mean of the study variate Y , based on ranked set sample obtained on the study variate Y .

Chacko and Thomas (2007) obtained the BLUE of the parameter involved in the study variate Y , under the assumption that (X,Y) follows bivariate Pareto distribution. Unbalanced RSS arising from Morgentern type bivariate exponential distribution have been considered by Chacko and Thomas (2008).

Chacko and Thomas (2006) used the concomitants of record values arising from a Morgenstern type bivariate logistic distribution to estimate some of its parameters. Sampling to get a given number of record values will require several selection (uncertain number) of units and moreover the obtained concomitants of record values are correlated, which makes one to determine the variance and covariance of concomitants of record values to use them for inference problem. However, in case of Stokes method of ranked set sampling, the number of units to be selected is definite and there exists no correlation between one observation to another as they are drawn from independent samples so that handling the observations in a ranked set sample for inference problem will be very easy. Papers by Shaibu and Muttlak (2004); Hossain and Khan (2006); Hossain and Muttalk (2006) and Al- Rawwash *et al.* (2010) are also of interest in this context.

Sacrificing the unbiasedness of an estimator Searls (1964) was first to propose an improved estimator of the population mean. Later motivated by Searls (1964); Singh *et al.* (1973) have suggested an improved estimator of the population variance which is revisited by Searls and Intarapanich (1990).

Chacko and Thomas (2009) considered the case when (X,Y) follows Morgenstern type bivariate logistic distribution (MTBLD) with cumulative distribution function (cdf) defined by (see Kotz *et al.* , 2000):

$$F_{X,Y}(x,y)=\left[\left\{1+\exp\left(-\frac{x-\theta_1}{\sigma_1}\right)\right\}^{-1}\left\{1+\exp\left(-\frac{y-\theta_2}{\sigma_2}\right)\right\}^{-1}\right] \times\left[1+\alpha\left\{\frac{\exp\left(-\frac{x-\theta_1}{\sigma_1}\right)}{1+\exp\left(-\frac{x-\theta_1}{\sigma_1}\right)}\right\}\left\{\frac{\exp\left(-\frac{y-\theta_2}{\sigma_2}\right)}{1+\exp\left(-\frac{y-\theta_2}{\sigma_2}\right)}\right\}\right] \quad (1)$$

$$-\infty<(x,y)<\infty;\ -\infty<(\theta_1,\theta_2)<\infty;\ (\sigma_1,\sigma_2)>0;\ -1\le\alpha\le1.$$

In this paper motivated by Searls (1964); Singh *et al.* (1973) and Searls and Intarapanich (1990), we have suggested some improved estimator of the parameters θ_2 and σ_2 involved in (1), using ranked set sampling and obtained their biases and mean squared errors. Efficiencies comparisons have been made with Chacko and Thomas (2009) estimators. Numerical illustrations are given in support of the present study.

2. UNBIASED ESTIMATORS OF θ_2 AND σ_2 BASED ON RANKED SET SAMPLE

Let (X,Y) be a bivariate random variable which follows a MTBLD with cdf defined by (1). Suppose n sampling units each of size n are taken. Let $X_{(r)r}$ be the rth order statistic of the auxiliary variate X in the rth sample and let $Y_{[r]r}$ be the measurement made on the variable associated with $X_{(r)r}, r = 1, 2, ..., n$. By using the approach of Scaria and Nair (1999) for obtaining means and variances of concomitants of order statistics arising from Morgenstern family of distributions, we get the mean and variance of $Y_{[r]r}$ for $1 \le r \le n$ as

$$E\left[Y_{[r]r}\right] = \theta_2 + \xi_r \sigma_2 \tag{2}$$

and

$$Var\left[Y_{[r]r}\right] = \delta_r \sigma_2^2 \tag{3}$$

where

$$\xi_r = -\alpha\left(\frac{n-2r+1}{n+1}\right) \tag{4}$$

and

$$\delta_r = \frac{\pi^2}{3} - \alpha^2\left(\frac{n-2r+1}{n+1}\right)^2. \tag{5}$$

Let $Y_{[r]r}, r = 1,2,...,n$ be the ranked set sample observations on a study variable Y obtained out of ranking made on an auxiliary variable X, when (X,Y) follows MTBLD as defined in (1).

Then Chacko and Thomas (2009) suggested unbiased estimators of θ_2 and σ_2 respectively as

$$\theta_2^* = \frac{1}{n}\sum_{r=1}^{n} Y_{[r]r} \tag{6}$$

and

$$\sigma_2^* = \frac{1}{\sum_{r=1}^{[n/2]} \xi_r} \sum_{r=1}^{[n/2]} T_r , \tag{7}$$

where $T_r = \dfrac{\left(Y_{[r]r} - Y_{[n-r+1]\overline{n-r+1}}\right)}{2}$, $\left[\dfrac{n}{2}\right]$ is the usual greatest integer function and $Y_{[n-r+1]\overline{n-r+1}}$ is measurement made on the variate associated with $x_{(n-r+1)\overline{n-r+1},} r = 1,2,...,n$. Chacko and Thomas

(2009) used $\left[\frac{n}{2}\right]$ in the summation on (7), for making an unbiased estimator of σ_2. The variances of θ_2^* and σ_2^* are respectively given by

$$Var\left[\theta_2^*\right] = MSE\left[\theta_2^*\right] = \sigma_2^2 V \tag{8}$$

and

$$Var\left[\sigma_2^*\right] = MSE\left[\sigma_2^*\right] = \sigma_2^2 V_1 , \tag{9}$$

where

$$V = \frac{1}{n}\left[\frac{\pi^2}{3} - \frac{\alpha^2}{n}\sum_{r=1}^{n}\left(\frac{n-2r+1}{n+1}\right)^2\right]$$

and

$$V_1 = \frac{1}{2\left(\sum_{r=1}^{[n/2]}\xi_r\right)^2}\sum_{r=1}^{[n/2]}\left[\frac{\pi^2}{3} - \alpha^2\sum_{r=1}^{n}\left(\frac{n-2r+1}{n+1}\right)^2\right] .$$

2.1. *Minimum mean squared error estimator of θ_2 based on unbiased estimator θ_2^**

We propose a class of estimators for the parameter θ_2 as

$$t_1 = \lambda\ \theta_2^* , \tag{10}$$

where λ is a suitably chosen constant such that mean squared error of t_1 is minimum.
The bias and mean squared error (MSE) of t_1 are respectively given by

$$B(t_1) = (\lambda - 1)\theta_2 \tag{11}$$

$$MSE(t_1) = \theta_2^2\left[\lambda^2\left(V R^2 + 1\right) - 2\lambda + 1\right] , \tag{12}$$

where $R = \sigma_2 / \theta_2$.
The $MSE(t_1)$ is minimum when

$$\lambda = \left(V R^2 + 1\right)^{-1}. \tag{13}$$

Substitution of (13) in (10), yields the minimum MSE estimator of θ_2 as

$$t_{1m} = \left(V R^2 + 1\right)^{-1}\frac{1}{n}\sum_{r=1}^{n} Y_{[r]r} . \tag{14}$$

It is to be mentioned that the value of the quantity $R = \sigma_2 / \theta_2$ can be made known either through past data or experienced gathered in due course of time.
The MSE of t_{1m} is given by

$$MSE(t_{1m}) = \frac{V\sigma_2^2}{(VR^2+1)}. \tag{15}$$

From (8) and (15) we have

$$Var[\theta_2^*] - MSE[t_{1m}] = \frac{V^2\sigma_2^2 R^2}{(VR^2+1)} > 0 \quad ,$$

which follows that the proposed estimator t_{1m} is more efficient than θ_2^* in presence of known R.

