

BIVARIATE VALUE-AT-RISK

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1. MOTIVATION

In its earlier stage of development the theory of financial risk management and portfolio selection was based on simple statistical measures of dispersion of the expected return (such as variance or coefficient of variation) as measures of the global risk of an asset or of the entire portfolio. Similarly the linear regression coefficient β of an asset, which measures the relative increase in portfolio risk due to an additional unitary investment in that asset, was considered a good measure of the market risk of that asset not due to general market fluctuation and hence a tool to assist the choice in building efficiently diversified portfolios.

It is now widely accepted that the variance of the expected returns measures a rather inaccurate notion of risk due to the frequent asymmetries and fat tails displayed by the asset's returns distributions (Afflect-Groves and McDonald, 1989). The concept of value-at-risk (VaR) has taken place as the new benchmark in the risk management techniques that takes downside risk into account (Jorion, 1997).

The VaR approach is based on a general (long known) notion of risk as the probability of exceeding (or not exceeding) a certain cut-off quantity perceived as dangerous. Such a definition is by no mean peculiar to finance and has found applications in many different fields such as insurance (Williams and Heins, 1976), technology (Hultzer *et al.*, 1983), environmental analysis (Simmons *et al.*, 1973), engeneering (Diamond, 1981) and many others (for a review see *e. g.* RSS, 1999).

Specifically, financial risk can be viewed as the probability of not exceeding a cut-off return (say R^*) on a random distribution (Cramer, 1930). Such a distribution can be an explicit random distribution, characterized by say, a density function $f_R(r)$, thus leading to a parametric (or model dependent) approach to VaR, or, alternatively, an implicit distribution based on the historical empirical distribution of returns (or model-free, see Artzner *et al.*, 1998).

At a given cut-off point R^* the risk can be evaluated as:

$$Risk = \int_{-\infty}^{R^*} f_R(u) du = F_R(R^*) = 1 - \alpha \quad (1)$$

with $F_R(R^*)$ the cumulative distribution function of the asset returns distribution evaluated in the point R^* . Conversely, at a maximum accepted level of risk $1-\alpha$ (e.g. $\alpha = 0.99$ as required by the Basle Committee on Banking Supervision, 1996), the cut-off return is found by solving equation (1) with respect to R . If we call $R(\alpha)$ such a solution, the VaR of a unitary investment is then evaluated as

$$VaR(\alpha) = -R(\alpha)$$

For reasons that will become clearer in the remainder of the paper we will henceforth refer to $-R(\alpha)$ as the Uni-VaR of an asset or of the entire portfolio.

The general definition of risk given in (1) is very intuitive and encompasses the traditional measure of variance as a special case when the asset returns are gaussian.

In fact, when the expected returns are normally distributed we have

$$Risk = F_R(r^*) = \int_{-\infty}^{R^*} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right] du \quad (2)$$

with $\mu = E(R)$ and $\sigma^2 = \text{Var}(R)$. Approximations for (2) are given e.g. in Abramoviz and Stegun (1965).

Figure 1 shows the plot of the risk as measured in equation (1), against the standard deviation for various values of the cut-off return expressed in standardized scores ranging from -4 to -0.5.

Figure 1 shows that in the normal case the general risk measure of equation (1) is a monotonically increasing function of σ . Although the function is non-linear, the choice between the two measures does not affect the ranking of risk, and this implicitly justifies the use of the standard deviation in past decades as a proxy to risk when empirical distributions do not depart dramatically from normality.

With the introduction of the VaR techniques it is recognised the crucial role of non-normal returns placing a greater emphasis on the left tail of the returns' distributions. This produces more accurate estimate leading to tremendous implications in terms of risk management, evaluation and ranking.

In this paper we claim that it is possible to extend the idea of VaR to "market risk" evaluation and portfolio management giving the right consideration to downside risk exposure.

The set up of the paper is the following. In the next section we will consider a general definition of market risk of an asset based on definition (1) that admits the traditional β measure as a particular case when returns are normal. The data used and the results are presented in section 3. Section 4 contains some conclusions and directions for further studies in the field.

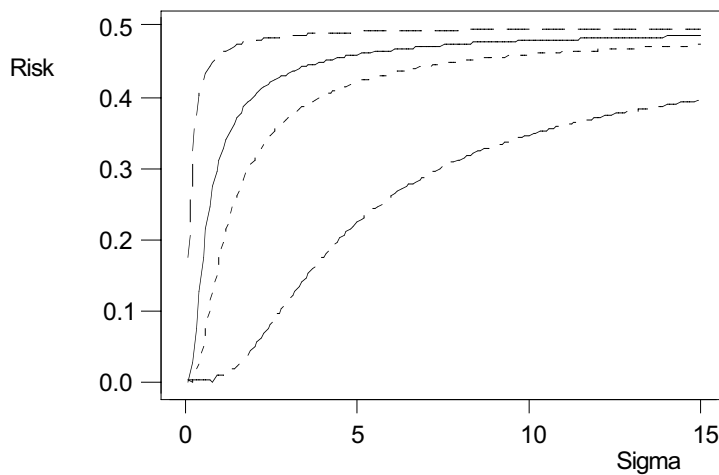


Figure 1 – Plot of the global risk measure of equation (1) as a function of the standard deviation of the returns for various values of the cut-off return expressed in standardized scores. Lower line refers to -4. Upper line corresponds to -0.5 The solid line considers a probability of 0.99 corresponding to the Basle requirement.

