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THE LACK OF MEMORY PROPERTY IN THE DENSITY FORM

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1. INTRODUCTION

1.1 Univariate Lack of Memory Property.

It has been reported in the literature that Lack of Memory Property (LMP) plays an important role in modeling of many real life situations (see Feller, 1965, Jurkat, 1965, De Bruijn, 1966, Fortet, 1977). This LMP for a continuous nonnegative random variable *X*, with $S(x)$ as its survival function, can be defined by the following condition

$$S(x+y)=S(x)S(y) \quad \forall x\geq 0, y\geq 0 \tag{1}$$

It is uniquely determines the exponential distribution and is known as a characterizing property of the exponential distribution.

Galambos and Kotz (1978) made an attempt to integrate this celebrated LMP property with a few other characterizing properties of the exponential distribution like constancy of failure rate, constancy of mean residual life, and the minimum exponential property (see Sethuraman, 1965). This integrated approach has given a greater insight into the structure of LMP and the basic nature of the survival function of the exponential distribution. In view of this special structure, the exponential distribution holds a fundamental position in classification of life distributions and provides with conservative reliability bounds.

1.2 Different variants of LMP.

Various researchers have tried to generalize LMP from various directions to widen the scope of this property in characterizing, classifying and approximating other life distributions. To increase the applicability of LMP for modeling purpose, Hoang (1968) studied the sensitivity of the LMP and introduced the idea of almost lack of memory property. Writing

$$h(x,y)=P(X\geq x+y\,/\,X\geq y)-P(X\geq x), \tag{2}$$

Hoang's result states that if $\sup\{h(x, y) : x \rightarrow \ \geq 0, y \geq 0\} < \in$, then there exists a constant $b > 0$ such that $\sup\{ | P(X < x) - (1 - e^{-bx}) | \} < 2\sqrt{\in}$.

Obretenov (1970) considered the following generalization of the LMP

$$P(X \geq Y + Z) = P(X \geq Y) \ P(X \geq Z), \tag{3}$$

where (3) holds for Y and Z with degenerate distribution. The exponential distribution comes out the as solution to the above property. Krishanji (1971) extended the result of Obretenov (1970) in the stochastic domain for Y and Z by dropping the condition that Y and Z are to be degenerate.

Marsaglia and Tubilla (1975) have ensued some further relaxation in LMP and had shown that if the condition in (1) is satisfied for all $x \geq 0$ and only two incommensurable values y_1 and y_2 ($0<y_1<y_2$) of y, then the same is equivalent to LMP.

Ramachandran and Lau (1991) and Rao and Shanbhag (1994) had suggested some semi-stochastic form of LMP leading to Strong version of LMP. Marshall and Olkin (1995) introduced the concept of partial LMP by getting the condition (1) to be satisfied for a subset of values of *x* and *y*.

Bivariate and multivariate generalizations of LMP have been attempted by Marshall and Olkin (1967) and Basu and Block (1975). One possible bivariate generalization of (1) is the following

$$S(x_1 + s,\ x_2 + t) = S(x_1, x_2)\ S(s,t) \quad \forall x_1, x_2, s, t \geq 0, \tag{4}$$

where $S(x_1, x_2)$ is the survival function of the bivariate random variables (X_1, X_2). Unfortunately, (4) results in characterization of the trivial bivariate distribution with independent univariate marginal exponential distributions. This means that the condition (4) is highly restrictive in nature and it is known as the very strong version of bivariate LMP (BLMP).

Marshall and Olkin (1967) suggested a weaker version of the condition (4), that is a weaker version of BLMP as

$$S(x_1 + t,\ x_2 + t) = S(x_1, x_2)\ S(t,t) \quad \forall x, y, t \geq 0. \tag{5}$$

Under the assumption of marginal exponential distributions, they have ensured from (5), the unique determination of a bivariate distribution, known as the Bivariate Exponential Distribution due to Marshall and Olkin (BVED-MO). Roy and Mukherjee (1989) presented equivalent stochastic versions of the condition (5). Recently, Roy (2002) has proposed an alternative definition for BLMP and uniquely determined the bivariate distribution due to Gumbel (1960) from the same. Roy (2004, 2005) also proposed two generalized versions of LMP in the univariate and bivariate set ups.

