STATISTICA, anno LXII, n. 4, 2002

HERMITE POLYNOMIALS EXPANSIONS FOR DISCRETE-TIME NONLINEAR FILTERING

G. Celant, G.B. Di Masi

1. INTRODUCTION

We consider the following discrete-time partially observable process $(x_t, y_t), x_t, y_t \in R$, with x_t the unobservable and y_t the observable components, given for t =0, 1, …, T on some probability space (**Ω**, *F*, *P*) by

$$x_{t+1} = a(x_t) + v_{t+1} \qquad x_0 = v_0 \tag{1a}$$

$$y_t = c(x_t) + w_t \qquad y_0 = w_0 \tag{1b}$$

where $\{v_t\}$and $\{w_t\}$ are independent standard white Gaussian noises.

Given a measurable function f, we shall be concerned with the solution to the filtering problem, namely the computation for each t = 1,…,T, assuming it exists, of the least squares estimate of $f(x_t)$ given the observations up to time t, namely

$$E\left\{f(x_t) \mid F_t^y\right\} \tag{2}$$

where $F_t^y := \sigma\{y_s \mid s \leq t\}$.

The filtering problem can be more generally described in terms of conditional distributions as follows. Given a Markov process x_t with known transition densities $p(x_t \mid x_{t-1})$ and an observable process y_t, characterized by a known conditional density $p(y_t, x_t)$, it is desired to compute for each t=1,…,T the filtering density $p(x_t, y^t)$ where $y^t := \{y_0, y_1, ..., y_t\}$.

A solution to this problem can be obtained by means of the recursive Bayes formula

Ricerca finanziata tramite MURST COFIN99 *Metodi di Inferenza Statistica per Problemi Complessi*.

$$p(x_t \mid y^t) = \frac{p(y_t \mid x_t) p(x_t \mid y^{t-1})}{\int p(y_t \mid x_t) p(x_t \mid y^{t-1}) dx_t}$$

$$= \frac{p(y_t \mid x_t) \int p(x_t \mid x_{t-1}) p(x_{t-1} \mid y^{t-1}) dx_{t-1}}{\int p(y_t \mid x_t) \int p(x_t \mid x_{t-1}) p(x_{t-1} \mid y^{t-1}) dx_{t-1} dx_t} \tag{3}$$

However, there is an inherent computational difficulty with this formula due to the fact that the integral

$$\int p(x_t \mid x_{t-1}) p(x_{t-1} \mid y^{t-1}) dx_{t-1}$$

is parametrized by $x_t \in R$. So that the problem is in general infinite-dimensional and cannot be solved in an explicitly computable way.

As it will be briefly reviewed in the next section (see also Di Masi *et al.*, 1986), this difficulty disappears in all those situations when $p(x_t \mid x_{t-1})$ is a combination of functions separated in the two variables, i.e.

$$p(x_t \mid x_{t-1}) = \sum_{i=0}^{n} \phi_i(x_t) \psi_i(x_{t-1}) \tag{4}$$

and for such situations an explicit finite-dimensional filter can be provided.

In Di Masi *et al.* (1986) the computational advantage resulting from (4) was exploited in order to approximate $p(x_t \mid y^t)$ by means of approximating densities $p_n(x_t \mid y^t)$, $n \geq 1$, that could be explicitly computed in a recursive way. Such $p_n(x_t \mid y^t)$ were obtained by means of the recursive Bayes formula (3) using approximations to $p(x_t \mid x_{t-1})$ given by suitable functions $p_n(x_t \mid x_{t-1})$ of the form (4). Furthermore, the approximation was such that an explicitly computable bound could be obtained for an appropriate approximation error. In addition, if $f(\cdot)$ does not grow too fast, then also $E\{f(x_t) \mid F_t^y\}$ could be approximated by $\int f(x_t) p_n(x_t \mid y^t) dx_t$ with a corresponding error bound (see Theorem 1 below).

The practically important problem of deriving explicit error bounds for the nonlinear filtering problem was also studied in Di Masi *et al.* (1982) for discrete-time problems and later in Di Masi *et al.* (1985) the results were extended to continuous-time problems (see also Clark, 1978; Kushner, 1977) for different techniques that however do not lead to explicit error bounds). While in Di Masi *et al.* (1982) the approximation is obtained by approximating the model (1), the method followed in Di Masi *et al.* (1986) consists in a direct approximation to the solution to the recursive Bayes formula.