2. VALUE-AT-RISK IN A BIVARIATE CONTEXT

Let us consider a portfolio P characterised by a random return R_p with expected value $E(R_p) = \mu_p$ and variance $Var(R_p) = \sigma_p^2$. Suppose now that we want to evaluate the market risk of introducing in this portfolio a new unitary investment in, say, asset-1.

We will define the market risk of asset-1 as the probability that the new investment does not exceed a certain cut-off (R_1^*) and simultaneously the portfolio as a whole does not exceed a certain cut-off (R_p^*). This is evaluated through the integral

$$Market - risk = \int_{-\infty}^{R_p^*} \int_{-\infty}^{R_1^*} f_{R_p, R_1}(u, v) du dv = F_{R_p, R_1}(R_p^*, R_1^*) \tag{3}$$

with $F_{R_p, R_1}(R_p^*, R_1^*)$ denoting the cumulative joint distribution function of the asset's returns and portfolio returns.

Remark 1. If the two random variables associated with the return of asset-1 and the return of the portfolio as a whole are jointly normally distributed, then definition (3) yields:

$$\text{Market - risk} = \int_{-\infty}^{R_1^*} \int_{-\infty}^{R_p^*} \frac{1}{2\pi\sigma_p\sigma_1\sqrt{1-\rho^2}} \cdot \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u-\mu_p}{\sigma_p}\right)^2 + \left(\frac{v-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{u-\mu_p}{\sigma_p}\right)\left(\frac{v-\mu_1}{\sigma_1}\right)\right]\right\} dudv$$

with $\mu_p = E(R_p)$; $\sigma_p^2 = \text{Var}(R_p)$; $\mu_1 = E(R_1)$ and $\sigma_1^2 = \text{Var}(R_1)$.

Let us now assume without loss of generality that $E(R_p) = E(R_1) = 0$ and, just for the purpose of illustration, $\sigma_1 = \sigma_p = 1$ and hence:

$$\beta = \frac{\text{Cov}(R_p, R_1)}{\sigma_p\sigma_1} = \text{Cov}(R_p, R_1) = \rho.$$

It follows that:

$$\text{Market - risk} = \int_{-\infty}^{R_1^*} \int_{-\infty}^{R_p^*} \frac{1}{2\pi\sqrt{1-\beta^2}} \exp\left\{-\frac{1}{2(1-\beta^2)}[u^2 + v^2 - 2\beta uv]\right\} dudv$$

Let us now set

$$w = \frac{R_1^* - \beta R_p^*}{\sqrt{1-\beta^2}}$$

and hence

$$R_1^* = w\sqrt{1-\beta^2} + \beta R_p^*$$

we have

$$\begin{aligned} \text{Market - risk} &= \int_{-\infty}^{w\sqrt{1-\beta^2} + \beta R_p^*} \int_{-\infty}^{R_p^*} \frac{1}{2\pi} \exp\left\{-\frac{1}{2}[w^2 + v^2]\right\} dw dv \\ &= \int_{-\infty}^{w\sqrt{1-\beta^2} + \beta R_p^*} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}w^2\right\} \int_{-\infty}^{R_p^*} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}v^2\right\} dv = \\ &= \int_{-\infty}^{w\sqrt{1-\beta^2} + \beta R_p^*} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}w^2\right\} F_{R_p^*}(R_p^*) \end{aligned}$$

Remark 2. Figure 2 plots the market risk associated with asset-1 as a function of β , showing that in the particular case examined the systematic risk of asset-1 is a monotonically increasing function of the β of the asset itself. This justifies the use of the traditional β of an asset as a proxy for the market risk of the asset itself, if the expected returns are jointly normally distributed.

However equation (3) provides a more accurate tool to justify the inclusion of an asset in the portfolio when the joint distribution departs dramatically from the bivariate gaussian case, as it is the case in most empirical situations. Obviously the joint distribution function can be fully specified or just evaluated non-parametrically from historical record counting.

Remark 3. If the expected returns distribution of asset-1 and the portfolio return distribution are stochastically independent than the joint distribution factorizes as:

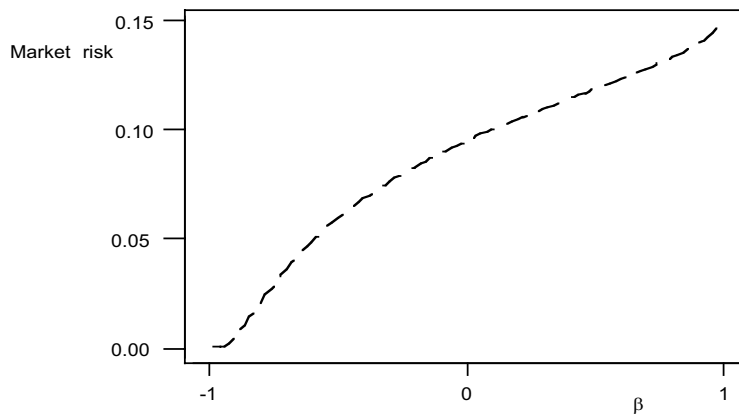


Figure 2 – Plot of the market risk measure of Equation 3 as a function of the β of the asset considering a cut-off return of - 0.5 (expressed in standardized scores) for both the portfolio and asset.

