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ON LAPLACIANNESS OF HERMITIAN QUADRATIC AND BILINEAR FORMS IN COMPLEX NORMAL VARIABLES

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1. INTRODUCTION

Hermitian quadratic and bilinear forms in complex normal variables and their distributions are considered in this article. A set of necessary and sufficient (NS) conditions for Hermitian yuadratic and bilinear expressions to be distributed as noncentral generalized Laplacian (NGL) is obtained in Section 2 which, as a special case, gives the NS conditions for Hermitian yuadratic and bilinear forms to be NGL. Mathai (1993a) has defined the NGL density and presented a set of NS conditions for these forms (for the real case) to be distributed as NGL and as a gamma difference. The density of linear combinations of independent gamma variables has been obtained by many authors. For example, Provost (1989a) has obtained the density of a sum of gamma variables using the inverse Mellin transform. Using this, Provost (1989b) expressed the density of a general linear combination of gamma variables. Sultan (1997) has obtained the explicit expressions for the cumulants of arbitrary order of a Hermitian bilinear form, joint cumulants of a quadratic and a bilinear form and of two IIermitian bilinear forms. A set of NS conditions for the independence of these forms is also given there.

In Section 3, a set of NS conditions is ohtained for the Hermitian bilinear forms to be distributed as an Erlang difference. Some definitions for the moment generating functions (m.g.f.) and the densities of NGL and Erlang difference are presented in Appendix.

Laplacian distribution is given by the density

$$f(x) = \frac{1}{2\beta} e^{-|x|/\beta}, \qquad -\infty < x < \infty \tag{1.1}$$

where $|x|$ is the absolute value of x. This density comes in a large variety of situations such as the formation of sand dunes, residual effect from an input-output type situation and so on. For example, if the input variable X, and output variable X_2 are independently and identically exponentially distributed with mean value β, then the residual effect $U = X, - X_2$ has the m.g.f., denoted by $M.(\cdot)$

$$M_U(t) = (1 - \beta t)^{-1}(1 + \beta t)^{-1} = (1 - \beta^2 t^2)^{-1} \tag{1.2}$$

which gives rise to Laplace density (1.1). A generalization of this situation is a general gamma type input and general gamma type output and the residual effect. Particular situations of this type are discussed in Mathai (1993b). Since the input-output type situation is arising in many practical problems, it is desirable to check the conditions under which a given bilinear torm or a quadratic form is distributed as a Laplacian as in (1.1) or its noncentral version delined in Mathai (1993a). As in the case ol quadratic form being distributed as a chisquare, central or noncentral, it is useful in many inference problems to check for bilinear and quadratic forms to be distributed as in (1.1). This paper explores the necessary and sufficient conditions under which a given bilinear form or quadratic lortn he distributed as in (1 1) or its noncentral version. A quadratic form being chisquare distributed is a very important result in statistical inference problems. This is the basis lor the tests of hypotheses in analysis of variance, regression and other related problems. Rilinear and quadratic iorms being Laplace distributed is then a parallel result and has similar potential in statistical inference problems. When the vectors of quadratic and bilinear forms are complex Gaussian, then the resulting Hermitian quadratic and bilinear forms have lots of applications in engineering, communication theory, radar and related areas. A large amount ol literature exists on these topics, see for example Iluang and Campbell (1991), Cavers and Ho (1992) and the references therein. Therefore, in this paper, we explored the NS conditions for Laplacianness of Hermitian bilinear and quadratic forms.

2. Laplacianness of Hermitian quadratic and bilinear forms

Let $\mathbf{x}$: $p \times 1$ and y: $q \times 1$ have a joint complex normal (CN) distribution, that is, $\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim CN_{p+q}(\mu, \Sigma)$, $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, $\Sigma = \Sigma^* > 0$, which denotes that C is Hermitian positive definite, $C = \bar{C}'$, $\bar{\Sigma}$ is complex conjugate of matrix C, p, and Σ_{11} are the mean vector and the variance-covariance matrix of vector $\mathbf{x}$, Σ_{12} is the covariance matrix of $\mathbf{x}$ and y and $CN_p(\cdot, \cdot)$ denotes a p-variate complex normal, matrices are denoted by capital letters (so are the random variables) and vectors by boldface letters.