Problems related with approximations for nonlinear filtering have been recently investigated (Arcudi, 1998, Goggin, 1996; Kannan *et al.*, 1998) also in connection with robustness analysis (Budhiraja *et al.*, 1997; Budhiraja *et al.*, 1998).

The aim of this paper is to study a particular case of the technique described in Di Masi *et al.* (1986), consisting in a Hermite polynomial expansion of $p(x_t \mid x_{t-1})$. This method provides an approximation to the nonlinear filtering problem with improved error bounds with respect to those given in Di Masi *et al.* (1986). In fact, it is in general well known the high degree of accuracy that can be obtained using Hermite polynomial approximations. Furthermore, as it will be apparent from the formulas in theorem 1 and proposition 2, the proposed approximation does converge to the exact nonlinear filter, thus providing an arbitrarily accurate suboptimal solution.

Finally, it is worth remarking that discrete-time nonlinear filtering problems can be used for the approximation of corresponding problems in continuous time (see e.g. Di Masi *et al.*, 1985; Kushner *et al.*, 1977). As a consequence, the method proposed in this paper can be interpreted as a useful tool for the derivation of approximate filters also for continuous-time problems.

In the next section 2 we shall review the results in Di Masi *et al.* (1986) that will be needed in the sequel, while in the following Section 3 the Hermite polynomial approximation will be examined in detail.

2. GENERAL APPROXIMATION RESULTS

As mentioned in the introduction, the computational difficulty due to the parametrization of the integrals in (3) disappears for transition densities of the form (4).

In fact, letting $\propto$ denote proportionality, it is easily seen that when (4) holds $p(x_t \mid y^t)$ can actually be computed by means of (3) resulting in

$$p(x_t \mid y^t) \propto \sum_{i=0}^{n} d_i(y^{t-1}) p(y_t \mid x_t) \phi_i(x_t); \quad t = 1,\ldots,T \tag{5}$$

where the vector $d(y^{t-1})$ of the coefficients in the combination can be recursively obtained as

$$d_i(y^0) = \int \psi_i(x_0) p(x_0) dx_0; \quad i = 0,\ldots,N \tag{6a}$$

$$d(y^t) = d(y^{t-1}) B(y_t); \quad t \geq 1 \tag{6b}$$

with $B(y_t) = \{b_{ij}(y_t)\}_{i,j=0,\ldots,n}$ where

$$b_{ij}(y_t) = \int \psi_j(x_t) p(y_t | x_t) \phi_i(x_t) dx_t. \tag{6c}$$

In this section we shall show how a suitably chosen approximation $p_n(x_t \mid x_{t-1})$ to $p(x_t \mid x_{t-1})$ produces, through the recursive Bayes formula (3), an approximation to the filtering density $p(x_t \mid y^t)$ as well as to the corresponding filter $E\{f(x_t) \mid y^t\}$, for which explicit upper bounds to the approximation error can be evaluated.

To this end it will be convenient to provide approximations to $p(x_t \mid y^t)$ in a suitable weighted norm of the type.

$$\|g\|_\alpha := \int \alpha(x) \mid g(x) \mid dx \tag{7}$$

In what follows we shall choose $\alpha(x) = \exp[\alpha|x|], \alpha > 0$, as this will enable us to approximate $E\{f(x_t) \mid F_t^y\}$ for all those $f(\cdot)$ for which $\left|\exp[-\alpha|x|]f(x)\right| \leq M$, for some $M > 0$; in particular, it will allow the approximation of all the conditional moments, as long as they exist.

The general approximation results are given in Di Masi *et al.* (1986) and summarized in Theorem1 below, whose proof is based on the following boundedress and uniform convergence assumptions.