In practice, $R = \sigma_2 / \theta_2$ is rarely known so that the estimator t_{1m} is of little utility. However if R has shown stability in repeated experiments, we can derive a guess value R_g of R [see, Hirano (1972); Srivastava (1974) and Srivastava *et al.* (1980)]. Replacing R in (14) by R_g, we obtain the following estimator for θ_2 as

$$t_{1g} = \left(VR_g^2+1\right)^{-1} \frac{1}{n}\sum_{r=1}^{n} Y_{[r]r} . \tag{16}$$

The bias and mean squared error of t_{1g} are respectively given by

$$B(t_{1g}) = -\frac{\theta_2 V R_g^2}{(1+VR_g^2)} \tag{17}$$

and

$$MSE(t_{1g}) = \frac{V\sigma_2^2}{R^2}\frac{(R^2+VR_g^4)}{(1+VR_g^2)^2} \quad , \tag{18}$$

which will be larger than the MSE of t_{1m}.

Interesting to note that the MSE of t_{1g} will be smaller than the variance of the usual unbiased estimator θ_2^* only when

$$R^2 > \frac{R_g^2}{\left(2+VR_g^2\right)} \quad or \quad R_g^2 < \frac{2R^2}{\left(1-VR^2\right)} \quad , \tag{19}$$

which may hold in some situation.

2.2. *Minimum mean squared error estimators of σ_2 based on unbiased estimator σ_2^**

We propose a class of estimators for the parameter σ_2 as

$$t_2 = \lambda_1 \sigma_2^* \ , \tag{20}$$

where λ_1 is a suitably chosen constant such that mean squared error of t_2 is minimum.
The bias and MSE of t_2 are respectively given as

$$B(t_2) = (\lambda_1 - 1)\sigma_2 , \tag{21}$$

$$MSE(t_2) = \sigma_2^2 \left[\lambda_1^2 (V_1 + 1) - 2\lambda_1 + 1\right]. \tag{22}$$

The MSE of t_2 is minimum for

$$\lambda_1 = [1 + V_1]^{-1} . \tag{23}$$

Thus the resulting minimum MSE estimator of σ_2^* is given by

$$t_{2m} = [1 + V_1]^{-1} \frac{1}{\sum_{r=1}^{[n/2]} \xi_r} \sum_{r=1}^{[n/2]} T_r \ . \tag{24}$$

The MSE of t_{2m} is given by

$$MSE[t_{2m}] = \sigma_2^2 \frac{V_1}{1 + V_1} . \tag{25}$$

From (9) and (25) we have

$$Var[\sigma_2^*] - MSE[t_{2m}] = \sigma_2^2 \frac{V_1^2}{1 + V_1} > 0 \ .$$

Thus the proposed estimator t_{2m} is more efficient than σ_2^*.

3. BEST LINEAR UNBIASED ESTIMATORS (BLUES) OF θ_2 AND σ_2

When the parameter α is known, Chacko and Thomas (2009) have suggested BLUEs $\hat{\theta}_2$ and $\hat{\sigma}_2$ of θ_2 and σ_2 respectively as

$$\hat{\theta}_2 = \sum_{r=1}^{n} \left\{ \frac{\delta_r^{-1}\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \xi_r \delta_r^{-1}\left(\sum_{i=1}^{n} \delta_i^{-1}\right)}{\left(\sum_{i=1}^{n} \delta_i^{-1}\right)\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \left(\sum_{i=1}^{n} \xi_i \delta_i^{-1}\right)^2} \right\} Y_{[r]r} \tag{26}$$

and

$$\hat{\sigma}_2 = \sum_{r=1}^{n} \left\{ \frac{\xi_r \delta_r^{-1}\left(\sum_{i=1}^{n} \delta_i^{-1}\right) - \delta_r^{-1}\left(\sum_{i=1}^{n} \xi_i \delta_i^{-1}\right)}{\left(\sum_{i=1}^{n} \delta_i^{-1}\right)\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \left(\sum_{i=1}^{n} \xi_i \delta_i^{-1}\right)^2} \right\} Y_{[r]r} . \tag{27}$$

The variances of $\hat{\theta}_2$ and $\hat{\sigma}_2$ are respectively given by

$$Var\left[\hat{\theta}_2\right] = MSE\left[\hat{\theta}_2\right] = V_2\, \sigma_2^2 \tag{28}$$

and

$$Var\left[\hat{\sigma}_2\right] = MSE\left[\hat{\sigma}_2\right] = V_3\, \sigma_2^2 , \tag{29}$$

where

$$V_2 = \frac{\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right)}{\left(\sum_{i=1}^{n} \delta_i^{-1}\right)\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \left(\sum_{i=1}^{n} \xi_i \delta_i^{-1}\right)^2}$$

and

$$V_3 = \frac{\left(\sum_{i=1}^{n} \delta_i^{-1}\right)}{\left(\sum_{i=1}^{n} \delta_i^{-1}\right)\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \left(\sum_{i=1}^{n} \xi_i \delta_i^{-1}\right)^2} .$$

3.1. Minimum mean squared error estimator of θ_2 based on BLUE $\hat{\theta}_2$

We consider a class of estimators for the parameter θ_2 as

$$t_3 = \lambda_2 \hat{\theta}_2, \tag{30}$$

where λ_2 is a suitably chosen constant such that mean squared error of t_3 is minimum. The bias and MSE of t_3 are respectively given by

$$B(t_3) = (\lambda_2 - 1)\theta_2, \tag{31}$$

$$MSE(t_3) = \theta_2^2 \left[\lambda_2^2 (V_2 R^2 + 1) - 2\lambda_2 + 1\right]. \tag{32}$$

The MSE of t_3 is minimum when

$$\lambda_2 = (V_2 R^2 + 1)^{-1} \tag{33}$$

Thus the resulting minimum MSE of θ_2 is given by

$$t_{3m} = (V_2 R^2 + 1)^{-1} \sum_{r=1}^{n} \left\{ \frac{\delta_r^{-1}\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \xi_r \delta_r^{-1}\left(\sum_{i=1}^{n} \delta_i^{-1}\right)}{\left(\sum_{i=1}^{n} \delta_i^{-1}\right)\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \left(\sum_{i=1}^{n} \xi_i \delta_i^{-1}\right)^2} \right\} Y_{[r]r} \tag{34}$$

$$MSE[t_{3m}] = \frac{V_2 \sigma_2^2}{(V_2 R^2 + 1)}. \tag{35}$$

From (28) and (35) we have

$$Var\left[\hat{\theta}_2\right] - MSE[t_{3m}] = \frac{V_2^2 \sigma_2^2 R^2}{(V_2 R^2 + 1)} > 0 .$$

Thus the proposed estimator t_{3m} is more efficient than the BLUE $\hat{\theta}_2$, when the ratio R is known.

Let R_g be a good guess of R, then as in section 2.1, we suggest an alternative estimator for θ_2 as

$$t_{3g} = (V_2 . R_g^2 + 1)^{-1} \sum_{r=1}^{n} \left\{ \frac{\delta_r^{-1}\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \xi_r \delta_r^{-1}\left(\sum_{i=1}^{n} \delta_i^{-1}\right)}{\left(\sum_{i=1}^{n} \delta_i^{-1}\right)\left(\sum_{i=1}^{n} \xi_i^2 \delta_i^{-1}\right) - \left(\sum_{i=1}^{n} \xi_i \delta_i^{-1}\right)^2} \right\} Y_{[r]r} . \tag{36}$$

The bias and MSE of t_{3g} are respectively given by

$$B(t_{3g}) = -\frac{\theta_2 V_2 R_g^2}{(1+V_2 R_g^2)} \tag{37}$$

and

$$MSE(t_{3g}) = \frac{V_2\sigma_2^2}{R^2}\frac{(R^2 + V_2 R_g^4)}{(1+V_2 R_g^2)^2}. \tag{38}$$

The MSE of t_{3g} is larger than the MSE of t_{3m}
We note from (28) and (38) that

$$MSE(t_{3g}) < Var(\hat{\theta}_2) \quad if \quad \frac{(R^2 + V_2 R_g^4)}{(1+V_2 R_g^2)^2} < R^2$$

i.e. if

$$R^2 > \frac{R_g^2}{(2+V_2 R_g^2)} \quad or \quad R_g^2 < \frac{2R^2}{(1-V_2 R^2)}, \tag{39}$$

which may hold in certain situation.