2.1. *Hermitian quadratic* expressions

Let

$$Q(\mathbf{z}) = \mathbf{z}^* A \mathbf{z} + \frac{1}{2}(\mathbf{a}^* \mathbf{z} + \mathbf{z}^* \mathbf{a}) + d_1 \tag{2.1}$$

where $A = A$ is a $(p+q)$ Ilermitian matrix of constants, a is a $(p+q)$ complex constant vector, d_1 is a real constant, then $Q(\mathbf{z})$ is a Hermitian quadratic expression. The m.g.f. of $Q(\mathbf{z})$ could be obtained as, see for example Mathai and Provost (1992, p. 40) for the real case

$$M_Q(t) = |I - t\Sigma^{1/2}A\Sigma^{1/2}|^{-1}\exp\Big\{t\Big(d_1 + \mu^{*}A\mu + \frac{1}{2}(\mathbf{a}^{*}p + p\ a)\Big)$$

$$+t^2\Big(\Sigma^{1/2}A\mu + \frac{1}{2}\Sigma^{1/2}\mathbf{a}\Big)^{*}(I - t\Sigma^{1/2}A\Sigma^{1/2})^{-1}\Big(\Sigma^{1/2}A\mu + \frac{1}{2}\Sigma^{1/2}\mathbf{a}\Big)\Big\} \tag{2.2}$$

where $\| t\Sigma^{1/2}A\Sigma^{1/2}\| < 1$, $\|.\|$ is a norm and $|(\cdot)|$ is the determinant of matrix $(\cdot)$, I is the identity matrix and $\Sigma^{1/2}$ denotes the Hermitian positive definite square root of the Hermitian positive definite matrix Σ. We will investigate the NS conditions for $Q(\mathbf{z})$ to be a NGL, that is NS conditions for *(2.2)*and (A.5) to be equal. This, then will be a result analogous to the chisquaredness of quadratic forms in real random variables.

Theorem 1 The NS conditions for $Q(\mathbf{z})$ of *(2 2)* to be a NGL of (A.5) with the parameters (α, β, A) are the following:

(I), $\alpha = n$ for some positive integer n;

$(\text{II})_a$ The eigenvalues of $\Sigma^{1/2}A\Sigma^{1/2}$ are such that exactly n of them are equal to β, exactly n of them are equal to $-\beta$ and the remaining p ι y $- 2n$ are equal to zeros, where $2n$ is the rank of the matrix $A\Sigma$, or $(\Sigma A)^2\Sigma = \dfrac{1}{\beta^2}(\Sigma A)^4\Sigma$ and $\operatorname{tr}(A\Sigma) = 0$;

$(\text{III})_a$ $d_1 + \mu^{*}A\mu + \dfrac{1}{2}(\mathbf{a}^{*}\mu + \mu^{*}\mathbf{a}) = 0 = \Big(\dfrac{1}{2}\Sigma\mathbf{a} + \Sigma A\mu\Big)^{*}A\Big(\dfrac{1}{2}\Sigma\mathbf{a} + \Sigma A\mu\Big)$;

$(\text{IV})_a$ $\lambda = \dfrac{1}{2\beta^2}\Big[\mu^{*}A\Sigma A\mu + \dfrac{1}{2}(\mathbf{a}^{*}\Sigma A\mu + \mu^{*}A\Sigma\mathbf{a}) + \dfrac{1}{4}\mathbf{a}^{*}\Sigma\mathbf{a}\Big]$;

$(\text{V})_a$ $d_1 = \Big(\dfrac{1}{4\beta^2}\Big)(\mathbf{a}^{*}\Sigma A\Sigma\mathbf{a})$;

and if the rank of A is less than (p ı y) then

$(\text{VI})_a$ $\mathbf{a}\ \Sigma\mathbf{a} = \dfrac{1}{B''}\mathbf{a}\ \Sigma A\Sigma A\Sigma\mathbf{a}$ or $a = \dfrac{1}{\beta^2}(A\Sigma)^2\mathbf{a}$

The following two lemmas are used in the proof of the above theorem:

Lemma *1* Let $p_i(x)$, i - 1, 2, 3, 4, be polynomials in x with rational coefficients such that

$$p_1(x)\, e^{p_2(x)/p_4(x)} = p_3(x)$$

for all x, where $p_2(0)/p_4(0) = 0$, $p_1(0) = p_3(0) = 1$. Then $p_1(x) = p_3(x)$ and $p_2(x) = p_4(x)$, see for example Ogawa (1950).