There exist a function $V(y_t)$ and constants U, W, Z, Z_n such that for all *t*:

A.1: $\inf\limits_{x_t} p(y_t \mid x_t) \geq V(y_t) > 0$

$\sup\limits_{x_t} p(y_t \mid x_t) \leq U$

A.2: $\int \inf\limits_{x_{t-1}} p_n(x_t \mid x_{t-1}) dx_t \geq W > 0$

A.3: $\sup\limits_{x_{t-1}} \|p_n(x_t \mid x_{t-1})\|_\alpha \leq Z$

A.4: $\sup\limits_{x_{t-1}} \|p(x_t \mid x_{t-1}) - p_n(x_t \mid x_{t-1})\|_\alpha \leq Z_n$

with $\lim\limits_{n\to\infty} Z_n = 0$

We then have

Theorem 1

Under A.1–A.4 we have for all $t \geq 1$

a) $\|p(x_t \mid y^t) - p_n(x_t \mid y^t)\|_\alpha \leq Z_n \sum\limits_{s=1}^{t} (2U^2W^{-1}Z^2)^s \prod\limits_{u=t-s+1}^{t} \cdot V^{-2}(y_u)$

b) $\left|E\{f(x_t) \mid F_t^y\} - \int f(x_t) p_n(x_t \mid y^t) dx_t\right| \leq MZ_n \sum\limits_{s=1}^{t} (2U^2W^{-1}Z^2)^s$

$\prod\limits_{u=t-s+1}^{t} \cdot V^{-2}(y_u)$

where $M > 0$ is such that $\left| f(x)e^{-\alpha|x|} \right| \le M$.

Notice that the theorem states the convergence of the approximate filtering density $p_n(x_t \mid y^t)$ to the exact density $p(x_t \mid y^t)$ and the convergence (in weighted norm) is strong enough to guarantee the approximation of the corresponding conditional mean $E\left\{ f(x_t) \mid F_t^y \right\}$ when f does not grow too fast.

3. HERMITE POLYNOMIALS APPROXIMATION

In this section we shall provide an approximate solution to the nonlinear filtering problem (1), (2), based on an approximation $p_n(x_t \mid x_{t-1})$ to $p(x_t \mid x_{t-1})$ of type (4) and given in terms of a Hermite polynomials expansion of $p(x_t \mid x_{t-1})$. For the validity of the results of the previous section it is necessary to show that assumptions A.1–A.4 are satisfied. To this end we shall need the following additional assumption on model (1):

A.5: There exist constants *A* and *C* such that

$$\sup_x |a(x)| \le A \tag{8}$$

$$\sup_x |c(x)| \le C \tag{9}$$

Taking into account that, due to the normalization in (3), we can take $p(y_t \mid x_t) = \exp -(y_t - c(x_t))^2/2$ we have that (9) implies A.1 with $U = 1$ and $V(y_t) = \exp\left[-\left(|y_t| + C \right)^2/2 \right]$.

We now recall some properties of Hermite polynomials that will be needed in the sequel. Denoting by $H_k(x) := (-1)^k e^{x^2/2} (d^k / dx^k) e^{-x^2/2}$ the *k*-th Hermite polynomial we have (Bourbaki, 1976), (Sansone, 1959)

P.1: For all *t* and *x*

$$e^{tx-t^2/2} = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}$$

$$= \sum_{k=0}^{\infty} e^{x^2/2} \frac{(-t)^k}{k!} \frac{d^k}{dx^k} e^{-x^2/2} \tag{10}$$

P.2: For all *x* and positive integer *k*

$$H_k(x) = k! \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(-1)^m}{2^m m!} \frac{x^{k-2m}}{(k-2m)!} \tag{11}$$

where $\lfloor k/2 \rfloor$ is the maximum integer not greater than *k/2*.

P.3: For all *x* and popsitive integer *k*

$$\left|H_k(x)\right| \le l\sqrt{k!}e^{x^2/4} \tag{12}$$

where *l* is known as Charlier's constant and $l \cong 1.0864$.