3.2. *Minimum mean squared error estimator of σ_2 based on BLUE $\hat{\sigma}_2$*

We define a class of estimators for the parameter σ_2 as

$$t_4 = \lambda_3 \hat{\sigma}_2 , \tag{40}$$

where λ_3 is a suitably chosen constant such that mean squared error of t_4 is minimum.
The bias and MSE of t_4 are given by

$$B(t_4) = (\lambda_3 - 1)\sigma_2 \tag{41}$$

$$MSE(t_4) = \sigma_2^2\left[\lambda_3^2(V_3+1) - 2\lambda_3 + 1\right] \tag{42}$$

The optimum value of λ_3 which will minimize the MSE of t_4 is given by

$$\lambda_3 = [1+V_3]^{-1} \tag{43}$$

This yields the minimum MSE estimator of σ_2 as

$$t_{4m}=\left[1+V_3\right]^{-1}\sum_{r=1}^{n}\left\{\frac{\xi_r\delta_r^{-1}\left(\sum_{i=1}^{n}\delta_i^{-1}\right)-\delta_r^{-1}\left(\sum_{i=1}^{n}\xi_i\delta_i^{-1}\right)}{\left(\sum_{i=1}^{n}\delta_i^{-1}\right)\left(\sum_{i=1}^{n}\xi_i^2\delta_i^{-1}\right)-\left(\sum_{i=1}^{n}\xi_i\delta_i^{-1}\right)^2}\right\}Y_{[r]r} \tag{44}$$

The MSE of t_{4m} is given by

$$MSE\left[t_{4m}\right]=\sigma_2^2\frac{V_3}{1+V_3} \tag{45}$$

From (29) and (45) we have

$$Var\left[\hat{\sigma}_2\right]-MSE\left[t_{4m}\right]=\sigma_2^2\frac{V_3^2}{1+V_3}>0.$$

Thus the proposed estimator t_{4m} is more efficient than the BLUE $\hat{\sigma}_2$ of σ_2 .

4. RELATIVE EFFICIENCY

To throw some light on the performances of various estimators $\theta_2^*, \hat{\theta}_2, \sigma_2^*, \hat{\sigma}_2$; t_{1m}, t_{2m}, t_{3m} and t_{4m} of the parameters θ_2 and σ_2, we have computed the relative efficiencies by using the formulae:

$$RE\left(\hat{\theta}_2,\theta_2^*\right)=\frac{V}{V_2};$$

$$RE\left(\hat{\sigma}_2,\sigma_2^*\right)=\frac{V_1}{V_3};$$

$$RE\left(t_{1m},\theta_2^*\right)=\left(1+VR^2\right);$$

$$RE\left(t_{1m},\hat{\theta}_2\right)=\frac{V_2\left(1+VR^2\right)}{V};$$

$$RE\left(t_{2m},\sigma_2^*\right)=\left(1+V_1\right);$$

$$RE\left(t_{2m},\hat{\sigma}_2\right)=\frac{V_3\left(1+V_1\right)}{V_1};$$

$$RE\left(t_{3m},\theta_2^*\right)=\frac{V\left(1+V_2R^2\right)}{V_2};$$

$$RE\left(t_{3m},\hat{\theta}_2\right)=\left(1+V_2R^2\right);$$

$$RE\left(t_{4m},\sigma_2^*\right)=\frac{V_1\left(1+V_3\right)}{V_3};\text{ and}$$

$$RE\left(t_{4m},\hat{\sigma}_2\right)=\left(1+V_3\right),$$

for $n=2(2)20; R=0.25(0.25)1.00(1.00)5.00$ and $\alpha=0.25(0.25)1.00$.

Findings are shown in Tables 1 to 9.

TABLE 1

The values of $RE\left(\hat{\theta}_2,\theta_2^*\right)$, $RE\left(t_{1m},\theta_2^*\right)$, $RE\left(t_{3m},\hat{\theta}_2\right)$, $RE\left(t_{3m},\theta_2^*\right)$ *and* $RE\left(t_{1m},\hat{\theta}_2\right)$ *when* $R=0.25$

n	α	$RE\left(\hat{\theta}_2,\theta_2^*\right)$	$RE\left(t_{1m},\theta_2^*\right)$	$RE\left(t_{1m},\hat{\theta}_2\right)$	$RE\left(t_{3m},\theta_2^*\right)$	$RE\left(t_{3m},\hat{\theta}_2\right)$
2	0.25	1.0000	1.1026	1.1026	1.1026	1.1026
	0.50	1.0000	1.1019	1.1019	1.1019	1.1019
	0.75	1.0000	1.1009	1.1009	1.1009	1.1009
	1.00	1.0000	1.0993	1.0993	1.0993	1.0993
4	0.25	1.0000	1.0512	1.0512	1.0512	1.0512
	0.50	1.0000	1.0506	1.0506	1.0506	1.0506
	0.75	1.0000	1.0496	1.0496	1.0496	1.0496
	1.00	1.0026	1.0483	1.0455	1.0508	1.0481
6	0.25	1.0000	1.0341	1.0341	1.0341	1.0341
	0.50	1.0000	1.0336	1.0336	1.0336	1.0336
	0.75	1.0019	1.0329	1.0309	1.0348	1.0328
	1.00	1.0059	1.0318	1.0257	1.0377	1.0316
8	0.25	1.0000	1.0256	1.0256	1.0256	1.0256
	0.50	1.0000	1.0252	1.0252	1.0252	1.0252
	0.75	1.0026	1.0246	1.0220	1.0271	1.0245
	1.00	1.0053	1.0237	1.0183	1.0290	1.0236
10	0.25	1.0000	1.0204	1.0204	1.0204	1.0204
	0.50	1.0000	1.0201	1.0201	1.0201	1.0201
	0.75	1.0032	1.0196	1.0164	1.0228	1.0196
	1.00	1.0067	1.0189	1.0121	1.0255	1.0188
12	0.25	1.0000	1.0171	1.0171	1.0171	1.0171
	0.50	1.0000	1.0168	1.0168	1.0168	1.0168
	0.75	1.0038	1.0163	1.0124	1.0202	1.0163
	1.00	1.0080	1.0157	1.0076	1.0237	1.0156
14	0.25	1.0000	1.0146	1.0146	1.0146	1.0146
	0.50	1.0000	1.0144	1.0144	1.0144	1.0144
	0.75	1.0000	1.0139	1.0139	1.0139	1.0139
	1.00	1.0047	1.0134	1.0086	1.0181	1.0133
16	0.25	1.0000	1.0128	1.0128	1.0128	1.0128
	0.50	1.0000	1.0126	1.0126	1.0126	1.0126
	0.75	1.0000	1.0122	1.0122	1.0122	1.0122
	1.00	1.0054	1.0117	1.0063	1.0171	1.0116
18	0.25	1.0000	1.0114	1.0114	1.0114	1.0114
	0.50	1.0000	1.0112	1.0112	1.0112	1.0112
	0.75	1.0000	1.0108	1.0108	1.0108	1.0108
	1.00	1.0061	1.0104	1.0043	1.0164	1.0103
20	0.25	1.0000	1.0103	1.0103	1.0103	1.0103
	0.50	1.0000	1.0101	1.0101	1.0101	1.0101
	0.75	1.0000	1.0098	1.0098	1.0098	1.0098
	1.00	1.0068	1.0093	1.0025	1.0161	1.0093