Lemma 2. Let

$$(1 - t^2\beta^2)^{-\alpha} = \prod_{j=1}^{p}(1 - t\lambda_j)^{-1} \tag{2.3}$$

for $|t^2\beta^2| < 1$, $|t\lambda_j| < 1$, $j = 1, 2, \ldots, p$, $\alpha > 0$, $\beta > 0$, λ_j's real; then

(I) $\alpha = n$ for some positive integer n;
(II) exactly n of the λ_j's are equal to β, exactly n of them are equal to $-\beta$ and the remaining $p - 2n$ of the λ_j's are zeros. See for example, Mathai (1993a) for the real case.

Proof of Theorem 1. Necessity. Applying Lemmas 1 and 2 to (A.5) and (2.2), conditions (I), and $(\text{II})_a$ are obtained. Equating exponential parts in (A.5) and (2.2) and expanding, we get

$$2\lambda\sum_{r=1}^{\infty}(\beta^2t^2)^r = t\left(d_1 + \mu^*A\mu + \frac{1}{2}(\mathbf{a}^*\mu + \mu^*\mathbf{a})\right) + t^2\left(\Sigma^{1/2}A\mu + \frac{1}{2}\Sigma^{1/2}\mathbf{a}\right)^*$$
$$\times\left[\sum_{r=0}^{\infty}(t\Sigma^{1/2}A\Sigma^{1/2})^r\right]\left(\Sigma^{1/2}A\mu + \frac{1}{2}\Sigma^{1/2}\mathbf{a}\right) \tag{2.4}$$

Comparing the coefficient of t, t^2, t^3, ..., on both sides of (2.4) and simplifying, conditions $(\text{III})_a$-$(\text{VI})_a$ are obtained. It is straightforward to see that these conditions are sufficient.

2.2. Hermitian quadratic form

When a is null vector and $d_1 = 0$, (2.1) reduces to $Q_1(\mathbf{z}) = \mathbf{z}^*A\mathbf{z}$, which is a Hermitian quadratic form.

Corollary 1. The NS conditions for $Q_1(\mathbf{z}) = \mathbf{z}^*A\mathbf{z}$, A =A", to be a NGL are:
$(\text{I})_b = (\text{I})_a$;
$(\text{II})_b = (\text{II})_a$;
$(\text{III})_b = \mu^*A\mu = 0$;
$(\text{IV})_b \quad \lambda = \left(\frac{1}{2\beta^2}\right)\mu^*A\Sigma A\mu.$

Corollary 2. The NS conditions for $Q_1(\mathbf{z}) = \mathbf{z}^*A\mathbf{z}$, $A = A^*$, to be distributed as noncentral chi-square are $(\text{I})_b$-$(\text{IV})_b$ of Corollary 1 with $\beta = 2$, $\alpha = m$, m = 1, 2, 3, ...

2.3. *Hermitian* bilinear expression

Let

$$B(\mathbf{x},\mathbf{y}) = \frac{1}{2}(\mathbf{x}^*A_1\mathbf{y} + \mathbf{y}^*A_1^*\mathbf{x}) + \frac{1}{2}(\mathbf{b}^*\mathbf{x} + \mathbf{x}^*\mathbf{b}) + \frac{1}{2}(\mathbf{c}^*\mathbf{y} + \mathbf{y}^*\mathbf{c}) + d_2 \tag{2.5}$$

where A_1: p X q is a complex matrix of constants, b: p X 1 and c: q X 1 are complex vectors of constants, d_2 is a real constant, then $B(\mathbf{x}, \mathbf{y})$ is a Hermitian bilinear expression, see for example Sultan (1997) for some aspects of Hermitian bilinear forms. Writing $B(\mathbf{x}, \mathbf{y})$ as a Hermitian quadratic form in $\mathbf{z} = \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$, we get