$$f_{R_p R_1}(u, v) = f_{R_p}(u) f_{R_1}(v)$$

and hence

$$Market - risk = \int_{-\infty}^{R_p^*} f_{R_p}(u) du \int_{-\infty}^{R_1^*} f_{R_1}(v) dv \tag{4}$$

so that market risk is simply the product of the risk evaluated through (1) for the asset-1 and for the entire portfolio. In this instance substituting (1) into (4) yields:

$$Market - risk = (1 - \alpha)^2 \tag{5}$$

Now let us revert the argument and assign $(1 - \alpha)^2$ as the maximum level of accepted market risk in Equation (3). If we substitute the cut-off return for the entire portfolio derived from the *Uni-VaR* technique ($R(\alpha)_p$) at the level of risk

$(1 - \alpha)$, we can solve Equation (3) with respect to R_1^* and obtain the solution $R(\alpha)_{\beta,1}$ such that

$$R(\alpha)_{\beta,1} : \int_{-\infty}^{R(\alpha)_p} \int_{-\infty}^{R(\alpha)_{\beta,1}} f_{R_p R_1}(u, v) du dv = (1 - \alpha)^2 \quad (6)$$

We finally define the Bivariate Value-at-Risk (or β -VaR) of an asset as:

$$\beta\text{-VaR}(\alpha) = -R(\alpha)_{\beta,1} \quad (7)$$

Since at a given probability $(1 - \alpha)^2$ market risk increases as $-R(\alpha)_{\beta,1}$ increases, then the β -VaR(α) of an asset is a measure of the market risk of the asset itself.

Thus β -VaR(α) is the negative of the $(1 - \alpha)^2$ quantile of the bivariate distribution of returns having fixed to $R(\alpha)_p$ the $(1 - \alpha)$ univariate quantile for the entire portfolio.

Remark 4. The bivariate Value-at-Risk is equivalent to a conditional Uni-VaR given the portfolio returns' distribution. In fact

$$f_{R_p R_1}(u, v) = f_{R_1 | R_p}(v | u) f_{R_p}(u)$$

and, integrating both sides of the equation with respect to u and v we obtain:

$$\int_{-\infty}^{R_1^*} \int_{-\infty}^{R_p^*} f_{R_p R_1}(u, v) du dv = \int_{-\infty}^{R_1^*} f_{R_1 | R_p}(v | u \leq R_p^*) dv \int_{-\infty}^{R_p^*} f_{R_p}(u) du$$

If we set R_p^* to the level of $R(\alpha)_p$ corresponding to the Uni-VaR, and furthermore the market risk to the level $(1 - \alpha)^2$ we obtain:

$$(1 - \alpha) = \int_{-\infty}^{R_1^*} f_{R_1 | R_p}(v | u \leq R(\alpha)_p) dv$$

and solving with respect to R_1^* we obtain again the solution $R(\alpha)_{\beta,1}$. Thus the β -VaR can also be viewed as the negative of the $(1 - \alpha)$ conditional quantile (Bassett and Könker, 1982) of the returns' distribution given that the portfolio return exceeds its cut-off point R^* .

From a comparison between the β -VaR and the Uni-VaR we can also derive a measure of dependence between the asset's returns and the portfolio returns. In fact when the portfolio returns and the asset-1 returns are independent then $R(\alpha)_1 = R(\alpha)_{\beta,1}$ and, hence:

$$\beta\text{-VaR}(\alpha) = \text{Uni-VaR}(\alpha)$$

In the case of positive dependence, however, when points tend to concentrate in the first and third quadrant, the term $R(\alpha)_1$ is greater than $R(\alpha)_{\beta,1}$ and hence Uni-VaR is smaller than β -VaR. Conversely in the case of negative dependence when points are concentrated in the second and fourth quadrants the term $R(\alpha)_1$ is smaller than $R(\alpha)_{\beta,1}$ and hence Uni-VaR is greater than β -VaR.

3. BIVARIATE VALUE-AT-RISK AND NEW MEASURES OF RISK AND MARKET DEPENDENCE

We now make use of the general concept of the positive quadrant dependence (Joe, 1997) measure between two random variables. Two random variables X_1 and X_2 are positive quadrant dependent if

$$P[X_1 \leq x_1; X_2 \leq x_2] \geq P[X_1 \leq x_1]P[X_2 \leq x_2]$$

In the case of positive dependency

$$\int_{-\infty}^{R(\alpha)_{\beta,1}} \int_{-\infty}^{R(\alpha)_p} f_{R_p, R_1}(u, v) du dv - \int_{-\infty}^{R(\alpha)_p} f_{R_p}(u) du \int_{-\infty}^{R(\alpha)_1} f_{R_1}(v) dv \geq 0$$

Since at a given α risk and cut-off returns are inversely related, in case of positive dependence

$$R(\alpha)_1 - R(\alpha)_{\beta,1} \geq 0$$

Thus the quantity

$$\xi(\alpha) = [\beta\text{VaR}(\alpha) - \text{UniVaR}(\alpha)] \geq 0 \tag{8}$$

is a general measure of non-linear dependence between R_1 and R_p restricted to the left tails of their joint distribution.

A further measure of market risk can be obtained by computing the expected returns of an asset when these returns do not exceed $R(\alpha)_{\beta,1}$

$$S(\alpha)_{\beta,1} = E(R_1 | R_1 \leq R(\alpha)_{\beta,1}) \tag{9}$$

This measure is the bivariate counter part of the measure suggested by Artzner *et al.* (1999).

