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USE OF INDEPENDENT RUNS FOR THE IMPROVEMENT OF THE RESPONSE OF SHEWHART CONTROL CHARTS

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1. INTRODUCTION

Standard Shewhart control charts, as is well known, provide out-of-control signals when a sample estimate of some parameter of the distribution of a quality characteristic falls outside its control limits. In particular, in the case of X-charts they are generally set at a 3-sigma (or 3.09-sigma) distance from the central line.

It has been observed by many authors, however, how this simple and most applied rule does not take into account the patterns of estimates represented in the chart and is quite slow in detecting small shifts in the process parameters. This is the reason why supplementary runs rules have been widely introduced in literature with particular reference to X-charts. Several runs rules have been suggested, for instance, by Page (1955), the Western Electric Company (1956), Roberts (1958), Bissell (1978), Wheeler (1983).

Supplementary runs rules are generally used in various combinations. Page (1955) and Bissell (1978) were the first to provide an evaluation of the average run length (ARL) of $\bar{X}$-charts for simpler combinations of runs rules.

An extensive review of the behaviour of X-charts when various combinations of runs rules are at work simultaneously has been provided by Champ and Woodall (1987). They have found that the use of these combinations of rules does improve the capability of a chart to detect small shifts in the process mean, but, as Montgomery (1996) observes, the in-control ARL or ARL_0, which indicates the number of samples expected before a false out-of-control signal is provided by the chart, is substantially degraded. This is the reason why Montgomery (1996) discourages the use of supplementary runs rules to detect small shifts in the process mean and favours the use of cumulative sums (CUSUM) or exponentially weighted moving average (EWMA) charts for the purpose.

However, as the common experience of statistical process control (SPC) analysts shows, and as has also been clearly reported in a Saniga and Shirland's (1977) survey, practitioners often tend to avoid more complicated, even if efficient, control charts in favour of the simpler Shewhart charts.

For this reason we deem it useful to remain in the field of Shewhart-type applications by introducing independent runs (IRs) of various length. We will show how these runs can significantly reduce the ARL of the corresponding standard Shewhart charts by simply finding appropriate lengths for the runs and convenient distances between control limits, without decreasing, or rather increasing the ARL_0, as is opportune.

It will also be shown how IK-based control charts can be considered as a generalisation of standard Shewhart charts.

2. INDEPENDENT RUNS. THE IR-CHART

Let us consider a Shewhart control chart with upper control limit U and lower control limit L, constructed to monitor sample estimates ϑ_n, obtained from independent samples of n items each, of a real parameter H of some process characteristic X. Let us also consider independent runs (IRs) of length h ≥ 2,

$$\hat{\vartheta}_{(i-1)h+1}, \ldots, \hat{\vartheta}_{ih}$$

with $i = 1, 2, \ldots$

The rule we apply in this article in order to obtain an out-of-control signal is the following. "an out-of-control signal is given when, for the first time, r or more points of the same run of length h fall in the same interval $(-\infty, L)$ or (U, ∞), while the remaining points fall in the interval $[L, U]$ where $r \leq h$ and $L < U$".

The rationale of this choice is that a given number of points are required to fall on the same side of the interval $[L, U]$ as proof of a consistent shift in the process parameter This choice, obviously, also reduces the probability of false alarms, when the process remains in the initial in-control state with respect to the case in which we consider r or more points falling on both sides of the interval [I., U].

We shall define these runs as 'independent runs of order r and h', and use the abbreviation IR(r, h) to denote them Therefore, standard control charts can also be defined as IR(1, 1)

If we now indicate by θ_0 the in-control value for 8, and by θ_1 ($H_1 \neq H_0$) an out-of-control value for the same parameter, both belonging to the space parameter Ω for 8, and if we denote, for each $\theta \in \Omega$, the probability that an estimate-point $\hat{\vartheta}_i$ falls in the interval $(-\infty, L)$ or in the interval (U, ∞) as respectively $\gamma_L(\theta)$ and $\gamma_U(\theta)$, that is

$$\gamma_L = \gamma_L(\theta) = P\{\hat{\vartheta}_n \in (-\infty, L) \mid \theta \in \Omega\}$$
$$\gamma_U = \gamma_U(\theta) = P\{\hat{\vartheta}_n \in (U, +\infty) \mid \theta \in \Omega\}\,, \tag{1}$$

the same probabilities for a process still in the initial control state θ_0 are given by $\gamma_{L0} = \gamma_L(\theta_0)$ and $\gamma_{U0} = \gamma_U(\theta_0)$, while ior a process in out-of control state H_1 they are given by $\gamma_{L1} = \gamma_L(\theta_1)$ and $\gamma_{U1} = \gamma_U(\theta_1)$.