$$B(\mathbf{x},\mathbf{y}) = Q(\mathbf{z}) = \mathbf{z}^* A\mathbf{z} + \frac{1}{2}(\mathbf{a}^*\mathbf{z} + \mathbf{z}^*\mathbf{a}) + d_1 \quad (2.6)$$

where $A = \begin{pmatrix} O & \frac{1}{2}A_1 \\ \frac{1}{2}A_1^* & O \end{pmatrix}$, $\mathbf{a}^* = (\mathbf{b}^*, \mathbf{c}^*)$, $d_1 = d_2$ and O is the null matrix.

Theorem 2. The NS conditions for $B(\mathbf{x}, \mathbf{y})$ *of (2.5)* **to** *be a NGL are*

$(\text{I})_c = (\text{I})_a$;

$(\text{II})_c$ $\left(\frac{1}{\beta^2}\right) A\Sigma A\Sigma A\Sigma A = A\Sigma A$

and $\operatorname{tr}(\Sigma^{1/2}A\,\Sigma^{1/2}) = \operatorname{tr}(\Sigma A) = 0$, *or*

$$A_1\Sigma_{22}A_1^* = \left(\frac{1}{4\beta^2}\right)\{[(A_1\Sigma_{21})^2 + A_1\Sigma_{22}A_1^*\Sigma_{11}](A_1\Sigma_{22}A_1^*) + [A_1\Sigma_{21}A_1\Sigma_{22} + A_1\Sigma_{22}A_1^*\Sigma_{12}](A_1^*\Sigma_{12}A_1^*)\}$$

$$A_1\,\Sigma_{21}A_1 = \left(\frac{1}{4\beta^2}\right)\{[(A_1\Sigma_{21})^2 + A_1\Sigma_{22}A_1^*\Sigma_{11}](A_1\Sigma_{21}A_1) + [A_1\Sigma_{21}A_1\Sigma_{22} + A_1\Sigma_{22}A_1^*\Sigma_{12}](A_1^*\Sigma_{11}A_1)\}$$

$$A_1^*\Sigma_{12}A_1^* = \left(\frac{1}{4\beta^2}\right)\{[(A_1^*\Sigma_{12})^2 + A_1^*\Sigma_{11}A_1\Sigma_{22}](A_1^*\Sigma_{12}A_1^*) + [A_1^*\Sigma_{11}A_1\Sigma_{21} + A_1^*\Sigma_{12}A_1^*\Sigma_{11}](A_1\Sigma_{22}A_1^*)\}$$

$$A_1^*\Sigma_{11}A_1 = \left(\frac{1}{4\beta^2}\right)\{[(A_1^*\Sigma_{12})^2 + A_1^*\Sigma_{11}A_1\Sigma_{22}](A_1^*\Sigma_{11}A_1) + [A_1^*\Sigma_{11}A_1\Sigma_{21} + A_1^*\Sigma_{12}A_1^*\Sigma_{11}](A_1\Sigma_{21}A_1)\}$$

and $\operatorname{tr}\left(\frac{1}{2}\Sigma_{12}A_1^* + \frac{1}{2}\Sigma_{21}A_1\right) = 0$;

$(\text{III})_c$ $$\frac{1}{2}(\mu_2^*A_1^*\mu_1 + \mu_1^*A_1\mu_1) + \frac{1}{2}(\mu_1^*A_1\mu_2) + \frac{1}{2}(\mu_1^*\mathbf{b} + \mathbf{b}^*\mu_1) + \frac{1}{2}(\mu_2^*\mathbf{c} + \mathbf{c}^*\mu_2) + d_2 = 0;$$