4. DATA ANALYSIS

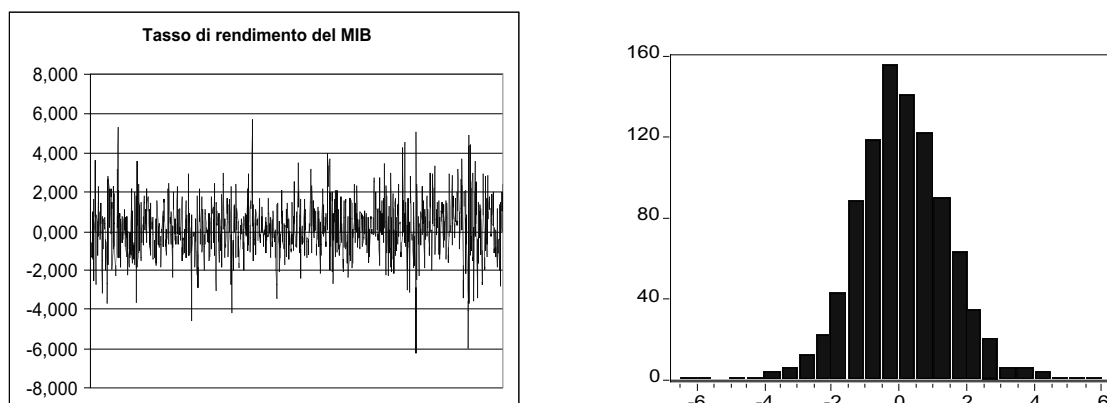
For the purpose of illustrating the methodology of bivariate Value-at-Risk in the present section we consider a set of Italian stock market data collected daily in the period ranging from the 14th of October 1994 to the 18th of August of 1998. The data set refers to a 960-day long time series of the general Milan stock market index (MIB - Milano Indice Borsa) and to six sectorial indices referring to assets classified as Industrial, Financial, Communications, Credit, Insurance and “Diverse”.

The 7 time series of returns are reported in figure 3 together with the corresponding histogram. All series display a market excess kurtosis and a negative skewness (with the exception of the “Insurance” series showing a positive skewness and of the MIB series showing no significant skewness).

Table 1 reports the comparison between two global risk measures: the standard deviation and the univariate Value-at-Risk (the traditional VaR), together with the different rankings deriving from the two measures. The value of $\alpha = 0.95$ has been chosen with respect to the length of the time series.

Figure 4 shows the joint distributions of each sector’s return and of the MIB index in terms of a 3-dimensional frequency distribution, and in terms of a scatter diagram. All graphs provide evidence of a positive dependence between all sectorial indices and the MIB index.

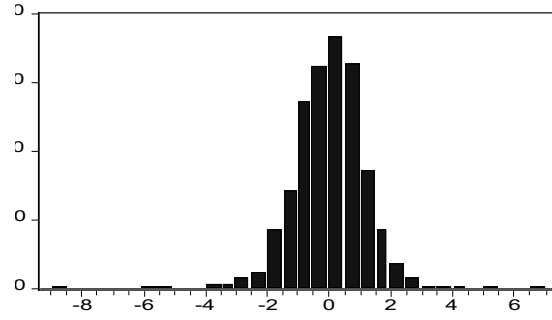
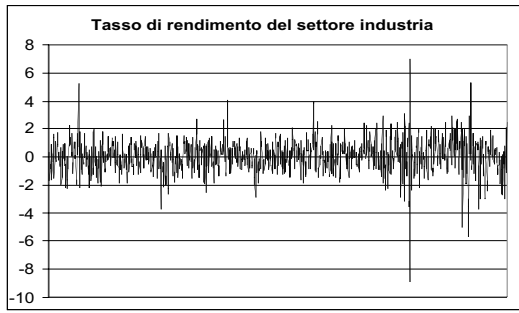
Table 2 compares the traditional β measure with the β -VaR obtained through Equation (7) with $\alpha = 0.95$, estimated as the quantile of the historical records distributions, making no distributional assumption. Table 2 also reports the non-linear dependency measure $\xi(\alpha)$ suggested in Equation (8). For most sectors the dependency in the left tail is positive; however, for the case of indices referring to “Insurance”, “Finance” and “Diverse” assets, the $\xi(\alpha)$ measure reveals a negative dependence. Finally table 2 reports the two different rankings of risk derived by using the traditional β measure and the β -VaR.



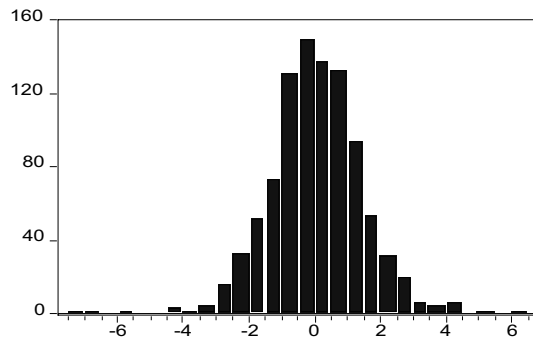
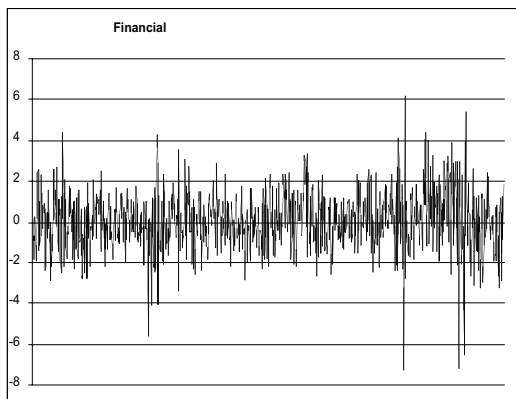
a) MIB