P.4: For all *x*, *y* and positive integer *k*

$$H_k(x+y) = \sum_{m=0}^{k} \binom{k}{m} H_{k-m}(x) y^m \tag{13}$$

The expansion in (10) suggests the following approximation

$$e^{tx-t^2/2} \cong \sum_{k=0}^{n-1} H_k(x) \frac{t^k}{k!} \tag{14}$$

with corresponding absolute error

$$R_n = \left| e^{tx-t^2/2} - \sum_{k=0}^{n-1} H_k(x) \frac{t^k}{k!} \right| = \left| \sum_{k=n}^{\infty} H_k(x) \frac{t^k}{k!} \right|.$$

Using P.3 we have

$$R_n \le \ell e^{x^2/4} \sum_{k=0}^{\infty} \frac{|t|^k}{\sqrt{k!}}$$

where the series, according to the ratio criterion, is convergent so that, denoting by K_t its sum, namely

$$K_t = \sum_{k=0}^{\infty} \frac{|t|^k}{\sqrt{k!}} \tag{15}$$

we have

$$R_n \le K_t \ell e^{x^2/4}. \tag{16}$$

With the notation introduced above, the approximation $p_n(x_t \mid x_{t-1})$ to $p(x_t \mid x_{t-1})$ which will be used in the sequel is given by

$$p_n(x_t \mid x_{t-1}) = \frac{1}{\sqrt{2\pi}} e^{-x_t^2/2} \sum_{k=0}^{n-1} H_k(x_t) \frac{a^k(x_{t-1})}{k!} \tag{17}$$

We now show that A.2 immediately follows from A.3. In fact, the latter implies convergence in the mean of $p_n(\cdot\,|\,x_{t-1})$ to $p(\cdot\,|\,x_{t-1})$ uniformly in x_{t-1} (by (8)) and uniformly on compact sets. Then, for n big enough $p_n(\cdot\,|\,x_{t-1})$ concentrates an arbitrarily large amount of probability mass on a compact set, where it is also positive, and A.2 easily follows.

It remains now to show that assumptions A.3 and A.4 are satisfied. This will be done in Propositions 1 and 2 below, for which we need some preliminary results.

Lemma 1

For any real $\alpha>0, \beta, \gamma$

$$\int_{-\infty}^{+\infty} e^{-(\alpha x^2+\beta x+\gamma)}dx = \sqrt{\pi/\alpha}\,e^{(\beta^2-4\alpha\gamma)/4\alpha} \tag{18}$$

and for any real α and β

$$\int_{-\infty}^{+\infty} e^{\alpha|x|+\beta x-x^2/2}dx \le \mu(\alpha,\beta) \tag{19}$$

where

$$\mu(\alpha,\beta)=\sqrt{2\pi}\,[e^{(\beta+\alpha)^2/2}+e^{(\beta-\alpha)^2/2}] \tag{20}$$

Proof. Completing the square in the exponent and using the change of variable $y = x+\beta/2\alpha$ we have

$$\int_{-\infty}^{+\infty} e^{-(\alpha x^2+\beta x+\gamma)}dx = e^{(\beta^2-4\alpha\gamma)/4\alpha}\int_{-\infty}^{+\infty} e^{-\alpha y^2}dy$$

from which (18) follows immediately.

Furthermore

$$\int_{-\infty}^{+\infty} e^{\alpha|x|+\beta x-x^2/2}dx = \int_{0}^{+\infty} e^{-(\beta-\alpha)x-x^2/2}dx + \int_{0}^{+\infty} e^{(\beta+\alpha)x-x^2/2}dx$$

so that (19) follows from (18).

We are now in the position to prove that with the choice made for $p_n(x_t\,|\,x_{t-1})$, given by (17), the assumptions required by Theorem 1 are satisfied so that it is possible to evaluate the error bound provided there.

Proposition 1

For any real α and positive integer n

$$\left\| p_n(x_t \mid x_{t-1}) \right\|_\alpha \leq Z \tag{21}$$

where Z does not depend on n and is given by

$$Z = 2\sqrt{2}\ell K_A e^{\alpha^2} \tag{22}$$

with K_A as in (15).

Proof. Using P.3 and Lemma 1 we have

$$\left\| p_n(x_t \mid x_{t-1}) \right\|_\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\alpha|x_t|} \left| \sum_{k=0}^{n-1} e^{-x_t^2/2} H_k(x_t) \frac{\alpha^k (x_{t-1})}{k!} \right| dx_t \leq$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\alpha|x_t| - x_t^2/2} \ell e^{x_t^2/4} \sqrt{k!} \sum_{k=0}^{n-1} \frac{A^k}{k!} dx_t \leq$$

$$\leq \frac{\ell K_A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\alpha|x_t| - x_t^2/4} dx_t \leq$$

$$\leq \frac{\ell}{\sqrt{2\pi}} K_A \sqrt{2} \mu(\sqrt{2}\alpha, 0) = 2\sqrt{2}\ell K_A e^{\alpha^2}$$

