STATISTICA, anno LXXIII, n. 4, 2013

BIVARIATE DISCRETE LINNIK DISTRIBUTION

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1. INTRODUCTION

Linnik (1963) proved that the function

$$\phi_\alpha(t) = \left(1+|t|^\alpha\right)^{-1}, \; 0 < \alpha \leq 2 \tag{1}$$

is the characteristic function of a random variable with support on $(-\infty, \infty)$. The corresponding distribution is called Linnik distribution. For $\alpha = 2$, the Linnik distribution coincides with the Laplace distribution with density

$$f(x) = e^{-\frac{|x|}{2}}$$

As a generalization to (1), Devroye (1993) introduced generalized Linnik distribution with characteristic function

$$\phi_{\alpha,\beta}(t) = \left(\frac{1}{1+|t|^\alpha}\right)^\beta, \; \beta > 0. \tag{2}$$

It was showed that the random variable with characteristic function (2) can be represented as $S_\alpha\left(V_\beta^{\frac{1}{\alpha\beta}}\right)$ where S_α is symmetric stable with characteristic function $\phi_{S_\alpha}(t) = e^{-|t|^\alpha}$ where V_β denotes a random variable following gamma distribution.

Pakes (1995) introduced positive Linnik distribution. A nonnegative random variable is said to follow positive Linnik distribution if its Laplace transform is

$$\phi(\lambda) = \left(\frac{1}{1+\lambda^\alpha}\right)^\upsilon, \; 0 < \alpha \leq 1, \upsilon > 0, \lambda \geq 0. \tag{3}$$

Jayakumar and Gadag (1999) obtained distribution function of the Laplace transform in (3) and called as quasi factorial gamma. A non negative random variable X is said to follow quasi factorial gamma distribution if its distribution function is

$$F_{\alpha,\upsilon}(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\mathrm{k}+\upsilon)\mathrm{x}^{\alpha(\mathrm{k}+\upsilon)}}{\Gamma(\upsilon)k!\Gamma(1+\alpha(\mathrm{k}+\upsilon))},\ x\geq 0,\ \upsilon>0,\ 0<\alpha\leq 1. \tag{4}$$

This is a rich family of distributions as it includes important distributions like gamma, Mittag-Leffler, exponential, etc. Jayakumar and Gadag (1999) studied various distributional properties of quasi factorial gamma. They have also developed first order stationary autoregressive process with marginals as quasi factorial gamma distribution.

Pillai and Jayakumar (1995) introduced discrete Mittag-Leffler distribution as a generalization to the geometric distribution. A random variable X on {0, 1, 2, …} is said to follow discrete Mittag-Leffler distribution if its probability generating function (p.g.f.) is

$$P(s)=\frac{1}{1+c(1-s)^{\alpha}},\ 0<\alpha\leq 1,\ c<0,\ |s|\leq 1 \tag{5}$$

The discrete Mittag-Leffler distribution can be viewed as the distribution of geometric sum of independently and identically distributed Sibuya random variables. In a sequence of independent Bernoulli trials, let $\frac{\alpha}{k}$ be the probability of success in k^{th} trial. Then the number of trials required to obtain the first success has Sibuya distribution (see, Devroye (1993)). When $\alpha=1$, we note that discrete Mittag-Leffler distribution coincides with geometric distribution.

Christoph and Schreiber (1998a) studied the discrete analogue of the positive Linnik distribution in (3). A non negative integer valued random variable is said to be discrete Linnik distributed with exponent $\alpha \in (0, 1]$ and a scale parameter 'c' if it has p.g.f.

$$P(s)=\begin{cases}\left(\frac{1}{1+c(1-s)^{\alpha}}\right)^{\upsilon} & for\, 0<\upsilon<\infty \\ e^{-c(1-s)^{\alpha}} & for\ \ \upsilon=\infty\end{cases} \tag{6}$$

Probabilities of the discrete Linnik distribution, some properties of the probabilities and characterization via survival function are investigated in Christoph and Schreiber (1998a). Bouzar (2002) obtained representations for discrete Linnik distribution using Poisson mixtures. Christoph and Schreiber (1998b) proved that discrete Linnik distribution belongs to the domain of discrete attraction of a discrete stable law as well as to the domain of attraction of non negative strictly stable law and obtained the rate of convergence in both cases.