1.3 Proposed work.

In section 2, we propose to extend this LMP in terms of probability density function and examine its characterizing property. The corresponding stability result will also be addressed. In this process the density version of the lack of memory property can be interlinked with reciprocal coordinate subtangent of the density curve and a few other derived measures. Section3 presents an empirical study and concluding remarks.

2. DEFINITIONS AND RESULTS

2.1. LMP

As already pointed out in Section 1, Galambos and Kotz (1978) integrated the celebrated LMP property with a few other characterizing properties of the exponential distribution. These are constancy of failure rate, constancy of mean residual life, and the minimum exponential property. We propose to add another property to this unified setup of Galambos and Kotz (1978) and like to refer it as the density version of the LMP (DLMP). The basic idea lies in the fact that the equilibrium distribution of an exponential distribution is also exponential with the same mean value. Writing $f_E(x)$ as the probability density function of the first order equilibrium distribution, whenever exists, $f_E(x)$ can be expressed as

$$f_E(x) = S(x) / \mu \, , \tag{6}$$

where $\mu(<\infty)$ is the mean of the variable X. For the exponential distribution, (1) holds uniquely and hence dividing both sides of the same by μ^2 and using the fact that $S(0)=1$, we can express (1) as

$$f_E(x+y) f_E(0) = f_E(x) f_E(y) \quad \forall x \geq 0, y \geq 0. \tag{7}$$

But $f_E(x)$ being identical with the original density $f(x)$ for an exponential distribution, we can finally propose the definition of DLMP in terms of $f(x)$ as follows.

DEFINITION 2.1. *A nonnegative random variable X follows DLMP if and only if the corresponding probability function f(x) satisfies the following condition:*

$$f(x+y) f(0) = f(x) f(y) \quad \forall x \geq 0, y \geq 0. \tag{8}$$

Definition 2.1 for DLMP may further lead to two classes of life distributions one based on the condition that

$$f(x+y) f(0) \leq f(x) f(y) \quad \forall x \geq 0, y \geq 0$$

and the other based on the condition

$$f(x+y) f(0) \geq f(x) f(y) \quad \forall x \geq 0, y \geq 0.$$

These classes are dual in nature and are different from DMRL and IMRL classes of life distributions that follow from similar deduction from the LMP.

2.2 Characterization results.

The following results characterize the DLMP in terms of the exponential distribution and other measures. We also touch upon a discrete analog.

THEOREM 2.1. *For a nonnegative continuous random variable X, DLMP holds if and only if it follows the exponential distribution.*

PROOF. If part is easy to establish. To prove the only if part let us consider (8) to be true. Integrating both sides of it with respect to y over the range 0 to ∞ we get

$$S(x)f(0) = f(x), \quad \forall x \geq 0. \tag{9}$$

This implies failure rate $r(x)=f(x)/S(x)$ is a constant, which in turn implies exponential distribution for X. Q.E.D.

The following is a discrete version of the DLMP that characterizes the geometric distribution.

THEOREM 2.2. *For a discrete integer valued nonnegative random variable X, DLMP holds if and only if it follows geometric distribution.*

PROOF is similar to that of theorem 2.1 with integration replaced by summation over y over the range 0 to ∞.

In fact, proofs being similar, for our subsequent presentations we shall restrict ourselves to continuous nonnegative random variables.

One can extend the definition 2.1 in the stochastic domain by replacing y by $y+U$, where U is a nonnegative random variable, distributed independently of X with

$$\mathrm{E}_{\mathrm{U}}\left(f(y+U)\right) = w(y) < \infty. \tag{10}$$

It is easy to note that $w(0) > 0$.