According to the rule introduced before, the probability for an out-of-control signal, for $\theta = H_1$, is given hy

$$p_{r,h}(1) = \sum_{l=r}^{h} \binom{h}{l} (\gamma_{L1}^l + \gamma_{U1}^l)(1 - \gamma_{L1} - \gamma_{U1})^{h-l} \tag{2}$$

while the probability of an out-of control signal for a process in control ($8 = \theta_0$), that is the probability of a false alarm, is given by

$$p_{r,h}(0) = \sum_{l=r}^{h} \binom{h}{l} (\gamma_{L0}^{l} + \gamma_{U0}^{l})(1 - \gamma_{L0} - \gamma_{U0})^{h-l} \tag{3}$$

The expected number of IR(r, h) necessary to obtain an out-of-control signal is obviously given by $1/p_{r,h}(I)$, (I= 0, 1), therefore the overall ARL of the proposed rule is given by

$$L_{r,h}(I) = \frac{h}{p_{r,h}(I)} \qquad (I = 0,1) \tag{4}$$

We deem it useful to consider also another indicator of the effectiveness of the proposed procedure. After fixing a value F $\in$ (0, 1), we denote by K the minimum number of independent runs for which the probability of not detecting a shift does not exceed E, that is

$$K(I) = \min\{k : [1 - p_{r,h}(I)]^{k} \leq \text{E} | \text{k E } N\} =$$

$$= \min\left\{k : k \geq \frac{\log \varepsilon}{\log[1 - p_{r,h}(I)]} | k \in N\right\} \qquad (I = 0,1) \tag{5}$$

Therefore, we can introduce the indicator $C_{r,h}(I)$, (I= 0, 1), which denotes the minimum number of sample points for which the probability of not detecting a shift does not exceed E, that is the 100(1 - E) percentile of the overall run length distribution, namely

$$C_{r,h}(I) = h \cdot K(I) \qquad (I = 0, 1) \tag{6}$$

We name this indicator 'confidence threshold for the run length (CTRL)', since it is very likely for an out-of-control signal to occur not later than indicated by it.

As can be easily seen, we have h $\leq C_{r,h}(I) < - L_{r,h}(I)$ log E (see, also, Iacobini, 1991, p. 113). So, for instance, if we choose $\varepsilon = 0.05$ we have that $C_{r,h}(I) < 3 L_{r,h}(I)$. Notice that, according to (5) and (6), $C_{r,h}(I)$ is always a multiple of h.

The corresponding values of ARL and CTRL for standard Shewhart charts can be obtained, for comparative purposes, by putting h = r = 1 in the previous expressions since, as noted above, Shewhart charts can be considered as IR(1, 1) charts.

3. METHODS FOR CHOOSING AN APPROPRIATE IR-CHART. THE CASE OF A $\bar{X}$-CHART

Several criteria can be applied to choose the dimension of the IRs introduced in the previous section. The common purpose of these criteria, however, should be a good reduction of the ARL for shifts in the process parameters which may cause noticeable increases in the nonconformity level of a process, especially when they are caused by small shifts, for which standard Shewhart charts are slow in providing out-of-control signals. On the other hand, the in-control ARL for IRs should not be lower, but rather higher than that of standard charts, since false out-of-control signals may add remarkable costs to the economy of a process. We will show how, in general, the proposed methods yield good results in terms of out-of-control ARL if compared with the supplementary runs rules mentioned in the introductory section, and even better results for in-control ARL values.