Figure 3 – Time series of the MIB Index (series a) and of the returns of the various assets classified within 6 categories (series b to g) in the time period ranging from the 14th October 1994 to the 18th August 1998. (continued)

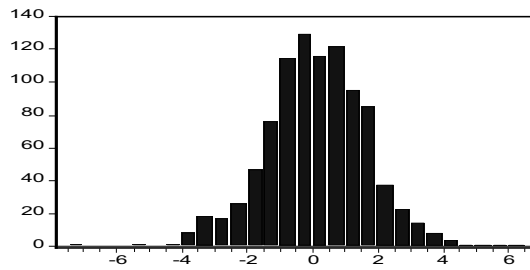
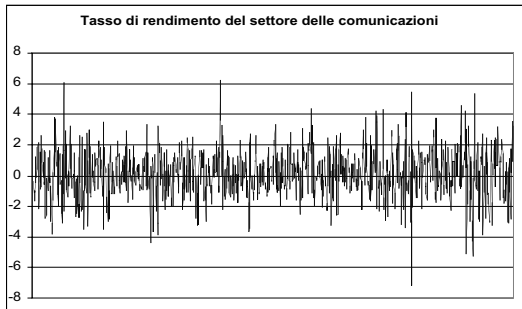
Figure 3 – (continued)



b) Industrial

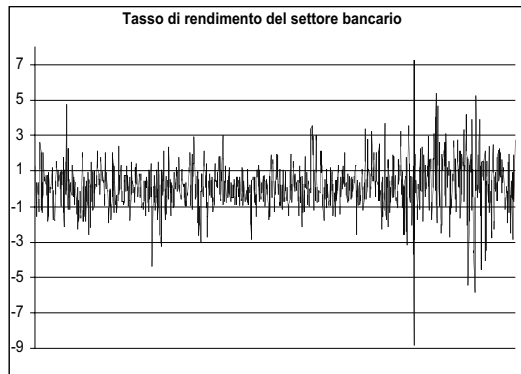


c) Financial

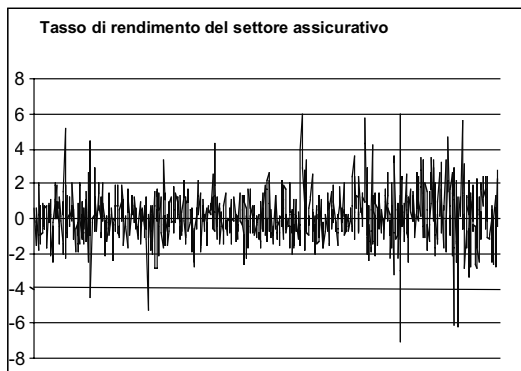
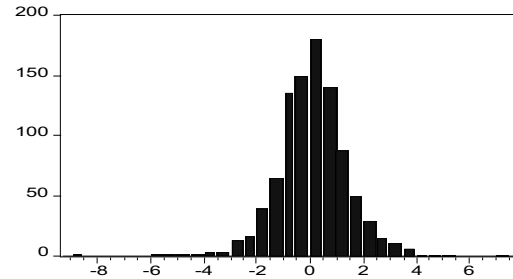


d) Communications

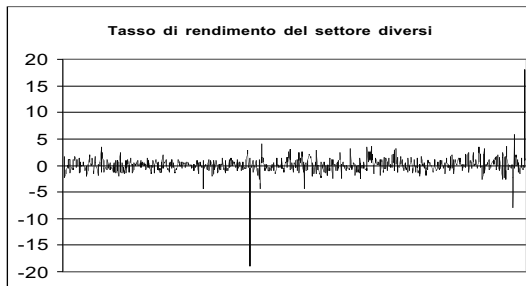
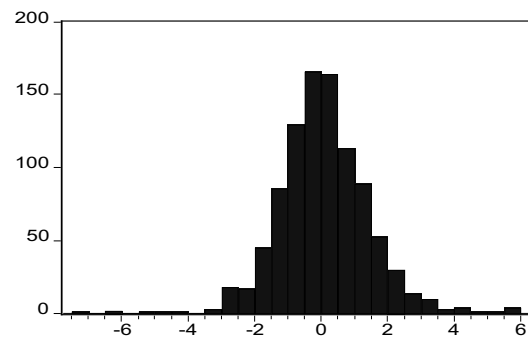
Figure 3 – (continued)



e) Credit



f) Insurance



g) Diverse

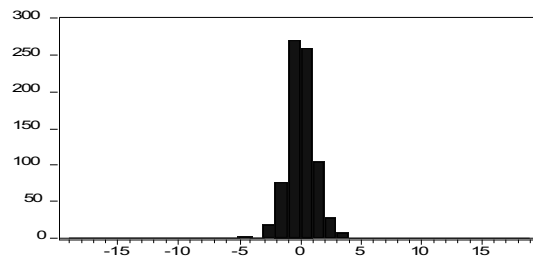


TABLE 1

Measures of global risk (σ and Uni-VaR) and relative rankings within parentheses.

	σ	Uni-VaR(α)
Communications	1.568412 (1°)	2.566 (1°)
Credit	1.338803 (6°)	1.905 (5°)
Diverse	1.487575 (2°)	1.566 (7°)
Financial	1.422402 (3°)	2.198 (2°)
Industry	1.206565 (7°)	1.828 (6°)
Insurance	1.374296 (5°)	1.931 (4°)
MIB	1.414148 (4°)	2.051 (3°)

Note: $\alpha = 0.95$

TABLE 2
 Measures of market risk: β , β -VaR(α), (and relative rankings within parentheses),
 Uni-VaR(α) and $\xi(\alpha)$ non-linear tail dependency measure.