TABLE 2

The values of $RE\left(\hat{\theta}_2,\theta_2^*\right)$, $RE\left(t_{1m},\theta_2^*\right)$, $RE\left(t_{3m},\hat{\theta}_2\right)$, $RE\left(t_{3m},\theta_2^*\right)$ *and* $RE\left(t_{1m},\hat{\theta}_2\right)$ *when* $R = 0.50$

n	α	$RE\left(\hat{\theta}_2,\theta_2^*\right)$	$RE\left(t_{1m},\theta_2^*\right)$	$RE\left(t_{1m},\hat{\theta}_2\right)$	$RE\left(t_{3m},\theta_2^*\right)$	$RE\left(t_{3m},\hat{\theta}_2\right)$
2	0.25	1.0000	1.4103	1.4103	1.4103	1.4103
	0.50	1.0000	1.4078	1.4078	1.4078	1.4078
	0.75	1.0000	1.4035	1.4035	1.4035	1.4035
	1.00	1.0000	1.3973	1.3973	1.3973	1.3973
4	0.25	1.0000	1.2048	1.2048	1.2048	1.2048
	0.50	1.0000	1.2025	1.2025	1.2025	1.2025
	0.75	1.0000	1.1985	1.1985	1.1985	1.1985
	1.00	1.0026	1.1930	1.1899	1.1956	1.1925
6	0.25	1.0000	1.1365	1.1365	1.1365	1.1365
	0.50	1.0000	1.1345	1.1345	1.1345	1.1345
	0.75	1.0019	1.1315	1.1293	1.1334	1.1313
	1.00	1.0059	1.1273	1.1206	1.1332	1.1265
8	0.25	1.0000	1.1023	1.1023	1.1023	1.1023
	0.50	1.0000	1.1008	1.1008	1.1008	1.1008
	0.75	1.0026	1.0983	1.0955	1.1008	1.0980
	1.00	1.0053	1.0948	1.0890	1.1001	1.0943
10	0.25	1.0000	1.0818	1.0818	1.0818	1.0818
	0.50	1.0000	1.0805	1.0805	1.0805	1.0805
	0.75	1.0032	1.0785	1.0751	1.0817	1.0783
	1.00	1.0067	1.0755	1.0684	1.0822	1.0750
12	0.25	1.0000	1.0683	1.0683	1.0683	1.0683
	0.50	1.0000	1.0670	1.0670	1.0670	1.0670
	0.75	1.0038	1.0653	1.0612	1.0691	1.0650
	1.00	1.0080	1.0628	1.0543	1.0708	1.0623
14	0.25	1.0000	1.0585	1.0585	1.0585	1.0585
	0.50	1.0000	1.0575	1.0575	1.0575	1.0575
	0.75	1.0000	1.0558	1.0558	1.0558	1.0558
	1.00	1.0047	1.0535	1.0486	1.0582	1.0533
16	0.25	1.0000	1.0510	1.0510	1.0510	1.0510
	0.50	1.0000	1.0503	1.0503	1.0503	1.0503
	0.75	1.0000	1.0488	1.0488	1.0488	1.0488
	1.00	1.0054	1.0468	1.0412	1.0521	1.0465
18	0.25	1.0000	1.0455	1.0455	1.0455	1.0455
	0.50	1.0000	1.0448	1.0448	1.0448	1.0448
	0.75	1.0000	1.0433	1.0433	1.0433	1.0433
	1.00	1.0061	1.0415	1.0352	1.0476	1.0413
20	0.25	1.0000	1.0410	1.0410	1.0410	1.0410
	0.50	1.0000	1.0403	1.0403	1.0403	1.0403
	0.75	1.0000	1.0390	1.0390	1.0390	1.0390
	1.00	1.0068	1.0373	1.0303	1.0440	1.0370

TABLE 3

The values of $RE\left(\hat{\theta}_2,\theta_2^*\right)$, $RE\left(t_{1m},\theta_2^*\right)$, $RE\left(t_{3m},\hat{\theta}_2\right)$, $RE\left(t_{3m},\theta_2^*\right)$ *and* $RE\left(t_{1m},\hat{\theta}_2\right)$ *when* $R = 0.75$

n	α	$RE\left(\hat{\theta}_2,\theta_2^*\right)$	$RE\left(t_{1m},\theta_2^*\right)$	$RE\left(t_{1m},\hat{\theta}_2\right)$	$RE\left(t_{3m},\theta_2^*\right)$	$RE\left(t_{3m},\hat{\theta}_2\right)$
2	0.25	1.0000	1.9231	1.9231	1.9231	1.9231
	0.50	1.0000	1.9174	1.9174	1.9174	1.9174
	0.75	1.0000	1.9079	1.9079	1.9079	1.9079
	1.00	1.0000	1.8938	1.8938	1.8938	1.8938
4	0.25	1.0000	1.4607	1.4607	1.4607	1.4607
	0.50	1.0000	1.4556	1.4556	1.4556	1.4556
	0.75	1.0000	1.4466	1.4466	1.4466	1.4466
	1.00	1.0026	1.4343	1.4305	1.4368	1.4331
6	0.25	1.0000	1.3071	1.3071	1.3071	1.3071
	0.50	1.0000	1.3026	1.3026	1.3026	1.3026
	0.75	1.0019	1.2959	1.2934	1.2978	1.2953
	1.00	1.0059	1.2863	1.2787	1.2922	1.2846
8	0.25	1.0000	1.2301	1.2301	1.2301	1.2301
	0.50	1.0000	1.2267	1.2267	1.2267	1.2267
	0.75	1.0026	1.2211	1.2180	1.2236	1.2205
	1.00	1.0053	1.2132	1.2068	1.2185	1.2121
10	0.25	1.0000	1.1839	1.1839	1.1839	1.1839
	0.50	1.0000	1.1811	1.1811	1.1811	1.1811
	0.75	1.0032	1.1766	1.1729	1.1798	1.1761
	1.00	1.0067	1.1699	1.1621	1.1765	1.1688
12	0.25	1.0000	1.1536	1.1536	1.1536	1.1536
	0.50	1.0000	1.1508	1.1508	1.1508	1.1508
	0.75	1.0038	1.1468	1.1424	1.1507	1.1463
	1.00	1.0080	1.1412	1.1321	1.1492	1.1401
14	0.25	1.0000	1.1316	1.1316	1.1316	1.1316
	0.50	1.0000	1.1294	1.1294	1.1294	1.1294
	0.75	1.0000	1.1254	1.1254	1.1254	1.1254
	1.00	1.0047	1.1204	1.1151	1.1251	1.1198
16	0.25	1.0000	1.1148	1.1148	1.1148	1.1148
	0.50	1.0000	1.1131	1.1131	1.1131	1.1131
	0.75	1.0000	1.1097	1.1097	1.1097	1.1097
	1.00	1.0054	1.1052	1.0993	1.1106	1.1046
18	0.25	1.0000	1.1024	1.1024	1.1024	1.1024
	0.50	1.0000	1.1007	1.1007	1.1007	1.1007
	0.75	1.0000	1.0973	1.0973	1.0973	1.0973
	1.00	1.0061	1.0934	1.0868	1.0994	1.0928
20	0.25	1.0000	1.0923	1.0923	1.0923	1.0923
	0.50	1.0000	1.0906	1.0906	1.0906	1.0906
	0.75	1.0000	1.0878	1.0878	1.0878	1.0878
	1.00	1.0068	1.0838	1.0765	1.0906	1.0833