The methods will be exposed with regard to $\bar{X}$-charts, that is the case in which the process parameter we want to monitor is the mean p of a process characteristic having the normal distribution X – N(μ; σ^2) and the sample estimates of p are the means $\bar{x}$ calculated in samples of size n. If we denote by μ_0 the in-control mean of the process, which is the central value of the interval [I,, U], and by ***p,*** an out-of-control value, so that the standardised shift of the process mean can be expressed by

$$\delta = \frac{|\mu_1 - \mu_0|\sqrt{n}}{\sigma},$$

using the out-of-control signal rule established in the previous section we have that

$$\gamma_{L1} = 1 - \Phi(z + \delta)\ ; \qquad \gamma_{U1} = 1 - \Phi(z - \delta)\ ;$$

$$\gamma_{L0} = \gamma_{U0} = 1 - \Phi(z) = \Phi(-z)\ ; \tag{7}$$

where $\Phi(z)$ is the c.d.f of Z – N(0; 1) and where

$$z = \frac{(U - \mu_0)\sqrt{n}}{\sigma} = \frac{(\mu_0 - L)\sqrt{n}}{\sigma}$$

We can now indicate the IR-chart previously defined as IR(r, h, z). The standard Shewhart 3-sigma X-chart is therefore IR(1, 1, 3).

A first approach *(method* 1) to the definition of convenient IRs is that of fixing values for L and U, or rather for z, starting from the corresponding values of a standard $\bar{X}$-chart, that is $z = 3$, and proceeding towards the central line of the chart.

Table 1 shows ARL and CTRL ($\varepsilon = 0.05$) values for standard X-charts and only for those IR(r, h, z)-charts for which the following conditions are satisfied:

$$L_{r,h}(0) \geq L_{1,1}(0) = 370.4 \qquad L_{r,h}(1) \leq L_{1,1}(1) = 44.0 \qquad L_{r,h}(2) \leq L_{1,1}(2) = 6.3 \tag{8}$$

These conditions, in fact, ensure that the IR-charts provide out-of control signals for small or medium shifts of the process mean more quickly than standard $\bar{X}$-charts, but, at the same time, the ARL for a false out-of-control signal is not smaller than that of a standard chart, as required. Of course, a more precise choice among various satisfactory IR-charts should be made case by case, once the maximum detectable shift has been selected after studying suitable values for the process capability ratio (PCR).

Notice that only IRs with $2 \leq h \leq 5$ have been considered, since IRs of greater length are scarcely manageable and, besides, as can be seen from (4) and (7),

$$\lim_{\delta \to \infty} L_{r,h}(\delta) = h$$

TABLE 1

ARL and CTRL (ε = 5%) values for Standard Shewhart Charts and Selected IR-Charts for Process Means

δ	Shewhart chart	$z = 1$		$z = 1.5$		$z = 2$		
		$h = 3$ $r = 3$	$h = 5$ $r = 4$	$h = 4$ $r = 3$	$h = 5$ $r = 3$	$h = 3$ $r = 2$	$h = 4$ $r = 2$	$h = 5$ $r = 2$
				ARL				
0.0	370.4	375.6	1,104.6	1,898.9	1,074.9	1,004.1	695.8	542.4
0.1	352.9	343.5	941.0	1,638.2	927.3	908.9	629.8	491.0
0.2	308.4	271.9	639.7	1,139.3	645.8	700.9	486.1	379.3
0.3	253.1	199.0	401.3	726.1	413.6	496.3	345.0	269.8
0.4	200.1	141.8	250.8	456.4	262.3	341.1	238.1	186.9
0.5	155.2	101.1	160.4	290.9	169.3	234.1	164.4	129.8
0.6	119.7	73.0	105.9	189.8	112.2	162.3	114.9	91.5
0.7	92.3	53.7	72.2	127.1	76.6	114.4	81.8	65.7
0.8	71.6	40.3	50.9	87.4	53.9	82.0	59.3	48.2
0.9	55.8	30.8	37.0	61.7	39.0	59.8	43.9	36.2
1.0	44.0	24.0	27.7	44.0	29.1	44.0	33.2	27.8
1.2	27.8	15.4	17.0	25.3	17.6	26.0	20.1	17.5
1.4	18.2	10.7	11.6	15.7	11.8	16.3	13.2	12.0
1.6	12.4	7.8	8.7	10.7	8.7	10.9	9.4	8.9
1.8	8.7	6.1	7.1	7.9	7.0	7.9	7.1	7.2
2.0	6.3	5.0	6.1	6.3	6.1	6.0	5.8	6.2
2.5	3.2	3.7	5.2	4.6	5.2	3.9	4.4	5.2
3.0	2.0	3.2	5.0	4.1	5.0	3.2	4.1	5.0
∞	1.0	3.0	5.0	4.0	5.0	3.0	4.0	5.0
				CTRL				
0.0	1,109	1,122	3,305	5,684	3,215	3,006	2,080	1,620
0.1	1,056	1,026	2,815	4,904	2,775	2,721	1,884	1,465
0.2	923	813	1,910	3,408	1,930	2,097	1,452	1,130
0.3	757	594	1,195	2,172	1,235	1,485	1,028	805
0.4	598	423	745	1,364	780	1,020	708	555
0.5	464	300	475	868	500	699	488	385
0.6	357	216	310	564	330	483	340	270
0.7	276	159	210	376	225	339	240	190
0.8	213	117	145	256	155	243	172	140
0.9	166	90	105	180	110	177	128	105
1.0	130	69	80	128	80	129	96	80
1.2	82	42	45	72	45	75	56	45
1.4	54	30	30	44	30	45	39	30
1.6	36	21	20	28	20	30	24	20
1.8	25	15	15	20	15	21	16	15
2.0	18	12	10	12	10	15	12	10
2.5	9	6	5	8	5	9	8	5
3.0	5	6	5	4	5	6	4	5
∞	1	3	5	4	5	3	4	5