DEFINITION 2.2. *A nonnegative random variable X follows a Stochastic version of DLMP (SDLMP) if and only if the corresponding probability function f(x) satisfies the following condition:*

$$w(x+y)f(0) = f(x)w(y) \quad \forall x \geq 0, y \geq 0. \tag{11}$$

One may be interested to know whether SDLMP still characterizes_the exponential distribution.

THEOREM 2.3. *For a nonnegative continuous random variable X, SDLMP holds if and only if DLMP holds, and if and only if X follows exponential distribution.*

PROOF. (Only if part) Let (11) be true. This implies for a choice of $y=0$,

$$w(x)f(0)=f(x)w(0) \quad \forall x\geq 0. \tag{12}$$

Replacing x by $x+y$ in the above, we observe that

$$w(x+y)f(0)=f(x+y)w(0). \tag{13}$$

Multiplying both sides of (11) with $f(0)$ and using (12) and (13) in it, we get

$$w(0)f(x+y)f(0)=w(0)f(x)f(y),$$

which implies DLMP as $w(0)$ can not be zero.

(If part) To prove the converse, let DLMP hold. Then for all nonnegative x, y and u we have

$$f(x+y+u)=f(x)f(y+u).$$

Integrating both sides of the above with respect to the distribution of U with u as its realization, we get

$$w(x+y)f(0)=f(x)w(y).$$

Thus, follows the SDLMP.

Unique determination of the exponential distribution from SDLMP follows from the above, and from theorem 2.1. Q.E.D.

It is important to mention that the only if part of the proof does not require independence of X and U. But for if part this condition cannot be relaxed.

Let us now introduce a property of importance for stochastic version of DLMP.

PROPERTY A. *A nonnegative random variable Z is said to follow property A if for each interval I, which is a subset of [0, ∞) with positive Lebesgue measure, P(Z ε I) > 0.*

We also refer to Reciprocal Coordinate Subtangent (RCST), a concept which will be of use in the present context since we are dealing with the structure of the density curve. If f(x) is the probability density of a nonnegative random variable X, then RCST of the density curve is given by

$$T(x)=-f'(x)/f(x), \qquad x>0$$

Mukherjee and Roy (1989) introduced the concept of RCST of the density curve and studied the related characterization and classification results. They also introduced Expected Increasing Average RCST (EIARCST) and Expected Decreasing Average RCST (EDARCST) classes based on derived measures of RCST.

THEOREM 2.4. *Let U and V be independent nonnegative random variables, distributed independently of X, and satisfying property A. Let X belong to either EIARCST class or EDARCST class. Then X possesses DLMP or equivalently follows the exponential distribution if and only if the following condition holds*

$$E_U E_V \{ f(U+V) f(0) - f(U) f(V) \} = 0. \tag{14}$$

PROOF. (If part) Let (14) be true. Then writing

$$D(x, y) = f(x+y) f(0) - f(x) f(y), \tag{15}$$

we have from the EDARCST property (see Mukherjee and Roy, 1989) for X, $D(x, y) \geq 0$ for all $x, y \geq 0$. In case the inequality is strict, for some choice of x and y, then from property *A* we can claim that

$$\begin{aligned} &E_U E_V \{ f(U+V) f(0) - f(U) f(V) \} \\ &= \int_0^\infty D(u, v)\, dF_U(u)\, dF_V(v) \\ &> 0 \end{aligned} \tag{16}$$

where $F_U(u)$ and $F_V(v)$ are the respective distribution functions of *U* and *V*. But (16) leads to a contradiction of (14). Hence, $D(x, y)$ cannot be positive and is necessarily equal to zero for all nonnegative choices of *x* and *y*. This implies DLMP for X. Proof for the EIARCST class is similar.

(Only if part) In view of DLMP we have

$$\begin{aligned} &E_U E_V \{ f(U+V) f(0) \} \\ &= \int_0^\infty f(u, v) f(0)\, dF_U(u)\, dF_V(v) \\ &= \int_0^\infty f(u) f(v)\, dF_U(u)\, dF_V(v) \\ &= E_U \{ f(U) \} E_V \{ f(V) \}. \end{aligned}$$

Hence follows (14). Q.E.D.