Remark: if in (8) $A \leq 1$ an alternative inequality (21) can be derived using P.2. In fact we have also used Lemma 1 and letting $S = \sum_{m=0}^{\infty} \frac{1}{2^m m!}$:

$$\left\| p_n(x_t \mid x_{t-1}) \right\|_\alpha \leq \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{n} A^k \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{1}{2^m m! (k-2m)!} \cdot$$

$$\cdot \int_{-\infty}^{+\infty} e^{\alpha|x_t| - x_t^2/2} |x_t|^{k-2m} dx_t \leq$$

$$\leq \frac{1}{2\pi} \sum_{k=0}^{n} A^k \sum_{k=0}^{\lfloor k/2 \rfloor} \frac{1}{2^m m!} \int_{-\infty}^{+\infty} e^{(1+\alpha)|x_t| - x_t^2/2} dx_t \leq$$

$$\leq \frac{1}{2\pi} \frac{1}{1-A} S \mu(1+\alpha, 0) = \frac{1}{\sqrt{2\pi}} \frac{1}{1-A} S 2\sqrt{2\pi} e^{(1+\alpha)^2/2} =$$

$$= \frac{2S e^{(1+\alpha)^2/2}}{1-A}$$

Proposition 2

For any real α and positive integer n

$$\left\| p(x_t \mid x_{t-1}) - p_n(x_t \mid x_{t-1}) \right\|_\alpha \le Z_n$$

where

$$Z_n = 2\sqrt{2} e^{\alpha^2} (K_A - K_{A,n})$$

with K_A as in (15) and

$$K_{A,n} = \sum_{k=0}^{n-1} A^k / \sqrt{k!}$$

Proof. Using P.3 and Lemma 1

$$\left\| p(x_t \mid x_{t-1}) - p_n(x_t \mid x_{t-1}) \right\|_\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\alpha |x_t| - x_t^2/2} |R_n| dx_t \le$$

$$\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\alpha |x_t| - x_t^2/2} \sum_{k=n}^{\infty} \frac{A^k}{\sqrt{k!}} \ell e^{x_t^2/4} =$$

$$= \frac{\ell}{\sqrt{2\pi}} \mu(\sqrt{2}\alpha, 0)(K_A - K_{A,n}) = 2\sqrt{2} \ell e^{\alpha^2} (K_A - K_{A,n})$$

where $K_{A,n} \to K_A$.

The results of this section allow the evaluation of the error bounds given in Theorem 1. It could be easily verified that these bounds are in many instances better than those obtained in Di Masi *et al.* (1986).

EXAMPLE

Here we shall briefly illustrate the proposed algorithm on a sort of the so called cubic sensor problem. This problem is in a sense considered as the prototype of difficult nonlinear filtering problems since its infinite dimensionality has been explicitly proved (Hazewinkel *et al.*, 1983).

In this problem we have:

$$a(x) = \begin{cases} -N & x < -N \\ x & -N \le x \le N \\ +N & x > N \end{cases}$$

$$c(x) = \begin{cases} -N^3 & x < -N \\ x^3 & -N \le x \le N \\ +N^3 & x > N \end{cases}$$

Then the filter is given by (5) and (6) with:

$$p(x_t \mid x_{t-1}) = e^{-\frac{x_t^2}{2}} \sum_i H_i(x_t) \frac{a^i(x_{t-1})}{i!}$$

$$\phi_i(x_t) = e^{-\frac{x_t^2}{2}} H_i(x_t)$$

$$\psi_i(x_{t-1}) = \frac{a^i(x_{t-1})}{i!}$$

$$d_i(y_0) = \int_{-\infty}^{+\infty} \frac{x_0^i}{i!} p(x_0) dx_0$$

$$b_{ij}(y_t) = \sum_{m=0}^{\lfloor i/2 \rfloor} \frac{(-1)^m i!}{m! 2^m j!} \int_{-N}^{+N} x^{i+j-2m} e^{-1/2[(y_t - x^3)^2 + x^2]} dx$$

where the integrals can be easily evaluated.