Discrete Linnik distribution is a rich family of distributions which includes many important distributions. It belongs to the class of discrete self decomposable distributions. When $\upsilon=1$, we get discrete Mittag-Leffler distribution and for $\alpha=1$, it coincides with negative binomial distribution. For $\alpha=1$ and $\upsilon=1$, we get the geometric distribution. Discrete stable distributions can be successfully applied to model discrete heavy tailed data sets. Pillai and Jayakumar(1995) developed autoregressive models with marginals as discrete Mittag-Leffler distribution. Jayakumar and Thomas Mathew (2008) have developed generalized INAR(1) models with discrete Linnik marginal and its extensions to the autoregressive-moving average situation. The models are applicable in modeling discrete variate time series when the marginals follow discrete Mittag-Leffler and discrete Linnik laws. Discrete Mittag-Leffler distribution provides an approximate median residual life time function which can be considered as an alternative to the mean residual life time function. The widespread applications of these distributions motivated us to introduce bivariate form of the discrete Linnik distribution. In Section 2, we define a bivariate discrete Linnik distribution. Various distributional properties of bivariate discrete Linnik distribution are studied in Section 3. In Section 4, characterizations of the distribution are obtained. First order autoregressive processes with marginals follow bivariate discrete Linnik distributions are developed in Section 5.

2. BIVARIATE DISCRETE LINNIK DISTRIBUTION

We define a bivariate discrete Linnik distribution as follows:

DEFINITION 2.1. A non negative integer valued random vector (X, Y) is said to follow bivariate discrete Linnik distribution if it has p.g.f.

$$P(s_1,s_2)=\left(\frac{1}{\left(1+c_1(1-s_1)^{\alpha_1}\right)\left(1+c_2(1-s_2)^{\alpha_2}\right)-\theta c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}\right)^{\upsilon} \tag{7}$$

$$c_1,\ c_2>0,\ \upsilon>0,\ 0<\alpha_1,\ \alpha_2\le 1,\ 0\le\theta\le 1,\ |s_1|,|s_2|\le 1.$$

We represent the distribution with the above p.g.f. as BDL$(c_1,\ c_2,\alpha_1,\ \alpha_2,\theta,\upsilon)$. Note that BDL$(c_1,\ c_2,\alpha_1,\ \alpha_2,\theta,\upsilon)$ generalizes many important distributions. When $\alpha_1=\alpha_2=1$,

$$P(s_1,s_2)=\left(\frac{1}{\left(1+c_1(1-s_1)\right)\left(1+c_2(1-s_2)\right)-\theta c_1c_2(1-s_1)(1-s_2)}\right)^{\upsilon} \tag{8}$$

represents the p.g.f. of bivariate negative binomial distribution. The distribution corresponding to (8) is denoted by BNBD$(c_1,\ c_2,\theta,\upsilon)$. When $\upsilon=1$, (7) becomes the p.g.f. of bivariate discrete Mittag-Leffler distribution denoted by BDML$(c_1,\ c_2,\alpha_1,\ \alpha_2,\theta)$. Moreover, $\alpha_1=\alpha_2=\upsilon=1$, $P(s_1,s_2)$ represents the p.g.f. of bivariate geometric distribution denoted by BDL$(c_1,\ c_2,\theta)$.

From (7), it is clear that p.g.f of the components of *(X, Y)* are that of univariate discrete Linnik distribution given in (6).

When $\theta=1$, (7) becomes

$$P(s_1,s_2)=\left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon} \tag{9}$$

3. DISTRIBUTIONAL PROPERTIES

We show that a random vector *(X, Y)* following BDL$(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ is normally attracted to bivariate positive stable law.