TABLE 4

The values of $RE\left(\hat{\theta}_2,\theta_2^*\right), RE\left(t_{1m},\theta_2^*\right), RE\left(t_{3m},\hat{\theta}_2\right), RE\left(t_{3m},\theta_2^*\right)$ *and* $RE\left(t_{1m},\hat{\theta}_2\right)$ *when* $R=1.00$

n	α	$RE\left(\hat{\theta}_2,\theta_2^*\right)$	$RE\left(t_{1m},\theta_2^*\right)$	$RE\left(t_{1m},\hat{\theta}_2\right)$	$RE\left(t_{3m},\theta_2^*\right)$	$RE\left(t_{3m},\hat{\theta}_2\right)$
2	0.25	1.0000	2.6410	2.6410	2.6410	2.6410
	0.50	1.0000	2.6310	2.6310	2.6310	2.6310
	0.75	1.0000	2.6140	2.6140	2.6140	2.6140
	1.00	1.0000	2.5890	2.5890	2.5890	2.5890
4	0.25	1.0000	1.8190	1.8190	1.8190	1.8190
	0.50	1.0000	1.8100	1.8100	1.8100	1.8100
	0.75	1.0000	1.7940	1.7940	1.7940	1.7940
	1.00	1.0026	1.7720	1.7674	1.7746	1.7700
6	0.25	1.0000	1.5460	1.5460	1.5460	1.5460
	0.50	1.0000	1.5380	1.5380	1.5380	1.5380
	0.75	1.0019	1.5260	1.5231	1.5279	1.5250
	1.00	1.0059	1.5090	1.5001	1.5149	1.5060
8	0.25	1.0000	1.4090	1.4090	1.4090	1.4090
	0.50	1.0000	1.4030	1.4030	1.4030	1.4030
	0.75	1.0026	1.3930	1.3895	1.3956	1.3920
	1.00	1.0053	1.3790	1.3717	1.3843	1.3770
10	0.25	1.0000	1.3270	1.3270	1.3270	1.3270
	0.50	1.0000	1.3220	1.3220	1.3220	1.3220
	0.75	1.0032	1.3140	1.3098	1.3172	1.3130
	1.00	1.0067	1.3020	1.2934	1.3087	1.3000
12	0.25	1.0000	1.2730	1.2730	1.2730	1.2730
	0.50	1.0000	1.2680	1.2680	1.2680	1.2680
	0.75	1.0038	1.2610	1.2562	1.2648	1.2600
	1.00	1.0080	1.2510	1.2410	1.2590	1.2490
14	0.25	1.0000	1.2340	1.2340	1.2340	1.2340
	0.50	1.0000	1.2300	1.2300	1.2300	1.2300
	0.75	1.0000	1.2230	1.2230	1.2230	1.2230
	1.00	1.0047	1.2140	1.2083	1.2187	1.2130
16	0.25	1.0000	1.2040	1.2040	1.2040	1.2040
	0.50	1.0000	1.2010	1.2010	1.2010	1.2010
	0.75	1.0000	1.1950	1.1950	1.1950	1.1950
	1.00	1.0054	1.1870	1.1807	1.1924	1.1860
18	0.25	1.0000	1.1820	1.1820	1.1820	1.1820
	0.50	1.0000	1.1790	1.1790	1.1790	1.1790
	0.75	1.0000	1.1730	1.1730	1.1730	1.1730
	1.00	1.0061	1.1660	1.1590	1.1721	1.1650
20	0.25	1.0000	1.1640	1.1640	1.1640	1.1640
	0.50	1.0000	1.1610	1.1610	1.1610	1.1610
	0.75	1.0000	1.1560	1.1560	1.1560	1.1560
	1.00	1.0068	1.1490	1.1413	1.1558	1.1480

TABLE 5

The values of $RE\left(\hat{\theta}_2,\theta_2^*\right), RE\left(t_{1m},\theta_2^*\right), RE\left(t_{3m},\hat{\theta}_2\right), RE\left(t_{3m},\theta_2^*\right)$ *and* $RE\left(t_{1m},\hat{\theta}_2\right)$ *when* $R = 2.00$

n	α	$RE\left(\hat{\theta}_2,\theta_2^*\right)$	$RE\left(t_{1m},\theta_2^*\right)$	$RE\left(t_{1m},\hat{\theta}_2\right)$	$RE\left(t_{3m},\theta_2^*\right)$	$RE\left(t_{3m},\hat{\theta}_2\right)$
2	0.25	1.0000	7.5640	7.5640	7.5640	7.5640
	0.50	1.0000	7.5240	7.5240	7.5240	7.5240
	0.75	1.0000	7.4560	7.4560	7.4560	7.4560
	1.00	1.0000	7.3560	7.3560	7.3560	7.3560
4	0.25	1.0000	4.2760	4.2760	4.2760	4.2760
	0.50	1.0000	4.2400	4.2400	4.2400	4.2400
	0.75	1.0000	4.1760	4.1760	4.1760	4.1760
	1.00	1.0026	4.0880	4.0774	4.0906	4.0800
6	0.25	1.0000	3.1840	3.1840	3.1840	3.1840
	0.50	1.0000	3.1520	3.1520	3.1520	3.1520
	0.75	1.0019	3.1040	3.0981	3.1059	3.1000
	1.00	1.0059	3.0360	3.0181	3.0419	3.0240
8	0.25	1.0000	2.6360	2.6360	2.6360	2.6360
	0.50	1.0000	2.6120	2.6120	2.6120	2.6120
	0.75	1.0026	2.5720	2.5655	2.5746	2.5680
	1.00	1.0053	2.5160	2.5027	2.5213	2.5080
10	0.25	1.0000	2.3080	2.3080	2.3080	2.3080
	0.50	1.0000	2.2880	2.2880	2.2880	2.2880
	0.75	1.0032	2.2560	2.2488	2.2592	2.2520
	1.00	1.0067	2.2080	2.1934	2.2147	2.2000
12	0.25	1.0000	2.0920	2.0920	2.0920	2.0920
	0.50	1.0000	2.0720	2.0720	2.0720	2.0720
	0.75	1.0038	2.0440	2.0362	2.0478	2.0400
	1.00	1.0080	2.0040	1.9880	2.0120	1.9960
14	0.25	1.0000	1.9360	1.9360	1.9360	1.9360
	0.50	1.0000	1.9200	1.9200	1.9200	1.9200
	0.75	1.0000	1.8920	1.8920	1.8920	1.8920
	1.00	1.0047	1.8560	1.8473	1.8607	1.8520
16	0.25	1.0000	1.8160	1.8160	1.8160	1.8160
	0.50	1.0000	1.8040	1.8040	1.8040	1.8040
	0.75	1.0000	1.7800	1.7800	1.7800	1.7800
	1.00	1.0054	1.7480	1.7387	1.7534	1.7440
18	0.25	1.0000	1.7280	1.7280	1.7280	1.7280
	0.50	1.0000	1.7160	1.7160	1.7160	1.7160
	0.75	1.0000	1.6920	1.6920	1.6920	1.6920
	1.00	1.0061	1.6640	1.6540	1.6701	1.6600
20	0.25	1.0000	1.6560	1.6560	1.6560	1.6560
	0.50	1.0000	1.6440	1.6440	1.6440	1.6440
	0.75	1.0000	1.6240	1.6240	1.6240	1.6240
	1.00	1.0068	1.5960	1.5853	1.6028	1.5920