while the ARL of standard $\bar{X}$-charts tends towards 1. But, as long as we limit the choice of h to 5, as suggested above, larger ARL values for large δ values are not of great inconvenience, in fact, when comparing different control charts, it is important to compare ARL values for small or medium shifts in the process mean, since such values are quite large. What happens in the case of large shifts is mostly irrel-

TABLE 2

ARL and CTRL ($\varepsilon = 5\%$) values for Standard Shewhart Charts and IR-Charts for Process Means ($2 \leq h \leq 5$)

δ	Shew--hart chart	$(h; r)$									
		(2; 2)	(3; 2)	(3; 3)	(4; 2)	(4; 3)	(4; 4)	(5; 2)	(5; 3)	(5; 4)	(5; 5)
		$z = 1.63$	$z = 1.78$	$z = 1.01$	$z = 1.863$	$z = 1.18$	$z = 0.61$	$z = 1.92$	$z = 1.29$	$z = 0.79$	$z = 0.34$
						ARL					
0.0	370.4	376.3	378.5	393.2	375.5	374.7	371.2	382.3	381.5	383.1	375.9
0.1	352.9	349.7	348.2	359.3	343.5	336.1	335.8	348.2	337.7	337.4	337.1
0.2	308.4	287.3	279.2	283.7	271.9	254.7	259.5	272.9	249.0	246.1	255.9
0.3	253.1	218.9	207.0	207.1	198.7	178.1	185.5	197.5	169.6	166.2	179.9
0.4	200.1	161.2	148.8	147.2	141.2	122.2	129.9	139.2	114.0	111.2	124.6
0.5	155.2	117.8	106.6	104.7	100.3	84.7	91.8	98.3	77.9	75.8	87.5
0.6	119.7	86.5	77.0	75.4	72.1	59.8	65.9	70.4	54.6	53.1	62.8
0.7	92.3	64.2	56.5	55.3	52.7	43.3	48.5	51.5	39.4	38.4	46.3
0.8	71.6	48.3	42.2	41.4	39.3	32.2	36.5	38.4	29.2	28.6	35.1
0.9	55.8	36.9	32.0	31.6	29.9	24.5	28.1	29.3	22.4	22.0	27.3
1.0	44.0	28.6	24.8	24.6	23.2	19.1	22.2	22.9	17.6	17.4	21.7
1.2	27.8	18.0	15.6	15.7	14.9	12.5	14.7	14.9	11.8	11.8	14.8
1.4	18.2	12.0	10.6	10.8	10.3	8.9	10.5	10.6	8.7	8.8	10.9
1.6	12.4	8.4	7.6	8.0	7.7	6.9	8.1	8.1	7.0	7.1	8.6
1.8	8.7	6.2	5.9	6.2	6.1	5.7	6.6	6.7	6.0	6.2	7.3
2.0	6.3	4.8	4.8	5.1	5.2	4.9	5.6	5.9	5.5	5.6	6.4
2.5	3.2	3.1	3.5	3.7	4.2	4.2	4.5	5.1	5.1	5.1	5.4
3.0	2.0	2.4	3.1	3.2	4.0	4.0	4.1	5.0	5.0	5.0	5.1
∞	1.0	2.0	3.0	3.0	4.0	4.0	4.0	5.0	5.0	5.0	5.0
						CTRL					
0.0	1,109	1,126	1,131	1,176	1,120	1,120	1,108	1,140	1,140	1,145	1,120
0.1	1,056	1,046	1,041	1,074	1,024	1,004	1,004	1,040	1,005	1,005	1,005
0.2	923	858	834	846	810	760	772	815	740	730	760
0.3	757	654	618	618	592	528	552	585	505	495	535
0.4	598	480	444	438	420	364	384	410	335	330	370
0.5	464	350	315	312	296	248	272	290	230	220	255
0.6	357	258	228	222	212	176	192	205	160	155	185
0.7	276	190	165	162	152	124	140	150	115	110	135
0.8	213	142	123	120	112	92	104	110	80	80	100
0.9	166	108	93	93	84	68	80	85	60	60	75
1.0	130	84	72	72	64	52	64	65	45	45	60
1.2	82	52	45	45	40	32	40	40	30	30	40
1.4	54	34	27	30	28	24	28	25	20	20	25
1.6	36	24	18	21	20	16	20	20	15	15	20
1.8	25	16	15	15	12	12	16	15	10	10	15
2.0	18	12	12	12	12	8	12	10	10	10	10
2.5	9	6	6	6	8	4	8	5	5	5	10
3.0	5	4	3	6	4	4	4	5	5	5	5
∞	1	2	3	3	4	4	4	5	5	5	5