$(\text{IV})_c$ $$\lambda = \left(\frac{1}{8\beta^2}\right)[(\mathbf{b}^*\Sigma_{12} + \mathbf{c}^*\Sigma_{22})A_1^*\mu_1 + (\mathbf{b}^*\Sigma_{11} + \mathbf{c}^*\Sigma_{21})A_1\mu_2 + \mu_1^*A_1(\Sigma_{12}\mathbf{b} + \Sigma_{22}\mathbf{c}) + \mu_2^*A_1^*(\Sigma_{11}\mathbf{b} + \Sigma_{12}\mathbf{c}) + \mu_2^*A_1(\Sigma_{12}A_1^*\mu_1 + \Sigma_{11}A_1\mu_2) + \mu_1^*A_1(\Sigma_{22}A_1^*\mu_1 + \Sigma_{21}A_1\mu_2) + \mathbf{b}^*\Sigma_{11}\mathbf{b} + \mathbf{b}^*\Sigma_{12}\mathbf{c} + \mathbf{c}^*\Sigma_{21}\mathbf{b} + \mathbf{c}^*\Sigma_{22}\mathbf{c}];$$

$$(\text{V})_c \quad d_2 = \left(\frac{1}{8\beta^2}\right)[(\mathbf{b}^*\Sigma_{11} + \mathbf{c}^*\Sigma_{21})A_1\Sigma_{21}\mathbf{b} + (\mathbf{b}^*\Sigma_{12} + \mathbf{c}^*\Sigma_{22})A_1^*\Sigma_{11}\mathbf{b}$$
$$+ (\mathbf{b}^*\Sigma_{11} + \mathbf{c}^*\Sigma_{21})A_1\Sigma_{22}\mathbf{c} + (\mathbf{b}^*\Sigma_{12} + \mathbf{c}^*\Sigma_{22})A_1^*\Sigma_{12}\mathbf{c}].$$

When the rank of matrix $\begin{pmatrix} O & \frac{1}{2}A_1 \\ \frac{1}{2}A_1^* & O \end{pmatrix}$ is less than $(p+q)$, then

$$(\text{VI})_c \quad \mathbf{b} = \left(\frac{1}{4\beta^2}\right)(\Sigma_{12}A_1^*\Sigma_{12}A_1^*\mathbf{b} + \Sigma_{11}A_1\Sigma_{22}A_1^*\mathbf{b} + \Sigma_{12}A_1^*\Sigma_{11}A_1\mathbf{c} + \Sigma_{11}A_1\Sigma_{21}A_1\mathbf{c});$$

$$\mathbf{c} = \left(\frac{1}{4\beta^2}\right)(\Sigma_{22}A_1^*\Sigma_{12}A_1^*\mathbf{b} + \Sigma_{21}A_1\Sigma_{22}A_1^*\mathbf{b} + \Sigma_{22}A_1^*\Sigma_{11}A_1\mathbf{c} + \Sigma_{21}A_1\Sigma_{21}A_1\mathbf{c}).$$

Proof. Substituting the values of A, a and d_1 given by (2.6) in Theorem 1 and simplifying, conditions $(\text{I})_c$-$(\text{VI})_c$ are obtained.

Corollary 3. When b and c are null vectors and $d_1 = 0$, (2.7) reduces to $B(\mathbf{x},\mathbf{y}) = \frac{1}{2}(\mathbf{x}^* A_1\mathbf{y} + \mathbf{y}^* A_1^*\mathbf{x})$ and the NS conditions for $B(\mathbf{x}, \mathbf{y})$ to be distributed as NGL could be obtained from Theorem 2.

3. HERMITIAN BILINEAR FORMS AND THE ERLANG DIFFERENCE

The NS conditions for

$$B(\mathbf{x},\mathbf{y}) = \frac{1}{2}(\mathbf{x}^* A_1\mathbf{y} + \mathbf{y}^* A_1^*\mathbf{x}) \tag{3.1}$$

to be distributed as an Erlang difference are obtained in this section. From (2.6)

$$B(\mathbf{x},\mathbf{y}) = Q(\mathbf{z}) = \mathbf{z}^* A\mathbf{z}, \qquad A = \begin{pmatrix} O & \frac{1}{2}A_1 \\ \frac{1}{2}A_1^* & O \end{pmatrix} \tag{3.2}$$