Suppose that *(X, Y)* follows BDL$(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ with p.g.f. in (9). Then the Laplace transform corresponding to (9) is

$$\phi(\lambda_1, \lambda_2) = \left(\frac{1}{1 + c_1\left(1 - e^{-\lambda_1}\right)^{\alpha_1} + c_2\left(1 - e^{-\lambda_2}\right)^{\alpha_2}} \right)^{\upsilon}. \tag{10}$$

Consider a sequence $\{(X_i, Y_i),\ i \geq 1\}$ of independently and identically distributed random vectors with Laplace transform in (10). Define

$$U_n = n^{-\frac{1}{\alpha_1}}\left(X_1 + X_2 + \ldots + X_n\right) \text{ and}$$

$$V_n = n^{-\frac{1}{\alpha_2}}\left(Y_1 + Y_2 + \ldots + Y_n\right).$$

Then $(U_n,\ V_n)$ has the Laplace transform

$$\begin{aligned}
\psi_{U_n, V_n}(\lambda_1, \lambda_2) &= E\left(e^{-(\lambda_1 U_n + \lambda_2 V_n)}\right) \\
&= \left(\phi(\lambda_1 n^{-\frac{1}{\alpha_1}},\ \lambda_2 n^{-\frac{1}{\alpha_2}}) \right)^n \\
&= \left(\frac{1}{1 + c_1\left(1 - e^{-\lambda_1 n^{-\frac{1}{\alpha_1}}}\right)^{\alpha_1} + c_2\left(1 - e^{-\lambda_2 n^{-\frac{1}{\alpha_2}}}\right)^{\alpha_2}} \right)^{n\upsilon}.
\end{aligned}$$

Also we have

$$\left(1 - e^{-\lambda_1 n^{-\frac{1}{\alpha_1}}}\right)^{\alpha_1} = \frac{\lambda_1^{\alpha_1}}{n}\left(1 + o(1/n)\right) \quad \text{and} \quad \left(1 - e^{-\lambda_2 n^{-\frac{1}{\alpha_2}}}\right)^{\alpha_2} = \frac{\lambda_2^{\alpha_2}}{n}\left(1 + o(1/n)\right).$$

Hence when $n \to \infty$, we get $\psi_{U_n, V_n}(\lambda_1,\ \lambda_2) \to e^{-c_1\upsilon\lambda_1^{\alpha_1} - c_2\upsilon\lambda_2^{\alpha_2}}$.

Hence BDL$(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ distribution is normally attracted to bivariate positive stable law.

We can obtain a bivariate form of the quasi factorial gamma distribution discussed in (3) as a limit of random vectors following bivariate discrete Linnik distribution. For this, consider a random vector *(X, Y)* with Laplace transform in (10). Replacing c_1 and c_2 by $c_1 n^{\alpha_1}$ and $c_2 n^{\alpha_2}$ respectively. Then the Laplace transform of $\left(\frac{X}{n}, \frac{Y}{n}\right)$ will be

$$\phi_n(\lambda_1, \lambda_2) = \left(\frac{1}{1 + c_1 n^{\alpha_1}\left(1 - e^{-\frac{\lambda_1}{n}}\right)^{\alpha_1} + c_2 n^{\alpha_2}\left(1 - e^{-\frac{\lambda_2}{n}}\right)^{\alpha_2}} \right)^{n\upsilon}$$

When $n \to \infty$, we get

$$\phi_n(\lambda_1, \lambda_2) = \left(\frac{1}{1 + c_1 \lambda^{\alpha_1} + c_2 \lambda^{\alpha_2}} \right)^{\upsilon}.$$

Now, we prove attraction of BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ towards bivariate discrete stable law. For this we consider the operator '$\oplus$', denoted in Jayakumar (1995) in bivariate set up. Let *(X, Y)* have p.g.f. $P(s_1, s_2)$, then $(p \oplus X, p \oplus Y)$ is defined (in distribution) by p.g.f. $P(1 - p + ps_1, 1 - p + ps_2)$.

Let $\{(X_i, Y_i),\ i \geq 1\}$ be a sequence of independently and identically distributed random vectors according to BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$. Define

$$U_n = n^{-\frac{1}{\alpha_1}} \oplus (X_1 + X_2 + \ldots + X_n) \qquad \text{and} \qquad V_n = n^{-\frac{1}{\alpha_2}} \oplus (Y_1 + Y_2 + \ldots + Y_n)$$

Then (U_n, V_n) is asymptotically distributed according to bivariate discrete stable.