evant, since corresponding ARL and CTRL values are small anyway. Another important reason why the maximum h is chosen quite small is that, in case the shift of the process mean μ occurs over time, the risk of having points in the same run obtained under different values of μ is reduced. ARL and CTRL values for IRs

TABLE 3

ARL for Shewhart Control Charts with Supplementary Runs Rules

δ	C_1	C_7	C_{12}	C_{78}	C_{15}	C_{13}	C_{14}	C_{79}	C_{16}	C_{123}	C_{156}	C_{124}	C_{789}	C_{134}	C_{1456}	C_{1234}
0.0	370.4	499.6	225.4	239.7	278.0	166.0	152.7	170.4	349.4	132.9	266.8	122.0	126.2	105.8	133.2	91.7
0.2	308.4	412.0	177.6	185.5	222.6	120.7	110.5	120.9	279.5	97.9	208.4	89.1	91.2	76.0	96.4	66.8
0.4	200.1	269.2	104.5	106.1	134.2	63.9	59.8	63.8	165.5	52.9	119.5	48.7	49.2	40.9	51.9	36.6
0.6	119.7	153.9	57.9	57.8	75.3	34.0	33.6	35.5	89.1	28.7	63.7	27.5	27.6	23.1	29.0	20.9
0.8	71.6	90.4	33.1	32.7	43.0	19.8	21.1	22.1	48.4	16.9	35.0	17.1	17.1	14.6	17.9	13.2
1.0	44.0	54.6	20.0	19.7	25.6	12.7	14.6	15.3	27.7	10.9	20.4	11.7	11.7	10.2	12.2	9.2
1.2	27.8	34.0	12.8	12.6	16.1	8.8	10.9	11.4	17.0	7.7	12.8	8.6	8.6	7.7	8.9	6.9
1.4	18.2	22.0	8.7	8.6	10.6	6.6	8.6	9.0	11.3	5.8	8.6	6.6	6.6	6.1	6.8	5.4
1.6	12.4	14.7	6.2	6.2	7.4	5.2	7.0	7.4	8.0	4.5	6.2	5.3	5.3	5.0	5.4	4.4
1.8	8.7	10.1	4.7	4.6	5.4	4.3	5.8	6.2	6.0	3.7	4.7	4.3	4.3	4.2	4.4	3.7
2.0	6.3	7.2	3.6	3.6	4.1	3.7	4.9	5.2	4.7	3.1	3.7	3.5	3.5	3.6	3.6	3.1
2.2	4.7	5.4	3.0	3.0	3.2	3.2	4.1	4.4	3.8	2.7	3.0	2.9	2.9	3.2	3.0	2.7
2.4	3.6	4.1	2.5	2.5	2.6	2.8	3.4	3.7	3.1	2.3	2.5	2.5	2.5	2.8	2.5	2.3
2.6	2.9	3.2	2.1	2.2	2.2	2.4	2.8	3.0	2.6	2.1	2.2	2.1	2.2	2.4	2.2	2.1
2.8	2.4	2.6	1.9	1.9	1.9	2.1	2.3	2.5	2.3	1.8	1.9	1.9	1.9	2.1	1.9	1.8
3.0	2.0	2.1	1.7	1.7	1.7	1.9	2.0	2.1	1.9	1.7	1.7	1.7	1.7	1.9	1.7	1.7