These last two theorems can be viewed as extensions of Krishnaji (1971)'s results in the density form.

Close with the concept of RCST comes the concept of Mean RCST (MRCST). By definition MRCST is the counterpart of the mean residual life in the density domain.

DEFINITION 2.3. *MRCST function, M(x), of a nonnegative random variable X with density function f(x) is defined as*

$$M(x) = \int_0^\infty \{f(x+u) / f(x)\} du \tag{17}$$

In terms of DLMP, constancy of RCST and constancy of MRCST, we can now introduce equivalent properties of the exponential distribution as done in Galambos and Kotz (1978) in terms of constancy of failure rate and constancy of mean residual life.

Theorem 2.5. For a nonnegative random variable X, the following Density Equivalent Properties (DEP) hold true:

i. (DEP 1): X follows the exponential distribution;

ii. (DEP 2): X follows DLMP;

iii. (DEP 3): MRCST, $M(x)$, of X is a constant independent of x;

iv. (DEP 4): RCST, $T(x)$, of X is a constant independent of x.

PROOF. That (*i*) implies (*ii*) follows from theorem 2.1 presented herein. That (*ii*) implies (*iii*) follows from (17) by making use of the fact that under (*ii*)

$$M(x) = \int_0^\infty \{f(x+u) / f(x)\} du = \int_0^\infty \{f(u) / f(0)\} du,$$

which is a independent of x. Further observing the fact that

$$T(x) = \frac{1 + \frac{d}{dx} M(x)}{M(x)} \tag{18}$$

$T(x)$ of X is a constant independent of x whenever $M(x)$ of X is a constant independent of x. Thus, (*iii*) implies (*iv*). Finally, (*iv*) implies, from the definition of RCST, that

$$-\frac{\frac{d}{dx} f(x)}{f(x)} = c$$

where c is a constant. This implies that $f(x)=c\cdot\exp(-cx)$, which in turn implies (*i*). this completes the proof of the theorem. Q.E.D.

2.3. Stability results.

We shall close our discussion with two stability results. One may refer to Azlarov and Volodin (1986) for early references. Our first result on stability will be on DLMP. The second one is on RCST. These results will be of use for empirical studies.

THEOREM 2.6. *If*

$$\sup_{x,y}\left|\frac{f(x+y)f(0)}{f(x)f(y)}-1\right|<\varepsilon \tag{19}$$

where ε is a small positive quantity, then

$$\sup_{x}\left|S(x)-e^{-f(0)x}\right|<\varepsilon/\{(1-2\varepsilon)e\} \tag{20}$$

PROOF. Writing $h(x, y)$ as a function absolute value of which is bounded by ε, a small positive quantity, we can write from (19),

$$f(x+y)/f(x)=\{1+h(x,y)\}f(y)/f(0) \tag{21}$$

Integrating both sides of (19) with respect to y over the full range, we get

$$S(x)/f(x)=E\{1+h(x,y)\}/f(0).$$

Thus, from the given condition on h(x, y), we can write

$$-\varepsilon<f(0)/r(x)-1<\varepsilon. \tag{22}$$

Algebraic simplification of (22) leads to following condition:

$$-f(0)\varepsilon/(1-\varepsilon)<r(x)-f(0)<f(0)\varepsilon/(1-\varepsilon). \tag{23}$$

Integrating all sides of (23) with respect to the argument ranging from 0 to x, we get after simplification

$$-\{\varepsilon/(1-\varepsilon)\}f(0)x\cdot\exp\{-f(0)x(1-2\varepsilon)/(1-\varepsilon)\}<S(x)-\exp\{-f(0)x\}<$$
$$\{\varepsilon/(1-\varepsilon)\}f(0)x\cdot\exp\{-f(0)x(1-2\varepsilon)/(1-\varepsilon)\}$$

Hence, from the fact that $\sup\{x\cdot\exp(-cx)\}$ for nonnegative x is $\{1/ce\}$, we finally get $\sup_{x}\left|S(x)-e^{-f(0)x}\right|<\varepsilon/\{(1-2\varepsilon)e\}$. Q.E.D.