The p.g.f. of the random vector $\{(X_i, Y_i),\ i \geq 1\}$ is

$$P(s_1, s_2) = \left(\frac{1}{1 + c_1(1 - s_1)^{\alpha_1} + c_2(1 - s_2)^{\alpha_2}} \right)^{\upsilon}.$$

The p.g.f. of (U_n, V_n) is

$$P_{U_n, V_n}(s_1, s_2) = \left(\frac{1}{\left(1 + \frac{c_1}{n}(1 - s_1)^{\alpha_1} + \frac{c_2}{n}(1 - s_2)^{\alpha_2}\right)} \right)^{n\upsilon}.$$

As $n \to \infty$, we get

$$P_{U_n, V_n}(s_1, s_2) = e^{-c_1\upsilon(1-s_1)^{\alpha_1} - c_2\upsilon(1-s_2)^{\alpha_2}} \tag{11}$$

We obtain BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ as a mixture of bivariate discrete stable distribution and gamma distribution with parameters β and υ.

Take the joint distribution of the random vector *(S, T)* as bivariate discrete stable having p.g.f. in (11), exponents α_1 and α_2 with parameters $c_1 = c_2 = W$. Suppose that *W* follows gamma distribution with parameters β and υ. Consider the unconditional distribution of *(S, T)*. Its p.g.f. is

$$P(s_1, s_2) = \int_0^\infty e^{-W((1-s_1)^{\alpha_1} + (1-s_2)^{\alpha_2})} \frac{\beta^\upsilon e^{-\beta w} w^{\upsilon-1}}{\Gamma(\upsilon)} dw$$

$$= \left(\frac{\beta}{\beta + (1-s_1)^{\alpha_1} + (1-s_2)^{\alpha_2}} \right)^\upsilon = \left(\frac{1}{1 + \frac{1}{\beta}((1-s_1)^{\alpha_1} + (1-s_2)^{\alpha_2})} \right)^\upsilon.$$

Hence $(S,\ T)$ follows BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ such that $c_1 = c_2 = \dfrac{1}{\beta}$.

4. CHARACTERIZATION OF BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ THROUGH NEGATIVE BINOMIAL COMPOUNDING.

In order to obtain characterizations of BDL $(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$, we consider the probability distributions of random sums of independently and identically distributed random vectors. Let $U_N = \sum_{i=1}^{N} X_i$, $V = \sum_{i=1}^{N} Y_i$ where $\{(X_i, Y_i),\ i \geq 1\}$ is a sequence of independently and identically distributed random vectors with p.g.f. $Q(s_1, s_2)$ and N follows negative binomial distribution with p.m.f.

$$P(N = n) = \binom{n-1}{\upsilon-1} p^\upsilon (1-p)^{n-\upsilon} \tag{12}$$

$$\upsilon = 1, 2, 3, \ldots;\ n = \upsilon, \upsilon+1, \upsilon+2, \ldots;\ 0 < p < 1.$$

The p.g.f. of N is

$$P(s) = \left(\frac{ps}{1-(1-p)s} \right)^\upsilon \tag{13}$$

Then the p.g.f. of (U_N, V_N) is

$$P(s_1,s_2)=E\left(S_1^{U_N} S_2^{V_N}\right)=\left(\frac{pQ(S_1,S_2)}{1-(1-p)Q(S_1,S_2)}\right)^{\upsilon}. \tag{14}$$

Using this negative binomial compounding, a characterization of BDL$(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ is obtained in the following theorem.

THEOREM 4.1. Let $\{(X_i,Y_i),\ i\geq 1\}$ be a sequence of independently and identically distributed random vectors and N independent of $(X_i,Y_i),\ i\geq 1,$ be a random variable following negative binomial distribution in (12). The $\left(p^{\frac{1}{\alpha_1}}\oplus U_N,\ p^{\frac{1}{\alpha_2}}\oplus V_N\right)$ follow BDL $(c_1,c_2,\alpha_1,\alpha_2,1,\upsilon)$ if and only if $\{(X_i,Y_i),\ i\geq 1\}$ follows BDML $(c_1,c_2,\alpha_1,\alpha_2,1)$.