, Woodall, W.H., "Exact Results for Shewhart Control Charts with Supplementary Runs Rules", *Technometrics*, vol. 29 (1987) charts in Table 1 are computed for $z = 2$; $z = 1.5$ and $z = 1$ (no satisfactory IRs have been found for $z = 2.5$). Of course, suitable IRs for intermediate values of z can be easily determined. We can observe that, for $z = 2$, the IRs which satisfy the conditions (8) are IR(2, 3), IR(2, 4) and IR(2, 5); for $z = 1.5$, they are IR(3, 4) and IR(3, 5); for $z = 1$, they are IR(3, 3) and IR(4, 5). The value of z for the standard $\bar{X}$-chart, whose ARL and CTRL values are also shown in Table 1, is 3, as recalled above.

IRs providing higher ARL values, even though always compatible with the conditions (8), can be chosen when out-of-control signals for smaller shifts of the process mean want to be delayed in order to avoid too frequent, unnecessary interruptions of the process. On the other hand, IRs providing smaller ARL values can be chosen when out-of-control signals for small/medium shifts (say $1 \leq \delta \leq 2$) want to be hastened.

A second approach (method 2) to the search of convenient IRs is that of determining, for each IR(r, h, z) with $2 < h < 5$, the value of z – for which an approximation of two decimals may be sufficient – that satisfy the conditions

$$L_{r,h}(0 \mid z^*) \geq L_{1,1}(0) = 370.4 \qquad L_{r,h}(0 \mid z^* - 0.01) < L_{1,1}(0) = 370.4 \qquad (9)$$

The IRs, which can be read in Table 2, guarantee an in-control ARL not smaller than that of the standard Shewhart charts and, at the same time, provide minimum ARL values, obtained by computing ARL values from different values of z and for all possible combination of r and h within the range established above, all obviously smaller than the corresponding values obtained for standard charts when $\delta > 0$, and compatible with conditions (9). Among these IRs it is possible to select the one which provides minimum ARL (and corresponding CTRL with $\varepsilon = 0.05$) values for single δ shifts, or intervals, which represent significant out-of-control situations which have to be detected swiftly.

From Table 2 it can be seen that, for $0 < \delta < 1$, the IR-chart which provides minimum ARL values is the one constructed with $h = 5$, $r = 4$ and $z' = 0.79$, therefore IR(4, 5, 0.79); similarly, for $\delta = 1.2$ and $\delta = 1.4$ we obtain IR(3, 5, 1.29); for $\delta = 1.6$ and $\delta = 1.8$ we obtain IR(3, 4, 1.18); finally, for $\delta = 2$ we find IR(2, 3, 1.78). Values greater than 2 are not taken into account, for the reasons previously mentioned. Besides, IRs with $r = 1$ are not considered, since they provide ARL values very similar to those of the standard Shewhart $\bar{X}$-chart chart with 3-sigma limits, so that there is no advantage in using them.

4. A COMPARISON BETWEEN IR-CHARTS AND SHEWHART $\bar{X}$-CHARTS WITH SUPPLEMENTARY RUNS RULES

As recalled in the introductory section, several supplementary 'dependent' runs rules have been provided over time in order to obtain a faster detection of shifts in the mean of a normal process characteristic while using Shewhart $\bar{X}$-charts.

Champ and Woodall (1987) have summarised these rules, stating them in the following form: "an out-of-control signal is given if r of the last h standardised sample means fall in the interval (L, U), where $r \leq$ h and L < U" (notice that the original notations used by the authors have been substituted with the same notations used in the present article).