TABLE 6

The values of $RE\left(\hat{\theta}_2,\theta_2^*\right), RE\left(t_{1m},\theta_2^*\right), RE\left(t_{3m},\hat{\theta}_2\right), RE\left(t_{3m},\theta_2^*\right)$ *and* $RE\left(t_{1m},\hat{\theta}_2\right)$ *when* $R = 3.00$

n	α	$RE\left(\hat{\theta}_2,\theta_2^*\right)$	$RE\left(t_{1m},\theta_2^*\right)$	$RE\left(t_{1m},\hat{\theta}_2\right)$	$RE\left(t_{3m},\theta_2^*\right)$	$RE\left(t_{3m},\hat{\theta}_2\right)$
2	0.25	1.0000	15.7690	15.7690	15.7690	15.7690
	0.50	1.0000	15.6790	15.6790	15.6790	15.6790
	0.75	1.0000	15.5260	15.5260	15.5260	15.5260
	1.00	1.0000	15.3010	15.3010	15.3010	15.3010
4	0.25	1.0000	8.3710	8.3710	8.3710	8.3710
	0.50	1.0000	8.2900	8.2900	8.2900	8.2900
	0.75	1.0000	8.1460	8.1460	8.1460	8.1460
	1.00	1.0026	7.9480	7.9274	7.9506	7.9300
6	0.25	1.0000	5.9140	5.9140	5.9140	5.9140
	0.50	1.0000	5.8420	5.8420	5.8420	5.8420
	0.75	1.0019	5.7340	5.7231	5.7359	5.7250
	1.00	1.0059	5.5810	5.5481	5.5869	5.5540
8	0.25	1.0000	4.6810	4.6810	4.6810	4.6810
	0.50	1.0000	4.6270	4.6270	4.6270	4.6270
	0.75	1.0026	4.5370	4.5255	4.5396	4.5280
	1.00	1.0053	4.4110	4.3877	4.4163	4.3930
10	0.25	1.0000	3.9430	3.9430	3.9430	3.9430
	0.50	1.0000	3.8980	3.8980	3.8980	3.8980
	0.75	1.0032	3.8260	3.8138	3.8292	3.8170
	1.00	1.0067	3.7180	3.6934	3.7247	3.7000
12	0.25	1.0000	3.4570	3.4570	3.4570	3.4570
	0.50	1.0000	3.4120	3.4120	3.4120	3.4120
	0.75	1.0038	3.3490	3.3362	3.3528	3.3400
	1.00	1.0080	3.2590	3.2330	3.2670	3.2410
14	0.25	1.0000	3.1060	3.1060	3.1060	3.1060
	0.50	1.0000	3.0700	3.0700	3.0700	3.0700
	0.75	1.0000	3.0070	3.0070	3.0070	3.0070
	1.00	1.0047	2.9260	2.9123	2.9307	2.9170
16	0.25	1.0000	2.8360	2.8360	2.8360	2.8360
	0.50	1.0000	2.8090	2.8090	2.8090	2.8090
	0.75	1.0000	2.7550	2.7550	2.7550	2.7550
	1.00	1.0054	2.6830	2.6687	2.6884	2.6740
18	0.25	1.0000	2.6380	2.6380	2.6380	2.6380
	0.50	1.0000	2.6110	2.6110	2.6110	2.6110
	0.75	1.0000	2.5570	2.5570	2.5570	2.5570
	1.00	1.0061	2.4940	2.4790	2.5001	2.4850
20	0.25	1.0000	2.4760	2.4760	2.4760	2.4760
	0.50	1.0000	2.4490	2.4490	2.4490	2.4490
	0.75	1.0000	2.4040	2.4040	2.4040	2.4040
	1.00	1.0068	2.3410	2.3253	2.3478	2.3320

TABLE 7

The values of $RE\left(\hat{\theta}_2,\theta_2^*\right)$, $RE\left(t_{1m},\theta_2^*\right)$, $RE\left(t_{3m},\hat{\theta}_2\right)$, $RE\left(t_{3m},\theta_2^*\right)$ *and* $RE\left(t_{1m},\hat{\theta}_2\right)$ *when* $R=4.00$

n	α	$RE\left(\hat{\theta}_2,\theta_2^*\right)$	$RE\left(t_{1m},\theta_2^*\right)$	$RE\left(t_{1m},\hat{\theta}_2\right)$	$RE\left(t_{3m},\theta_2^*\right)$	$RE\left(t_{3m},\hat{\theta}_2\right)$
2	0.25	1.0000	27.2560	27.2560	27.2560	27.2560
	0.50	1.0000	27.0960	27.0960	27.0960	27.0960
	0.75	1.0000	26.8240	26.8240	26.8240	26.8240
	1.00	1.0000	26.4240	26.4240	26.4240	26.4240
4	0.25	1.0000	14.1040	14.1040	14.1040	14.1040
	0.50	1.0000	13.9600	13.9600	13.9600	13.9600
	0.75	1.0000	13.7040	13.7040	13.7040	13.7040
	1.00	1.0026	13.3520	13.3174	13.3546	13.3200
6	0.25	1.0000	9.7360	9.7360	9.7360	9.7360
	0.50	1.0000	9.6080	9.6080	9.6080	9.6080
	0.75	1.0019	9.4160	9.3981	9.4179	9.4000
	1.00	1.0059	9.1440	9.0901	9.1499	9.0960
8	0.25	1.0000	7.5440	7.5440	7.5440	7.5440
	0.50	1.0000	7.4480	7.4480	7.4480	7.4480
	0.75	1.0026	7.2880	7.2695	7.2906	7.2720
	1.00	1.0053	7.0640	7.0267	7.0693	7.0320
10	0.25	1.0000	6.2320	6.2320	6.2320	6.2320
	0.50	1.0000	6.1520	6.1520	6.1520	6.1520
	0.75	1.0032	6.0240	6.0048	6.0272	6.0080
	1.00	1.0067	5.8320	5.7934	5.8387	5.8000
12	0.25	1.0000	5.3680	5.3680	5.3680	5.3680
	0.50	1.0000	5.2880	5.2880	5.2880	5.2880
	0.75	1.0038	5.1760	5.1562	5.1798	5.1600
	1.00	1.0080	5.0160	4.9760	5.0240	4.9840
14	0.25	1.0000	4.7440	4.7440	4.7440	4.7440
	0.50	1.0000	4.6800	4.6800	4.6800	4.6800
	0.75	1.0000	4.5680	4.5680	4.5680	4.5680
	1.00	1.0047	4.4240	4.4033	4.4287	4.4080
16	0.25	1.0000	4.2640	4.2640	4.2640	4.2640
	0.50	1.0000	4.2160	4.2160	4.2160	4.2160
	0.75	1.0000	4.1200	4.1200	4.1200	4.1200
	1.00	1.0054	3.9920	3.9707	3.9974	3.9760
18	0.25	1.0000	3.9120	3.9120	3.9120	3.9120
	0.50	1.0000	3.8640	3.8640	3.8640	3.8640
	0.75	1.0000	3.7680	3.7680	3.7680	3.7680
	1.00	1.0061	3.6560	3.6340	3.6621	3.6400
20	0.25	1.0000	3.6240	3.6240	3.6240	3.6240
	0.50	1.0000	3.5760	3.5760	3.5760	3.5760
	0.75	1.0000	3.4960	3.4960	3.4960	3.4960
	1.00	1.0068	3.3840	3.3613	3.3908	3.3680