The m.g.f. of $Q(\mathbf{z})$ from (2.2) becomes (with $\mu = \mathbf{0}$)

$$M_Q(t) = |I - t\Sigma^{1/2}A\Sigma^{1/2}|^{-1}. \tag{3.3}$$

3.1. Equicorrelated Case

Let $q = p$, $A_1 = A$, $\Sigma_{11} = I = \Sigma_{22}$, $\Sigma_{12} = \rho I = C$, ρ is the coefficient of correlation between the components of vectors x and y. Then (3.3), after some simplification, using the results on the determinant of a matrix, could be written as

$$M_Q(t) = \prod_{j=1}^{p} [(1 - a_1\lambda_j t)^{-1}(1 + a_2\lambda_j t)^{-1}] \tag{3.4}$$

where $a_1 = \frac{1+\rho}{2}$, $a_2 = \frac{1-\rho}{2}$. If $\Sigma_{11} = \sigma_1^2 I$, $\Sigma_{22} = \sigma_2^2 I$, $\Sigma_{12} = \sigma_1\sigma_2\rho I$, then a_1 and a_2 would become $a_1 = \frac{\sigma_1\sigma_2(1+\rho)}{2}$, $a_2 = \frac{\sigma_1\sigma_2(1-\rho)}{2}$. Let A_1 be a non-null matrix, then we state the following result:

Theorem 3 The NS conditions for $B(\mathbf{x}, \mathbf{y})$ given by (3.2), to be distributed as a central Erlang difference of (A.3) with parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are the following:

(I) $\alpha_1 = \alpha_2 = n$, where n is the number of nonzero eigenvalues of A_1;

(II) all the nonzero eigenvalues of A_1 are positive and equal such that $\frac{\beta_1}{\beta_2} = \frac{(1+\rho)}{(1-\rho)}$ or all the nonzero eigenvalues of A_1 are negative and equal such that $\frac{\beta_1}{\beta_2} = \frac{(1-\rho)}{(1+\rho)}$.

If $\rho = 0$, then all eigenvalues are equal, either all positive or all negative and under (I) and (II) this Erlang difference will be a generalized Laplacian.

Proof. Similar as for the real case, see for example, Mathai *et al. (1995).*

Corollary 4. The NS conditions for $B(\mathbf{x}, \mathbf{y})$, as defined in (3.1), to be distributed as an Erlang difference with the parameters n, n, $\sigma_1\sigma_2(1+\rho)\beta$, $\sigma_1\sigma_2(1-\rho)\beta$ is that $\beta A_1 = A_1^2$ and of rank n, where β is the common repeated eigenvalue of A_1.

Theorem 4. The NS conditions for the Hermitian quadratic form $\mathbf{x}^* A_1\mathbf{x}$, $A_1 = A_1^*$, $\mathbf{x} \sim CN_p(\mathbf{0}, \Sigma_{11})$, $\Sigma_{11} = \Sigma_{11}^* > 0$ to be distributed as an Erlang difference of *(A.1)* are the following:

(I) $\alpha_1 = \nu_1$, $\alpha_2 = \nu_2$ for some positive integers ν_1 and ν_2;

(II) all the positive eigenvalues of $\Sigma_{11}A_1$ are equal to β_1 and all the negative eigenvalues are equal to $-\beta_2$, at least one of them is there in each set.

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APPENDIX A

DENSITIES OF GENERALIZED LAPLACIAN AND ERLANG DIFFERENCE

Central Case

Let X_1 and X_2 be two independent real gamma variables with parameters (α_1, β_1) and (α_2, β_2) respectively, $\alpha_j > 0$, $\beta_j > 0$, $j = 1, 2$. Let $U = X_1 - X_2$, then the m.g.f.s of X_1, X_2 and U are

$$M_{X_1}(t) = (1-\beta_1 t)^{-\alpha_1}, \qquad M_{X_2}(t) = (1-\beta_2 t)^{-\alpha_2}$$

$$M_U(t) = (1-\beta_1 t)^{-\alpha_1}(1+\beta_2 t)^{-\alpha_2} \tag{A.1}$$

for $|\beta_1 t| < 1$, $|\beta_2 t| < 1$. For $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, (A.1) becomes

$$M_U(t) = (1-\beta^2 t^2)^{-\alpha} \tag{A.2}$$

and the density corresponding to (A.2) is a special case of the generalized Laplace density with $\alpha > 0$. For $\alpha = 1$, U has the Laplace density. Mathai (1993a) has obtained the density of U in (A.1) for all positive values of α_1 and α_2, which for $\alpha_1 = m_1$ and $\alpha_2 = m_2$, where m_1 and m_2 are positive integers, reduces to the density of central Erlang difference as given below