Let us notice that these runs of length h are 'moving' runs, therefore not independent like the ones introduced in this article.

Champ and Woodall (1987) denote by $T(r, h$, L, U) the runs rule defined above. The standard $\bar{X}$-chart is therefore denoted by {T(1, 1, – ∞, – 3), T(l, 1, 3, m)}. The rules taken into account by the two authors in the mentioned article are:

Rule 1: C, = {T(1, 1, – ∞, – 3), T(1, 1, 3, ∞)}

Rule 2: C, = {T(2, 3, – 3, – 2), T(2, 3, 2, 3)}

Rule 3: C_3 = {T(4, 5, – 3, – 1), T(4, 5, 1, 3)}

Rule 4: C_4 = {T(8, 8, – 3, 0), T(8, 8, 0, 3)}

Rule 5: C_5 = {T(2, 2, – 3, – 2), T(2, 2, 2, 3)}

Rule 6: C_6 = {T(5, 5, – 3, – 1), T(5, 5, 1, 3)}

Rule 7: C_7 = {T(1, 1, – ∞, – 3.09), T(1, 1, 3.09, ∞)}

Rule 8: C_8 = {T(2, 3, – 3.09, – 1.96), T(2, 3, 1.96, 3.09)}

Rule 9: C, = {T(8, 8, – 3.09, 0), T(8, 8, 0, 3.09)}

The first four rules are the well known Western Electric Company (1956) rules. Champ and Woodall (1987) point out that "these nine rules can be combined to form most of the control charts suggested in the literature". The two authors use the notation

$$C_{ij,\ldots,k} = C_i \cup C_j \cup \ldots \cup C_k$$

to denote a generic combination (ij, ..., k) of the nine rules. The results obtained by the authors are the 15 combinations reported in Table 3 for comparative purposes, among which one of the rules 1 or 7 is always present in order to include in each combination the standard 3-sigrna or 3.09-sigma $\bar{X}$-chart, the last one more common in British practice.

As can be seen, the combinations that have the lowest ARL values for $\delta > 0$ (C_{13}, C,,, C_{79}, C_{123}, C_{124}, C_{789}, C_{134}, C_{1456}, C_{1234}) have too low values for the ARL,,, which is a very poor property for a control chart. As Montgomery (1996, p. 150) observes, the authors "found that the use of these rules does improve the ability of the control chart to detect smaller shifts, but the in-control ARL can be substantially degraded... Thus, the sensitising rules need to be used with considerable caution, as an excessive number of false alarms can be harmful to an effective SPC program".

It can also be seen that the combinations providing higher ARL values for 6 > 0 (C,,, C,,, C_{15}, C,,, C_{156}) often perform poorly if compared with the IRs introduced in the previous section; besides, their in-control AKL values are still considerably too small. For instance, the only combination that provides an in-control ARL similar that of a standard chart is C_{16}, for which the in-control AKL is 349.4 (still lower than $L_{1\ 1}(0) = 370.4$). If we compare the performance of this combination with that of IR(4, 5, 0.79) we see that the latter has $L_{4,5}(0) = 383.1$ while, for $0 < \delta < 1.6$, L, $_5(\delta)$ values are lower than the corresponding values for C_{16}. Only for 6 > 1.6 the situation is reversed, but AKL values, in this case, are very low anyway, and the difference between the two procedures is small.

Other comparisons can be made between combinations $C_{ij,\ k}$ and IRs(r, h, z) having similar ARL values for given values of 6 corresponding to standardised out-of-control shifts of the process mean one expects to detect quickly, but without having to interrupt the process too frequently in case of smaller shifts, not relevant for the quality of the product. For instance, IR(4, 5, 0.79) and C_{78} have similar ARL for $\delta = 1$ (respectively, 17.41 and 19.70) but, as can be seen, C_{78} has smaller ARL values for δ up to 0.4, while IR(4, 5, 0.79) has lower ARL values for 6 values from 0 6 until past 1. That means that the expected time of detection of shifts much smaller than the value corresponding to a fixed out-of-control situation (6=1, in the example) will be higher if we use IR(4, 5, 0.79), while shifts near $\delta = 1$ are expected to be detected more quickly. The use of IR(4, 5, 0.79), in this case, provides an ARL curve steeper than the one provided by C_{78}, and this seems to be a very good property for process control.