PROOF. Suppose that $\{(X_i,Y_i),\ i\geq 1\}$ follows BDML $(c_1,c_2,\alpha_1,\alpha_2,1)$ distribution with p.g.f. $Q(s_1,s_2)$. Then the p.g.f. of $\left(p^{\frac{1}{\alpha_1}}\oplus U_N,\ p^{\frac{1}{\alpha_2}}\oplus V_N\right)$ is

$$P(s_1,s_2)=E\left(s_1^{p^{\frac{1}{\alpha_1}}\oplus U_N} s_2^{p^{\frac{1}{\alpha_2}}\oplus V_N}\right).$$

From (14),

$$P(s_1,s_2)=\left(\frac{pQ\left(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1,\ 1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2\right)}{1-(1-p)Q\left(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1,\ 1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2\right)}\right). \tag{15}$$

In (15) substituting the p.g.f. of $(X_i,Y_i),\ i\geq 1$, we get

$$P(s_1,s_2)=\left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon}.$$

To prove the converse, suppose that $\left(p^{\frac{1}{\alpha_1}}\oplus U_N,\ p^{\frac{1}{\alpha_2}}\oplus V_N\right)$ follows BDL$(c_1,c_2,\alpha_1,\alpha_2,1,\upsilon)$ distribution. Substituting its p.g.f. in (15)

$$\left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon}=\left(\frac{pQ\left(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1,1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2\right)}{1-(1-p)Q\left(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1,1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2\right)}\right)^{\upsilon}.$$

On simplification we get

$$Q(s_1,s_2)=\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}.$$

REMARK 4.1 *Let* $\{(X_i,Y_i),\ i\geq 1\}$ *be a sequence of independently and identically distributed random vectors. Then* $(p\oplus U_N,\ p\oplus V_N)$ *follows BNBD* $(c_1,\ c_2,1,\upsilon)$ *distribution if and only if* $\{(X_i,Y_i),\ i\geq 1\}$ *have BGD* $(c_1,\ c_2,1)$ *where N is independent of* $\{(X_i,Y_i),\ i\geq 1\}$ *and follows negative binomial distribution.*

Proof of Remark 4.1 is omitted as it is obvious.

Now we introduce BDL$(c_1,c_2,\alpha_1,\alpha_2,\theta,\upsilon)$ using negative binomial sum of independent random vectors in which the components are independently distributed as Mittag-Leffler.

THEOREM 4.2. *Suppose that* $\{(X_i,Y_i),\ i\geq 1\}$ *is a sequence of independently and identically distributed random vectors and N, independent of* $(X_i,Y_i),\ i\geq 1$ *has the negative binomial distribution in (12). Then* $\left(p^{\frac{1}{\alpha_1}}\oplus U_N,\ p^{\frac{1}{\alpha_2}}\oplus V_N\right)$ *follows* BDL$(c_1,\ c_2,\alpha_1,\ \alpha_2,1-p,\upsilon)$ *distribution if and only if the components* $X_i^{'}s$ *and* $Y_i^{'}s$ *are independently distributed according to discrete Mittag-Leffler with parameters* (α_1,c_1) *and* (α_2,c_2) *respectively.*

PROOF Assume that the components of $(X_i,Y_i),\ i\geq 1$ are independently and distributed according to discrete Mittag-Leffler with parameters (α_1,c_1) and (α_2,c_2) respectively. Therefore the joint p.g.f. of $(X_i,Y_i),\ i\geq 1$ is

$$Q(s_1,s_2)=\frac{1}{\left(1+c_1(1-s_1)^{\alpha_1}\right)\left(1+c_2(1-s_2)^{\alpha_2}\right)}$$

From (15) the p.g.f. of $\left(p^{\frac{1}{\alpha_1}}\oplus U_N,\ p^{\frac{1}{\alpha_2}}\oplus V_N\right)$ is

$$P(s_1,s_2)=\left(\frac{p}{\left(1+pc_1(1-s_1)^{\alpha_1}\right)\left(1+pc_2(1-s_2)^{\alpha_2}\right)-1+p}\right)^{\upsilon}$$

$$=\left(\frac{p}{pc_1(1-s_1)^{\alpha_1}+pc_2(1-s_2)^{\alpha_2})+p^2c_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}+p}\right)^{\upsilon}$$

$$=\left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})+pc_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}\right)^{\upsilon}.$$

Comparing with (7), we get $\theta=1-p$.

Conversely, suppose that $\left(p^{\frac{1}{\alpha_1}}\oplus U_N,\ p^{\frac{1}{\alpha_2}}\oplus V_N\right)$ follows BDL$(c_1,\ c_2,\alpha_1,\ \alpha_2,1-p,\upsilon)$.