	β	β -VaR(α)	Uni-VaR(α)	$\xi(\alpha)$
MIB	-	-	2.051	-
Industry	0.5698 (5°)	1.898 (4°)	1.828	0.070
Financial	0.6101 (4°)	2.084 (2°)	2.198	-0.114
Communications	0.7639 (1°)	2.658 (1°)	2.566	0.092
Credit	0.6264 (3°)	1.943 (3°)	1.905	0.038
Insurance	0.6289 (2*)	1.848 (5°)	1.931	-0.083
Diverse	0.3813 (6°)	1.080 (6°)	1.566	-0.486

Note: $\alpha = 0.95$

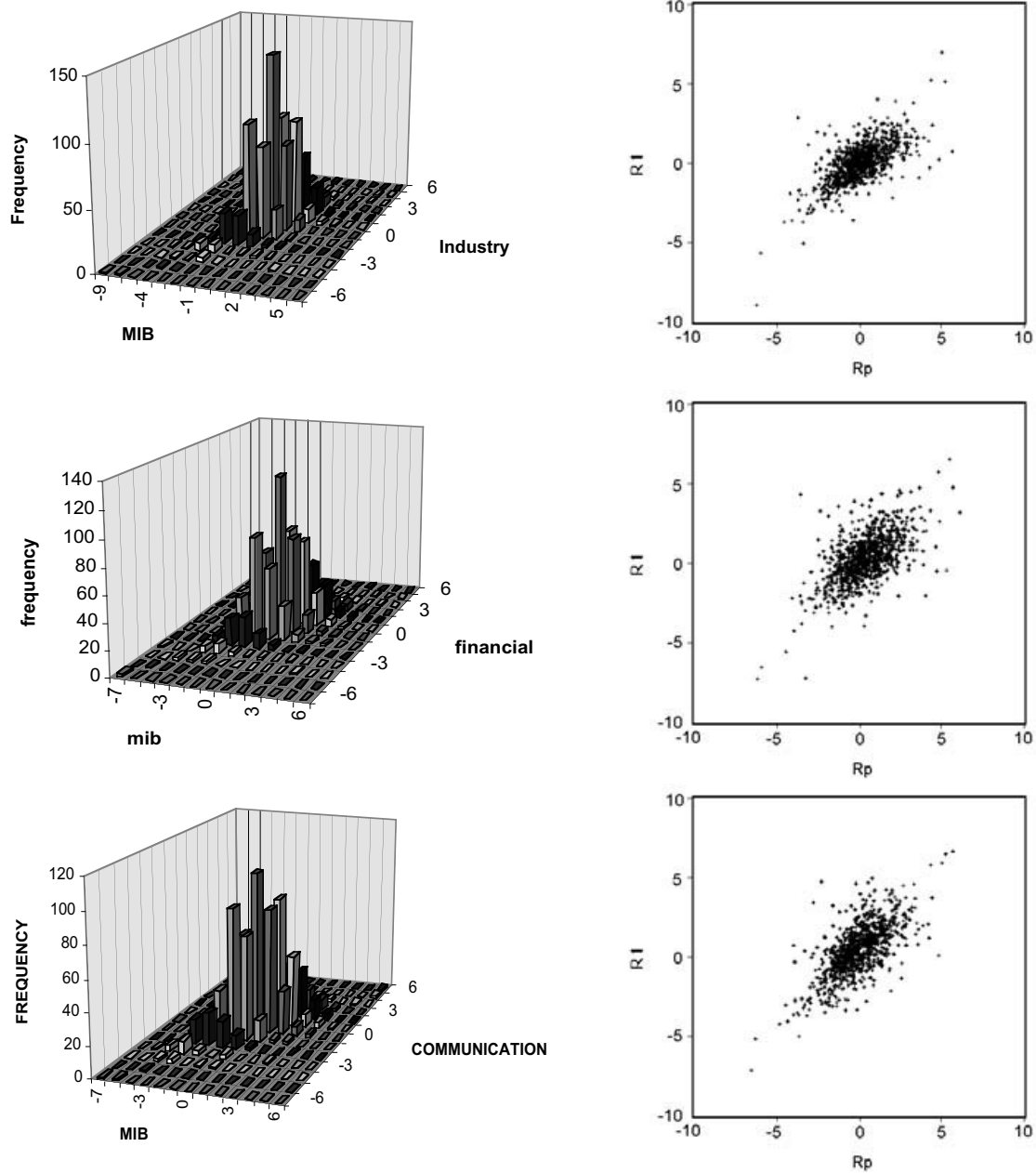
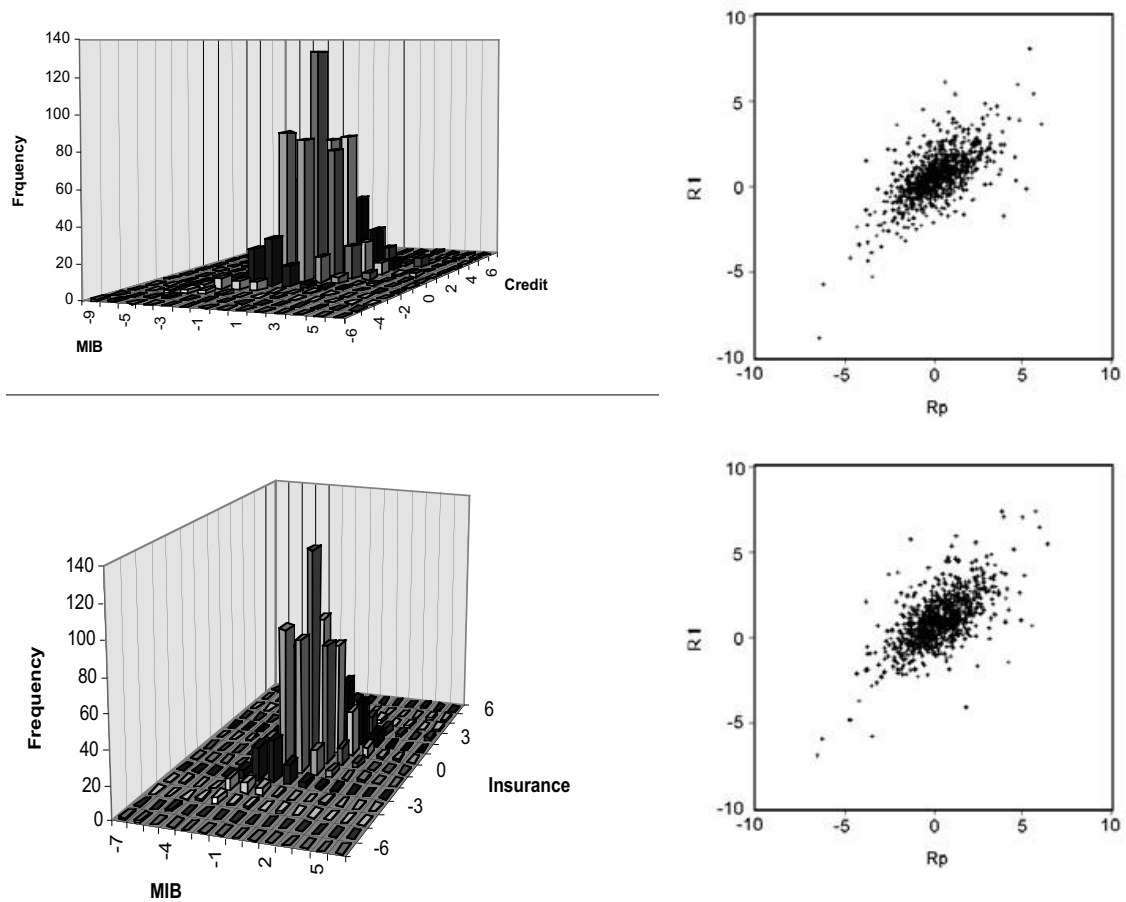