Finally, a'e recall that, as stated above, by using IRs selected according to method 1. that is IR(3, 3, 1), IR(4, 5, 1), IR(3, 4, 1.5), IR(3, 5, 1 5), IR(2, 3, 2), IR(2, 4, 2) and IR(2, 5, 2), we obtain lower values for low-medium range standardised shifts of the mean (1 < 6 < 2) with respect to a standard Shewhart $\bar{X}$-chart, even though they arc mostly larger than the ones obtained with the combinations of rules, while in-control ARL_0 values and ARL values for small shifts are larger. Therefore, the use of the latter IR-charts is recommended especially when one wants to avoid frequent out-of-control signals for insignificantly small shifts of the mean, since they would increase the costs of control and the costs due to unnecessary interruptions of the process.

5. CONCLUSIONS

'She consideration that Shewhart control charts are much more frequently used in the practice of process control with respect to other types of charts, due to their simplicity of construction, has suggested a further exploration of the possibility of using runs rules and avoiding, at the satne time, the defects so clearly pointed out by Montgomery (1996). The result has been the introduction of independent runs IRs, which can also be seen as a generalisation of standard Shewhart charts.

A more detailed study of the relevant case of the $\bar{X}$-charts has shown that selected IR-charts enjoy good properties, namely, allow a fast detection of small and medium shifts in the mean of the process characteristic under control, while providing longer expected times to obtain false alarms. 'She results are in many cases satisfactory when compared with those of the most common combinations of supplementary runs rules introduced in the literature and practice of process control.

We point out once more that IR-charts can be easily associated with standard control charts, and do not require the construction of different, more complicated charts, like CUSUM and EWMA charts, whose parameterisation often causes remarkable difficulties to practitioners, even though it has to be acknowledged that cumulative procedures are generally more efficient than Shewhart-type ones. But the scope of the present paper is that of using the very same points drawn in a Shewhart chart in a more efficient way, overcoming some of the problems raised by the use of the supplementary runs rules.

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RIASSUNTO

Uso di sequenze indipendenti per il miglioramento della risposta delle carte di controllo Shewhart

L'articolo si occupa di una estensione delle tradizionali carte di controllo di tipo Shewhart, che consiste nell'introduzione di sequenze indipendenti di punti-stima di assegnata lunghezza, allo scopo di individuare con maggiore rapidità sregolamenti di piccola e media entità nella media di una caratteristica di processo sotto osservazione.

L'autore sceglie di proposito di restare nel campo delle osservazioni campionarie indipendenti, perché sono quelle più frequentemente impiegate nella pratica del controllo statistico di processo (SPC), rispetto ad altri tipi di carte, a motivo della loro semplicità, e mostra come un'oculata scelta di sequenze indipendenti delle stesse medie campionarie usate in una carta-$\overline{X}$ può portare ad una riduzione del tempo medio di attesa di un segnale di 'fuori controllo', o ARL. La procedura proposta non presenta l'inconveniente principale delle corrispondenti procedure a sequenze dipendenti, cioè quelle note come 'regole della Western Electric Company' ed altre richiamate nell'articolo, che è quello per cui il tempo medio di

attesa di segnali di 'fuori controllo' non necessari risulta troppo basso, conducendo verosimilmente, in tal modo, a troppo freyuenti falsi allarmi.

SUMMARY

Use of independent runs for the improvement of the response of Shewhart control charts

The paper deals with an extension of standard Shewhart control charts which provide independent runs of given length, or IR-charts, with the purpose of improving the performance of the former by allowing a faster detection of small and medium shifts in a process mean.

The author purposely chooses to remain in the field of independent sample observations, which are much more frequently used in SPC compared to other types of charts due to their simplicity, and shows how a careful choice of independent runs, formed by the same sample means used in a standard $\bar{X}$-chart, can be adopted in order to reduce ARL values and, therefore, the expected time of detection of out-of-control situations. The result is obtained without the main drawback of dependent runs procedures as summarised in the Western Electric Company runs rules and others recalled in the paper, which lies in the fact that their 'in-control' ARL is too low, thus providing a too short expected time before a false 'out-of control' alarm occurs.