From (15) we get

$$\left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2})+pc_1c_2(1-s_1)^{\alpha_1}(1-s_2)^{\alpha_2}}\right)^{\upsilon}$$

$$=\left(\frac{pQ\left(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1,1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2\right)}{1-(1-p)Q\left(1-p^{\frac{1}{\alpha_1}}+p^{\frac{1}{\alpha_1}}s_1,1-p^{\frac{1}{\alpha_2}}+p^{\frac{1}{\alpha_2}}s_2\right)}\right)^{\upsilon}.$$

Solving, we obtain that $X_i^{'}$s and $Y_i^{'}$s are independently distributed according to discrete Mittag-Leffler with parameters (α_1,c_1) and (α_2,c_2) respectively.

Jayakumar and Mundassery (2007) obtained BNBD $(c_1,\ c_2,\theta,\upsilon)$ distribution as the negative binomial sum of independently and identically distributed random vectors.

REMARK 4.2. Consider a sequence $\{(X_i,Y_i),\ i\geq 1\}$ *of independently and identically distributed random vectors. Let N be independent of* $(X_i,Y_i),\ i\geq 1$ *and follow negative binomial distribution stated in (12). Then* $(p\oplus U_N,\ p\oplus V_N)$ *follows BNBD*$(c_1,\ c_2,1-p,\upsilon)$ *distribution if and only if* $(X_i,Y_i),\ i\geq 1$ *are BGD*$(c_1,\ c_2,0)$ *random vectors.*

Proof of the remark 4.2 follows easily.

Now, we obtain a characterization of the negative binomial distribution.

THEOREM 4.3. Consider a sequence $\left\{\left(X_i, Y_i\right),\ i \geq 1\right\}$ *of independently and identically distributed random vectors following BDML* $(c_1,\ c_2, \alpha_1,\ \alpha_2, 1)$ *distributions. Then for* $0 < p < 1$, $\left(p^{\frac{1}{\alpha_1}} \oplus U_N,\ p^{\frac{1}{\alpha_2}} \oplus V_N\right)$ *has BDL* $\left(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon\right)$ *distribution if and only if N follows the negative binomial distribution in (12).*

PROOF. The necessary part of the theorem is already discussed in Theorem 4.1.
To prove the sufficiency part, without loss of generality, take $c_1 = c_2 = 1$. Assume that $\left(p^{\frac{1}{\alpha_1}} \oplus U_N,\ p^{\frac{1}{\alpha_2}} \oplus V_N\right)$ follows BDL $\left(1, 1, \alpha_1, \alpha_2, 1, \upsilon\right)$ distributions.
Therefore,

$$P\left(s_1, s_2\right) = \left(\frac{1}{1 + \left(1 - s_1\right)^{\alpha_1} + \left(1 - s_2\right)^{\alpha_2}}\right)^{\upsilon}.$$

By definition, the p.g.f. of $\left(p^{\frac{1}{\alpha_1}} \oplus U_N,\ p^{\frac{1}{\alpha_2}} \oplus V_N\right)$ is

$$P\left(s_1, s_2\right) = \sum_{n=\upsilon}^{\infty} Q\left(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2\right) P\left(N = n\right)$$

where $P\left(s_1, s_2\right)$ represents the p.g.f. of $\left(X_i, Y_i\right),\ i \geq 1$ which follows BDML $(c_1,\ c_2, \alpha_1,\ \alpha_2, 1)$.
Therefore

$$\sum_{n=\upsilon}^{\infty} Q\left(1 - p^{\frac{1}{\alpha_1}} + p^{\frac{1}{\alpha_1}} s_1, 1 - p^{\frac{1}{\alpha_2}} + p^{\frac{1}{\alpha_2}} s_2\right) P\left(N = n\right) = \left(\frac{1}{1 + \left(1 - s_1\right)^{\alpha_1} + \left(1 - s_2\right)^{\alpha_2}}\right)^{\upsilon}$$

Expanding both sides and comparing coefficients of $\left(\left(1 - s_1\right)^{\alpha_1} + \left(1 - s_2\right)^{\alpha_2}\right)^j$, for j = 1, 2, 3,...