TABLE 8

The values of $RE\left(\hat{\theta}_2,\theta_2^*\right), RE\left(t_{1m},\theta_2^*\right), RE\left(t_{3m},\hat{\theta}_2\right), RE\left(t_{3m},\theta_2^*\right)$ *and* $RE\left(t_{1m},\hat{\theta}_2\right)$ *when* $R = 5.00$

n	α	$RE\left(\hat{\theta}_2,\theta_2^*\right)$	$RE\left(t_{1m},\theta_2^*\right)$	$RE\left(t_{1m},\hat{\theta}_2\right)$	$RE\left(t_{3m},\theta_2^*\right)$	$RE\left(t_{3m},\hat{\theta}_2\right)$
2	0.25	1.0000	42.0250	42.0250	42.0250	42.0250
	0.50	1.0000	41.7750	41.7750	41.7750	41.7750
	0.75	1.0000	41.3500	41.3500	41.3500	41.3500
	1.00	1.0000	40.7250	40.7250	40.7250	40.7250
4	0.25	1.0000	21.4750	21.4750	21.4750	21.4750
	0.50	1.0000	21.2500	21.2500	21.2500	21.2500
	0.75	1.0000	20.8500	20.8500	20.8500	20.8500
	1.00	1.0026	20.3000	20.2474	20.3026	20.2500
6	0.25	1.0000	14.6500	14.6500	14.6500	14.6500
	0.50	1.0000	14.4500	14.4500	14.4500	14.4500
	0.75	1.0019	14.1500	14.1231	14.1519	14.1250
	1.00	1.0059	13.7250	13.6441	13.7309	13.6500
8	0.25	1.0000	11.2250	11.2250	11.2250	11.2250
	0.50	1.0000	11.0750	11.0750	11.0750	11.0750
	0.75	1.0026	10.8250	10.7975	10.8276	10.8000
	1.00	1.0053	10.4750	10.4197	10.4803	10.4250
10	0.25	1.0000	9.1750	9.1750	9.1750	9.1750
	0.50	1.0000	9.0500	9.0500	9.0500	9.0500
	0.75	1.0032	8.8500	8.8218	8.8532	8.8250
	1.00	1.0067	8.5500	8.4934	8.5567	8.5000
12	0.25	1.0000	7.8250	7.8250	7.8250	7.8250
	0.50	1.0000	7.7000	7.7000	7.7000	7.7000
	0.75	1.0038	7.5250	7.4962	7.5288	7.5000
	1.00	1.0080	7.2750	7.2170	7.2830	7.2250
14	0.25	1.0000	6.8500	6.8500	6.8500	6.8500
	0.50	1.0000	6.7500	6.7500	6.7500	6.7500
	0.75	1.0000	6.5750	6.5750	6.5750	6.5750
	1.00	1.0047	6.3500	6.3203	6.3547	6.3250
16	0.25	1.0000	6.1000	6.1000	6.1000	6.1000
	0.50	1.0000	6.0250	6.0250	6.0250	6.0250
	0.75	1.0000	5.8750	5.8750	5.8750	5.8750
	1.00	1.0054	5.6750	5.6447	5.6804	5.6500
18	0.25	1.0000	5.5500	5.5500	5.5500	5.5500
	0.50	1.0000	5.4750	5.4750	5.4750	5.4750
	0.75	1.0000	5.3250	5.3250	5.3250	5.3250
	1.00	1.0061	5.1500	5.1190	5.1561	5.1250
20	0.25	1.0000	5.1000	5.1000	5.1000	5.1000
	0.50	1.0000	5.0250	5.0250	5.0250	5.0250
	0.75	1.0000	4.9000	4.9000	4.9000	4.9000
	1.00	1.0068	4.7250	4.6933	4.7318	4.7000

TABLE 9

The values of $RE\left(\hat{\sigma}_2, \sigma_2^*\right)$, $RE\left(t_{2m}, \sigma_2^*\right)$, $RE\left(t_{2m}, \hat{\sigma}_2\right)$, $RE\left(t_{4m}, \sigma_2^*\right)$ *and* $RE\left(t_{4m}, \hat{\sigma}_2\right)$

n	α	$RE\left(\hat{\sigma}_2, \sigma_2^*\right)$	$RE\left(t_{2m}, \sigma_2^*\right)$	$RE\left(t_{2m}, \hat{\sigma}_2\right)$	$RE\left(t_{4m}, \sigma_2^*\right)$	$RE\left(t_{4m}, \hat{\sigma}_2\right)$
2	0.25	1.0000	237.370	237.371	237.370	237.371
	0.50	0.9908	59.1780	59.7273	59.1688	59.7180
	0.75	1.0000	26.8190	26.8190	26.8190	26.8190
	1.00	0.9811	15.0340	15.3232	15.0151	15.3040
4	0.25	1.2531	82.9340	66.1850	83.1871	66.3870
	0.50	1.2626	21.2490	16.8300	21.5116	17.0380
	0.75	1.2793	9.8260	7.6807	10.1053	7.8990
	1.00	1.3052	5.8280	4.4652	6.1332	4.6990
6	0.25	1.3006	48.5480	37.3269	48.8486	37.5580
	0.50	1.3142	12.7250	9.6829	13.0392	9.9220
	0.75	1.3390	6.0910	4.5488	6.4300	4.8020
	1.00	1.3783	3.7690	2.7345	4.1473	3.0090
8	0.25	1.3175	34.1460	25.9170	34.4635	26.1580
	0.50	1.3332	9.1630	6.8731	9.4962	7.1230
	0.75	1.3620	4.5370	3.3312	4.8990	3.5970
	1.00	1.4103	2.9180	2.0691	3.3283	2.3600
10	0.25	1.3254	26.3450	19.8775	26.6704	20.1230
	0.50	1.3424	7.2370	5.3909	7.5794	5.6460
	0.75	1.3742	3.6990	2.6917	4.0732	2.9640
	1.00	1.4258	2.4600	1.7254	2.8858	2.0240
12	0.25	1.3297	21.4820	16.1550	21.8117	16.4030
	0.50	1.3478	6.0380	4.4800	6.3858	4.7380
	0.75	1.3811	3.1780	2.3011	3.5591	2.5770
	1.00	1.4371	2.1770	1.5148	2.6141	1.8190
14	0.25	1.3324	18.1700	13.6376	18.5024	13.8870
	0.50	1.3507	5.2210	3.8653	5.5717	4.1250
	0.75	1.3860	2.8240	2.0375	3.2100	2.3160
	1.00	1.4428	1.9840	1.3751	2.4268	1.6820
16	0.25	1.3341	15.7730	11.8225	16.1071	12.0730
	0.50	1.3528	4.6310	3.4232	4.9838	3.6840
	0.75	1.3888	2.5680	1.8490	2.9568	2.1290
	1.00	1.4494	1.8450	1.2729	2.2944	1.5830
18	0.25	1.3353	13.9590	10.4539	14.2943	10.7050
	0.50	1.3543	4.1840	3.0894	4.5383	3.3510
	0.75	1.3907	2.3740	1.7071	2.7647	1.9880
	1.00	1.4529	1.7410	1.1983	2.1939	1.5100
20	0.25	1.3361	12.5400	9.3854	12.8761	9.6370
	0.50	1.3558	3.8350	2.8286	4.1908	3.0910
	0.75	1.3929	2.2230	1.5959	2.6159	1.8780
	1.00	1.4547	1.6590	1.1404	2.1137	1.4530

5. CONCLUSION

It is observed from Tables 1 to 8 that

- for fixed sample size n and the ratio R, the values
- of $RE\left(t_{1m},\theta_2^*\right)$, $RE\left(t_{3m},\hat{\theta}_2\right)$, $RE\left(t_{3m},\theta_2^*\right)$ and $RE\left(t_{1m},\hat{\theta}_2\right)$ decrease as α increases.
- for fixed α and the ratio R, the values of $RE\left(t_{1m},\theta_2^*\right), RE\left(t_{3m},\hat{\theta}_2\right), RE\left(t_{3m},\theta_2^*\right)$ and $RE\left(t_{1m},\hat{\theta}_2\right)$ also decrease as sample size n increases.
- for fixed values of (n,α) the values of $RE\left(t_{1m},\theta_2^*\right), RE\left(t_{3m},\hat{\theta}_2\right), RE\left(t_{3m},\theta_2^*\right)$ and $RE\left(t_{1m},\hat{\theta}_2\right)$ increase considerably as the ratio R increases.
- no trend is observed for $RE\left(\hat{\theta}_2,\theta_2^*\right)$. However the value of $RE\left(\hat{\theta}_2,\theta_2^*\right)$ are almost equal to 'unity'. So both the estimators θ_2^* and $\hat{\theta}_2$ are almost aqually efficient.