Dipartimento di Scienze statistiche GIORGIO CELANT
Università degli studi di Padova

Istituto di Matematica pura e applicata GIOVANNI B. DI MASI
Università degli studi di Padova

REFERENCES

O. ARCUDI, (1998), *Convergence of conditional expectations given the random variables of a Skorokhod representation*, "Statistics & Probability Letters", 40, pp. 1-8.

O. ARCUDI, (1998), *Martingale and skorokhod techniques for approximations of nonlinear filters and conditional expectations*, "Boll. Unione Mat. Ital. Sez. A Mat. Soc. Cult. (8)", 1, suppl., pp. 77-80.

N. BOURBAKI, (1976), *Eléments de mathématique. livre IV: fonctions d'une variable réelle*, Hermann, Paris.

A. BUDHIRAJA, D. OCONE, (1997), *Exponential stability of discrete-time filters for bounded observation noise*, "System & Control Letters", 30, pp. 185-193.

A. BUDHIRAJA, H.J. KUSHNER, (1998), *Robustness of non linear filters over the infinite time interval*, "SIAM J. Control Optim.", 36, pp. 1618-1637.

J.M.C. CLARK, (1978), *The design of robust approximations to the stochastic differential equations of nonlinear filtering, in J.K. Skwirzynski (ed.)*, "Communication systems and random process theory", pp. 721-734, Sijthoff and Noordhoff.

G.B. DI MASI, W.J. RUNGGALDIER, (1982), *Approximations and bounds for discrete-time nonlinear filtering in A. Bensoussan, J.L. Lions (eds.)*, "Analysis and Optimization of Systems", L.N. in Control and Info. Sci. 44, pp. 191-202, Springer-Verlag.

G.B. DI MASI, M. PRATELLI, W.J. RUNGGALDIER, (1985), *An approximation for the nonlinear filtering problem, with error bound*, "Stochastics 14", pp. 247-271.

G.B. DI MASI, W.J. RUNGGALDIER, B. ARMELLIN, (1986), *On recursive approximations with error bounds in nonlinear filtering in V.I. Arkin, A. Shiraev, R. Wets (eds.)*, "Stochastic Optimization", L.N. in Control and Info. Sci. 81, pp. 127-135, Springer-Verlag.

E. GOGGIN, (1996), *An L^1 approximation for conditional expectations*, "Stochastics", 60, pp. 85-106.

M. HAZEWINKEL, S.I. MARCUS, H.J. SUSSMANN, (1983), *Nonexistence of finite-dimensional filters for conditional statistics of the cubic sensor problem*, "Systems Control Lett. 3", n.6, pp. 331-340.

D. KANNAN, J.N. RUSHTON, Q. ZHANG, (1998), *Finite dimensional exact filters for linear systems in mixed time I: filters*, "Dynamic Systems and Applications", 7, pp. 145-163.

D. KANNAN, J.N. RUSHTON, Q. ZHANG, (1998), *Finite dimensional exact filters for linear systems in mixed time II: parameter estimation*, "Dynamic Systems and applications", 7, pp. 165-185.

H.J. KUSHNER, (1977), *Probability methods for approximations in stochastic control and for elliptic equations*, Academic Press.

G. SANSONE, (1959), *Orthogonal functions*, Wiley Interscience, New York.

RIASSUNTO

Sviluppo tramite polinomi di Hermite di un filtro non lineare a tempo discreto

Nel lavoro si propone un metodo di calcolo di un filtro non lineare a tempo discreto tramite uno sviluppo in serie di Hermite. Si precisano le varie maggiorazioni degli errori relative al filtro stesso. Infine si dà un esempio di applicazione ad un problema infinito-dimensionale di rilevanza in ingegneria.

SUMMARY

Hermite polynomials expansions for discrete-time nonlinear filtering

A finite-dimensional approximation to general discrete--time nonlinear filtering problems is provided. It consists in a direct approximation to the recursive Bayes formula, based on a Hermite polynomials expansion of the transition density of the signal process. The approximation is in the sense of convergence, in a suitable weighted norm, to the conditional density of the signal process given the observations. The choice of the norm is in turn made so as to guarantee also the convergence of the conditional moments as well as to allow the evaluation of an upper bound for the approximation error.