$$\sum_{n=\upsilon}^{\infty} n\left(n+1\right)\left(n+2\right)\ldots\left(n+j-1\right) P\left(N = n\right) = \frac{\upsilon\left(\upsilon+1\right)\left(\upsilon+2\right)\ldots\left(\upsilon+j-1\right)}{p^j}$$

Therefore

$$E\left(N\right) = \frac{\upsilon}{p},\ E\left(N\left(N+1\right)\right) = \frac{\upsilon\left(\upsilon+1\right)}{p^2} \textit{ and so on}$$

Consider

$$\begin{aligned} E(1-t)^{-N} &= 1+\frac{t}{1!}E(N)+\frac{t^2}{2!}E(N(N+1)+\frac{t^3}{3!}E(N(N+1)(N+2)+\ldots \\ &= \left(\frac{p}{p-t}\right)^{\upsilon} \\ &= \left(\frac{p}{1-p}\right)^{\upsilon}\sum_{n=\upsilon}^{\infty}\binom{n-1}{\upsilon-1}\left(\frac{1-p}{1-t}\right)^{n} \\ &= p^{\upsilon}\sum_{n=\upsilon}^{\infty}\binom{n-1}{\upsilon-1}(1-t)^{-n}(1-p)^{n-\upsilon} \end{aligned}$$

But

$$E(1-t)^{-N} = \sum_{n=\upsilon}^{\infty}(1-t)^{-n}P(N=n)$$

Therefore,

$$\sum_{n=\upsilon}^{\infty}(1-t)^{-n}P(N=n)=p^{\upsilon}\sum_{n=\upsilon}^{\infty}\binom{n-1}{\upsilon-1}(1-t)^{-n}(1-p)^{n-\upsilon}\ .$$

Comparing both sides, we get

$$P(N=n)=\binom{n-1}{\upsilon-1}p^{\upsilon}(1-p)^{n-\upsilon},\quad n=\upsilon,\upsilon+1,\upsilon+2,\ldots.$$

Hence N follows the negative binomial distribution in (12).

5. AUTOREGRESSIVE PROCESS WITH BDL $(c_1,c_2,\alpha_1,\alpha_2,1,\upsilon)$ MARGINALS

In the following theorem we obtain a necessary and sufficient condition for a first order autoregressive process with marginals have BDL $(c_1,c_2,\alpha_1,\alpha_2,1,\upsilon)$ to be stationary

THEOREM 5.1. Let $\{(X_n,Y_n),\ n\geq 1\}$ *constitute a first order autoregressive process with structure*

$$\{X_n,Y_n\}=(\rho^{\frac{1}{\alpha_1}}\oplus X_{n-1}+\epsilon_n),\quad \rho^{\frac{1}{\alpha_1}}\oplus Y_{n-1}+\psi_n,\quad 0\leq\rho<1 \tag{16}$$

where $\{(\epsilon_n,\psi_n),\ n\geq 1\}$ *is a sequence of independently and identically distributed random vectors. Then the process (16) is stationary with BDL* $(c_1,c_2,\alpha_1,\alpha_2,1,\upsilon)$ *marginals if and only if innovation random vectors,* $(\epsilon_n,\psi_n),\ n\geq 1$ *have p.g.f.*

$$P_{\epsilon_n,\psi_n}(s_1,s_2)=\left(\rho+\frac{1-\rho}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon} \tag{17}$$

Provided (X_0,Y_0) *has the BDL* $(c_1,c_2,\alpha_1,\alpha_2,1,\upsilon)$ *distribution.*

PROOF. The p.g.f. of (16) is

$$P_{X_n,Y_n}(s_1,s_2)=P_{X_{n-1},Y_{n-1}}\left(1-\rho^{\frac{1}{\alpha_1}}+\rho^{\frac{1}{\alpha_1}}s_1,1-\rho^{\frac{1}{\alpha_2}}+\rho^{\frac{1}{\alpha_2}}s_2\right)P_{\epsilon_n,\psi_n}(s_1,s_2). \tag{18}$$

Suppose that the process is stationary with BDL $(c_1,c_2,\alpha_1,\alpha_2,1,\upsilon)$ marginals. Then from (18) we have

$$\left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon}=\left(\frac{1}{1+\rho c_1(1-s_1)^{\alpha_1}+\rho c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon}P_{\epsilon,\psi}(s_1,s_2). \tag{19}$$