In general the values $RE\left(t_{1m},\theta_2^*\right), RE\left(t_{3m},\hat{\theta}_2\right), RE\left(t_{3m},\theta_2^*\right)$ and $RE\left(t_{1m},\hat{\theta}_2\right)$ are greater than 'unity' for all values of (n,R,α). Thus the proposed estimators t_{1m} and t_{3m} are efficient than the usual unbiased estimator θ_2^* and BLUE $\hat{\theta}_2$.

Table 9 exhibits that the values of $RE\left(\hat{\sigma}_2,\sigma_2^*\right), RE\left(t_{2m},\sigma_2^*\right), RE\left(t_{4m},\hat{\sigma}_2\right), RE\left(t_{4m},\sigma_2^*\right)$ and $RE\left(t_{2m},\hat{\sigma}_2\right)$ are greater than 'unity' except when $n = 2, 0.25 \le \alpha \le 1.00$ for $RE\left(\hat{\sigma}_2,\sigma_2^*\right)$. So the proposed MMSE estimators t_{2m} and t_{4m} are more efficient than Chacko and Thomas (2009) estimators σ_2^* and $\hat{\sigma}_2$ with considerable gain in efficiency. We also note that the values of of $RE\left(\hat{\sigma}_2,\sigma_2^*\right), RE\left(t_{2m},\sigma_2^*\right), RE\left(t_{4m},\hat{\sigma}_2\right), RE\left(t_{4m},\sigma_2^*\right)$ and $RE\left(t_{2m},\hat{\sigma}_2\right)$ decrease as sample size n and α increase.

REMARK. If we have a situation with α unknown, we introduce an estimator (moment type) for α as follows. For MTBLD the correlation coefficient between the two variables is given by $\rho = 3\alpha / \pi^2$. If r is the sample correlation coefficient between $X_{(i)i}$ and $Y_{[i]i}$, $i = 1, 2, ..., n$ then the moment type estimator for α is obtained by equating with the population correlation coefficient ρ and is obtained as [see Chacko and Thomas (2009, p. 82)]

$$\hat{\alpha} = \begin{cases} -1 & if \quad r \le \dfrac{-3}{\pi^2} \\ 1 & if \quad r > \dfrac{3}{\pi^2} \\ \dfrac{r\pi^2}{3} & otherwise \end{cases} \tag{46}$$

6. AN APPLICATION

To see the performance of the suggested estimators $(t_{1m}, t_{2m}, t_{3m}, t_{4m})$ over Chacko and Thomas (2009) estimators $\left(\theta_2^*, \hat{\theta}_2, \sigma_2^*, \hat{\sigma}_2\right)$ and usual traditional estimator $\overline{Y}$ involved in MTBLD whose variance is $\dfrac{\sigma_2^2\pi^2}{3n}$, we consider the same data set as given in Chacko and Thomas (2009, p. 82). We reproduced the same as given below:

We consider a bivariate data set from Platt *et al.* (1969) relating to 396 Confir (Pinus Palustrine) trees. In Chen *et al.* (2004) also, the above bivariate data set is reproduced in which the first component X of a bivariate observation represents the diameter in centimetres of the Confir tree at breast height and the second component Y represents height in feet of the tree. Clearly X can be measured easily but it is somewhat difficult to measure Y. Also observations, such ar girth (function of diameter) or height follows normal distribution. It is well known that logistic distribution is having more or less similar properties of a normal distribution (Malik ,1985, P. 123) and hence it is known as an alternative model to normal distribution. Assume that (X,Y) follows Morgenstern type bivariate logistic distribution. We select 10 random samples each of size 10 from the 396 tree data and rank the sampling units of each sample according to the X variate values (diameter of the tree). From the i^{th} sample, we measure the Y variate (height of the tree) corresponding to the i^{th} order statistic of the X variate. The obtained RSS observations are reported in Table 10.

TABLE 10
Ranked set sample (RSS) observations

i	$X_{(i)i}$	$Y_{[i]i}$
1	6.3	11
2	10.1	28
3	3.8	6
4	4.5	10
5	6.0	16
6	15.9	28
7	38.6	42
8	17.8	38
9	41.4	177
10	51.7	219

The sample correlation between X and Y is 0.883.Thus, from (46), an estimate of α is taken as 1 (see Chacko and Thomas (2009)).

We have computed the estimates of $R = \dfrac{\sigma_2}{\theta_2}$:

(i) as $\hat{R}_1 = \dfrac{\sigma_2^*}{\theta_2^*} = 1.6628$ used in equations (14) and (15),

(ii) as $\hat{R}_2 = \dfrac{\hat{\sigma}_2}{\hat{\theta}_2} = 1.7789$ used in equations (34) and (35).

Table 11 presents the values of RSS estimators $\left(\theta_2^*, \hat{\theta}_2, \sigma_2^*, \hat{\sigma}_2\right)$ and $\left(t_{1m}, t_{2m}, t_{3m}, t_{4m}\right)$, along with their MSEs.

TABLE 11
Estimators and their MSEs

Estimator	Estimate	$MSE \times \sigma_2^{-2}$	Suggested Estimator	Estimate	$MSE \times \sigma_2^{-2}$	$Var\left(\overline{Y}\right) \times \sigma_2^{-2}$
θ_2^*	57.500	0.3017	t_{1m}	31.349	0.1645	0.3287
$\hat{\theta}_2$	60.745	0.2997	t_{3m}	31.177	0.1538	
σ_2^*	95.260	1.9443	t_{2m}	32.473	0.6603	
$\hat{\sigma}_2$	*108.062*	*1.0236*	t_{4m}	*53.401*	*0.5058*	

It is observed from Table 11 that there is substantial reduction in MSEs of θ_2 over the usual estimator $\overline{Y}$ and Chacko and Thomas (2009) estimators $\left(\theta_2^*, \hat{\theta}_2\right)$. We also note that the proposed estimators $\left(t_{2m}, t_{4m}\right)$ of σ_2 have smaller MSEs than the Chacko and Thomas (2009) estimators $\left(\sigma_2^*, \hat{\sigma}_2\right)$. So the proposed estimators $\left(t_{1m}, t_{3m}\right)$ and $\left(t_{2m}, t_{4m}\right)$ are to be preferred over existing estimators $\left(\bar{Y}, \theta_2^*, \hat{\theta}_2\right)$ and $\left(\sigma_2^*, \hat{\sigma}_2\right)$ respectively. Thus the present study establish the advantage of estimating the mean height of trees more closely to the true value of the mean using RSS.

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SUMMARY

An improved estimation of parameters of Morgenstern type bivariate logistic distribution using ranked set sampling

In this paper we have suggested some improved estimator of parameters of Morgenstern type bivariate logistic distribution (MTBLD) using ranked set sampling. We have shown the superiority of the proposed estimators over Chacko and Thomas (2009) estimators.

Keywords: Best linear unbiased estimator; concomitants of order statistics; minimum mean squared error estimator; Morgenstern type bivariate logistic distribuion; ranked set sampling.