This result ensures that even for small deviation from DLMP the underlying distribution will nearly follow the exponential distribution with failure rate nearly constant as evident from (23). This ensures stability of propertyDEP-2.

THEOREM 2.7. If

$$\sup_x \left| -\frac{\frac{df(x)}{dx}}{f(x)} - c \right| < \varepsilon \tag{24}$$

where ε is a small positive quantity, then Matusita's affinity (Matusita, 1964) between f(x) and exponential density $f_o(x)$ with c as failure rate, $m(f, f_o)$, satisfies:

$$\left(c(c-\varepsilon)\right)^{1/2} / (c+\varepsilon/2) < m(f, f_o) < \left(c(c+\varepsilon)\right)^{1/2} / (c-\varepsilon/2) \tag{25}$$

and tends to one as ε tends to zero.

PROOF. From (24) we get

$$-c - \varepsilon < f'(x)/f(x) < -c + \varepsilon.$$

Integration of the above with respect to x and making necessary simplification we get

$$(c-\varepsilon)e^{-(c+\varepsilon)x} < f(x) < (c+\varepsilon)e^{-(c-\varepsilon)x}$$

Then Matusita's affinity works out as one lying between $(c(c-\varepsilon))^{1/2}/(c+\varepsilon/2)$ and $(c(c+\varepsilon))^{1/2}/(c-\varepsilon/2)$. Hence follows (25). Q.E.D.

For a recent discussion on stability results one may refer to Roy and Roy (2013)

3. EMPIRICAL STUDY AND CONCLUDING REMARKS

3.1. Empirical study.

For numerical presentation, we present below 20 observations from a population distribution. We would like to apply Theorem 2.7 to check whether an exponential distribution fits well. Simulated observations from an exponential distribution along with estimated density values and estimated RCST values are given in the Table 1 with standard normal pdf as the kernel of the kernel estimator for the density function.
With c value chosen as the midpoint of the range of RCST value, we have c=0.06256. Hence ε works out as, via (24), ε=0.06256.. Thus, ε is same as the value of c. The corresponding Matusita's affinity, $m(f, f_o)$ is greater than 0.94281 As a result, we conclude that exponential distribution with failure rate 0.06256 may be fitted to the given data.

3.2. Concluding remarks.

Basically this paper aims to propose and study a density version of the celebrated lack of memory property. Different variations of this density version have also been covered. Characterization results along with stability properties have been presented to interlink with exponential distribution, RCST and other derived measures. At the end, an empirical study has been carried out to demonstrate how one can draw definite conclusion on model fitting.

TABLE 1.
Empirical fitting of RCST

Serial no.	Observation	Estimated density	Estimated RCST
1	46.31107	0.02840	0.03101
2	232.26730	0.02820	0.00000
3	210.86240	0.02820	0.00000
4	254.58190	0.02820	0.00000
5	7.82351	0.02855	0.05068
6	24.61292	0.02820	0.00000
7	44.08397	0.02841	0.02953
8	51.93804	0.02820	0.00000
9	88.91750	0.02820	0.00000
10	41.21026	0.02821	0.00149
11	351.08770	0.02820	0.00000
12	64.17916	0.05596	0.12511
13	3.46933	0.02838	0.02767
14	5.72486	0.02872	0.02303
15	175.46420	0.02820	0.00000
16	35.00495	0.02820	0.00000
17	116.53150	0.02820	0.00000
18	70.55176	0.02820	0.00000
19	64.30527	0.05596	0.12411
20	285.31870	0.02820	0.00000

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SUMMARY

The lack of memory property in the density form

The celebrated lack of memory property is a unique property of the exponential distribution in the continuous domain. It is expressed in terms of equality of residual survival function with the survival function of the original distribution. We propose to extend this lack of memory property in terms of probability density function and examine therefrom its characterization and stability properties. In this process the density version of the lack of memory property can be interlinked with reciprocal coordinate subtangent of the density curve and a few other derived measures.