Hence we get,

$$P_{\epsilon_n,\psi_n}(s_1,s_2)=\left(\rho+\frac{1-\rho}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon}.$$

Proof of the converse is obtained by induction method. When $n=1$, from (18), we get

$$P_{X_1,Y_1}(s_1,s_2)=P_{X_0,Y_0}\left(1-\rho^{\frac{1}{\alpha_1}}+\rho^{\frac{1}{\alpha_1}}s_1,1-\rho^{\frac{1}{\alpha_2}}+\rho^{\frac{1}{\alpha_2}}s_2\right)P_{\epsilon_n,\psi_n}(s_1,s_2).$$

Suppose that $\{(\epsilon_n,\psi_n),\ n\geq 1\}$ have the p.g.f. given in (17). Therefore,

$$\begin{aligned}P_{X_1,Y_1}(s_1,s_2)&=\left(\frac{1}{1+\rho c_1(1-s_1)^{\alpha_1}+\rho c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon}\left(\rho+\frac{1-\rho}{1+\rho c_1(1-s_1)^{\alpha_1}+\rho c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon}\\&=\left(\frac{1}{1+c_1(1-s_1)^{\alpha_1}+c_2(1-s_2)^{\alpha_2}}\right)^{\upsilon}.\end{aligned}$$

By mathematical induction, it follows that the process $\{(X_i,Y_i),\ i\geq 1\}$ is stationary with BDL $(c_1,c_2,\alpha_1,\alpha_2,1,\upsilon)$ marginals.
Now we develop a first order autoregressive process with BNBD $(c_1,c_2,1,\upsilon)$ marginals.

REMARK 5.1.Ssuppose that an autoregressive process $\{(X_i,Y_i),\ i\geq 1\}$ *has the structure*

$$\{X_n, Y_n\} = (\rho \oplus X_{n-1} + \epsilon_n),\ \ \rho \oplus Y_{n-1} + \psi_n,\ \ 0 \le \rho < 1$$

Where (ϵ_n, ψ_n), $n \geq 1$ *is a sequence of independently and identically distributed random vectors with p.g.f.*

$$P_{\epsilon_n, \psi_n}(s_1, s_2) = \left(\rho + \frac{1-\rho}{1 + c_1(1-s_1) + c_2(1-s_2)}\right)^{\upsilon}. \tag{20}$$

Assume that (X_0, Y_0) *have BNBD*$(c_1, c_2, 1, \upsilon)$ *distribution. Then the process* $\{(X_n, Y_n),\ n \geq 1\}$ *is stationary with BNBD*$(c_1, c_2, 1, \upsilon)$ *marginals if and only if* $(c_1, c_2, 1, \upsilon)$ *have the p.g.f. in (20).*

We omit the proof or Remark 5.1 as it is obvious.

6. CONCLUSION

In the present study we have introduced a bivariate form of discrete Linnik distribution and studied various distributional properties. Its characterizations were obtained using negative binomial compounding and first order autoregressive process is developed with marginal follows BDL$(c_1, c_2, \alpha_1, \alpha_2, 1, \upsilon)$ distribution. Random summation technique can be applied in modeling practical problems in which the respective mathematical models are sums of random number of independent random variables. The bivariate discrete Linnik distribution introduced in this paper can be applied to model bivariate data sets which are closed under negative binomial compounding. The autoregressive model developed is applicable in modeling discrete variate time series when the marginals follow bivariate discrete Linnik distribution. The parameters of BDL$(c_1, c_2, \alpha_1, \alpha_2, \theta, \upsilon)$ may be estimated along the lines of Remillard and Theodorescu (2000).

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SUMMARY

Bivariate discrete Linnik distribution

Christoph and Schreiber (1998a)studied the discrete analogue of positive Linnik distribution and obtained its characterizations using survival function. In this paper, we introduce a bivariate form of the discrete Linnik distribution and study its distributional properties. Characterizations of the bivariate distribution are obtained using compounding schemes. Autoregressive processes are developed with marginals follow the bivariate discrete Linnik distribution.

Keywords: Discrete Linnik distribution; Discrete Mittag-Leffler distribution; Linnik distribution; Negative Binomial compounding; Quasi factorial gamma distribution.