$$g(u) = \begin{cases} k_1 u^{m_1-1} e^{-\frac{u}{\beta_1}} \sum_{r=0}^{m_1-1} \left[\frac{(m_2)_r (\beta_0 u)^r}{r!(m_1-1-r)!} \right], & u > 0 \\ k_2 (-u)^{m_2-1} e^{\frac{u}{\beta_2}} \sum_{r=0}^{m_2-1} \left[\frac{(m_1)_r (-\beta_0 u)^r}{r!(m_2-1-r)!} \right], & u \le 0 \end{cases} \tag{A.3}$$

where $k_1 = \beta_0^{-m_2}\beta_1^{-m_1}\beta_2^{-m_2}$, $k_2 = \beta_0^{-m_1}\beta_1^{-m_1}\beta_2^{-m_2}$, $\beta_0 = \frac{1}{\beta_1} + \frac{1}{\beta_2}$ and for example, $(a)_r = a(a+1)\ldots(a+r-1)$, $(a)_0 = 1$, $a \neq 0$. When $\beta_1 = \beta_2 = 2$ in (3.7), it reduces to the density of the difference of two independent chi-squares with $2m_1$ and $2m_2$ degrees of freedom (d.f.).

Noncentral case

The m.g.f. of a random variable U distributed as a noncentral gamma difference is (see for example, Mathai, 1993a)

$$M_U(t) = (1-\beta_1 t)^{-\alpha_1}(1+\beta_2 t)^{-\alpha_2} \exp\{-(\lambda_1+\lambda_2) + \lambda_1(1-\beta_1 t)^{-1} + \lambda_2(1+\beta_2 t)^{-1}\} \tag{A.4}$$

where $\beta_j > 0$, $\alpha_j > 0$, $|\beta_j t| < 1$, $\lambda_j > 0$, $j = 1, 2$. With $\lambda_1 = \lambda_2 = \lambda$, $\beta_1 = \beta_2 = \beta$, $\alpha_1 = \alpha_2 = \alpha$, (A.4) reduces to

$$M_U(t) = (1-\beta^2 t^2)^{-\alpha} \exp\{-2\lambda + 2\lambda(1-\beta^2 t^2)^{-1}\} \tag{A.5}$$

Then U in (A.5) is a special case of a noncentral generalized Laplace variable with parameters (α, β, Λ). The density of U in (A.4) for $\alpha_1 = m_1$ and $\alpha_2 = m_2$ is given as

$$f(u) = e^{-(\lambda_1+\lambda_2)} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \left[\frac{(\lambda_1)^{r_1}(\lambda_2)^{r_2}}{r_1!\, r_2!} g_{r_1,r_2}(u) \right] \tag{A.6}$$

where $f(u)$ is the density of U of (A.4) and $g_{r_1, r_2}(u)$ is the $g(u)$ of (A.3) with m_j replaced by $m_j + r_j$, $j = 1, 2$. The density for a chi-square difference with $2m_j$ d.f. could be obtained from (A.6)by replacing, β_j by 2 and α_j by m_j, $j = 1, 2$.

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RIASSUNTO

Distribuzione laplaciana di forme hermitiane e quadratiche di variabili normali complesse

Si ottengono risultati che danno le condizioni necessarie e sufficienti perché un'espressione quadratica Hermitiana o un'espressione bilineare Hermitiana si distribuiscano come una Laplaciana generalizzata non-centrale. Si ottengono anche le condizioni necessarie e sufficienti affinché forme bilineari Hermitiane siano distribuite come differenza di Erlang.

SUMMARY

On Laplacianness of Hermitian quadratic and bilinear forms in complex normal variables

A srt of results are obtained which will give the necessary and sufficient (NS) conditions for a Hermitian quadratic expression or a Hermitian bilinear expression to be distributed as a noncentral generalized Laplacian. The NS conditions are also obtained for Hermitian bilinear forms to be distributed as an Erlang difference.