Figure 4 – Joint distributions of each sectors' returns and MIB returns (14th October 1994 - 18th August 1998). (continued)

Figure 4 – (continued)



Apart from the “Communications” sector, which appears to be the more exposed to market risk in both cases, by focusing on the left-tail behaviour of the return, we obtain two remarkably different rankings. In particular the “Insurance” sector appears the least risky with the β -VaR criterion, whereas it is in the second position according to the β criterion. This is due to the negative dependency with MIB in the left tails shown in table 2 and to the positive skewness observed above. Conversely “Financial” and “Industrial” assets appear to be more exposed to market risk if evaluated through the β -VaR than through β . Table 3 reports the ranking of risk obtained by employing the restricted expected shortfall. The ranking obtained is not substantially different from that reported in Table 2 based on the β -VaR.

In order to investigate the dynamics of market risk in the period examined we divided the time series into two subsamples referring to years 1994 to 1996 on one side, and to years 1997 and 1998 on the other (see table 4 and 5). The two sample periods contain a different number of observations, but this has negligible influence on our analysis. The second subperiod includes a period of greater volatility of all indices (see figures 3) due to the instability of stock markets registered in the summer of 1998. In the first subperiod the ranking obtained with the two measures is not very different with “Communications” and “Financial” assets

by far at the top, followed by the other sectors that do not display significant differences in risk. In contrast in the second subperiod “Communications” is followed by “Credit” if we rely on the β criterion, and by “Insurance” if we adopt the β -VaR role. The entire ranking appears to be very different in the two cases. If we compare this evidence with that of table 2, it appears that the differences in ranking of the two measures are greater in periods of greater instability.

TABLE 3
Rank of the restricted expected shortfall of the various asset categories for returns above the β -VaR

	$S(\alpha)_{\beta,1}$
Communications	3.6792
Financial	3.3957
Credit	3.4121
Insurance	3.2685
Industry	3.1611
Diverse	2.3671

TABLE 4
Measures of market risk in two subperiods.

	10.14.1994 - 12.31.1996			1.2.1997 - 8.18.1998		
	β	β -VaR(α)	$\xi(\alpha)$	β	β -VaR(α)	$\xi(\alpha)$
Industry	0.5370	1.7746	0.0017	0.5959	2.4119	0.1512
Financial	0.6001	1.9379	0.0007	0.6138	2.8908	0.0229
Communications	0.8084	2.4954	0.0228	0.7268	2.9245	0.0168
Credit	0.5715	1.7794	0.0066	0.6647	2.4585	0.2251
Insurance	0.5836	1.6952	0.0045	0.6609	2.8989	0.0388
Diverse	0.3961	1.3555	0.0021	0.3470	1.5356	- 0.0154

TABLE 5
Ranking of assets category according to the two market risk measures in the two subperiods considered

FIRST SUBPERIOD (10.14.1994 - 12.31.1996)

	β		β -VaR(α)
Communications	0.8084	Communications	2.4954
Financial	0.6001	Financial	1.93791
Insurance	0.5836	Credit	1.77944
Credit	0.5715	Industry	1.7746
Industry	0.537	Insurance	1.69519
Diverse	0.3961	Diverse	1.35556

Note: $\alpha = 0.95$

SECOND SUBPERIOD (1.2.1997 - 8.18.1998)

	β		β -VaR(α)
Communications	0.7268	Communications	2.9245
Credit	0.6647	Insurance	2.8989
Insurance	0.6609	Financial	2.8908
Financial	0.6138	Credit	2.4585
Industry	0.5959	Industry	2.4119
Diverse	0.3470	Diverse	1.5356

Note: $\alpha = 0.95$

5. CONCLUSIONS AND DIRECTIONS FOR FURTHER RESEARCH

The aim of this paper is to define measures to properly rank financial risks when the return distributions are not Gaussian and present heavy tails and are skewed. Such measures are needed to emphasize downside risk exposure, which is the main feature to look at in optimal asset allocation and portfolio management. To achieve this aim we extend the VaR measure proposed for global risk evaluation to market risk evaluation by introducing the concept of bivariate Value-at-Risk (β -VaR). The method introduced is applied to daily time series of Italian stock market data in the period from 9.94 to 8.98. By using the proposed methodology as opposed to the traditional criterion based on the popular linear regression coefficient β , we show that we end up with a very different market risk evaluation and hence with a very different ranking of the assets with respect to risk. We also show that the difference in the two measures is particularly dramatic in a period of higher volatility and market instability like that experienced in the summer of 1998 due to the stock market world crisis.

The work reported here is preliminary and requires further investigation and extensions in various directions.

First of all, in the present paper we did not consider the sampling distribution of the bivariate Value-at-risk. Further work in this area should rely upon the well known results on the inference on conditional quantiles (see *e. g.* Bassett and Könker, 1982; Steinhorst and Bowden, 1977 and Kabe, 1976) and on the literature about order statistics related to the L-estimators (Sarhan and Greenberg, 1962; David, 1981). The work on bootstrap interval estimation for quantiles by Efron (1981) should also provide a theoretical basis for further work.

Secondly, here we only consider a particular specification of non-linear dependence in the left tails of the bivariate distributions of returns. Various alternatives that can prove useful in this field (*e. g.*, left tail dependence or general bivariate tail dependence) can be found in the literature on multivariate extreme order statistics (Galambos, 1987; Deuhauvels, 1983, Joe, 1994). For a recent review see Joe (1997). For application to finance see Embrechts et al. (1998).

A third area that deserves special attention is linked to the idea of combining past records and simulation techniques to create worst-case scenarios of market risk through autoregressive conditional models, as indicated in Barone-Adesi et al. (1998) for the standard univariate Value-at-Risk measure, and by Koul and Saleh (1995) in a more general setting. The work on quantile regression (Hogg, 1975; Griffith and Willcox, 1978; Stone, 1977; Casady and Cryer, 1976; and for an application to economic data Buchinski, 1995) and the work of Rousseeuw and Hubert (1999) on conditional median modelling could also be of use in this field.

Finally, in the present paper we propose a measure of market risk referring to a unitary investment of each asset individually taken. A natural extension of the method described here seems to be the derivation of a multivariate VaR for the entire portfolio that takes into account the proportion of investment for each asset within the portfolio. The results on multivariate quantile estimation (de Haan and Huang, 1995) and the general measures of dependence between order

statistics in a multivariate context, like the orthant dependency concept (Joe, 1997), should prove helpful in this field.

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RIASSUNTO

Value-at-Risk bivariato

Nel presente lavoro il concetto di Value-at-risk (VaR) viene esteso al caso di distribuzioni bivariate dei rendimenti di titoli finanziari allo scopo di ottenere una misura del loro rischio di mercato la quale tenga conto di caratteristiche aggiuntive rispetto la pratica corrente legate all'esposizione al rischio nelle code delle distribuzioni stesse. Il lavoro prende le mosse da una definizione generale di rischio come probabilità di un evento indesiderato misurata su una distribuzione aleatoria. Successivamente viene introdotta una misura del rischio di mercato di un titolo (β -VaR) la quale ammette il tradizionale coefficiente β nella gestione di portafoglio come un caso particolare quando la distribuzione congiunta dei rendimenti del titolo stesso e dell'indice di mercato è Gaussiana bivariata. Infine il lavoro presenta alcune elaborazioni relative alla stima non parametrica della misura proposta, basate su dati azionari del mercato italiano nel periodo dal 1994 al 1998. I risultati evidenziano le diverse graduatorie dei rischi di mercato dei titoli alle quali si giunge utilizzando il coefficiente β (e quindi implicitamente assumendo la normalità dei rendimenti) ed utilizzando la misura alternativa suggerita (β -VaR) che, al contrario, non necessita tale assunzione.

SUMMARY

Bivariate Value-at-Risk

In this paper we extend the concept of Value-at-risk (VaR) to bivariate return distributions in order to obtain measures of the market risk of an asset taking into account additional features linked to downside risk exposure. We first present a general definition of risk as the probability of an adverse event over a random distribution and we then introduce a measure of market risk (β -VaR) that admits the traditional β of an asset in portfolio management as a special case when asset returns are normally distributed. Empirical evidences are provided by using Italian stock market data.