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A NOTE ON GOODNESS OF FIT TEST USING MOMENTS

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1. INTRODUCTION

An important problem in statistical applications is to test whether or not an assumed model gives good fit to the data. Among the most commonly used procedures for testing goodness of fit of a parametric family is the Pearson-Fisher chi-square test (see Fisher, 1922, 1924). The Pearson-Fisher chi-square test involves partitioning the real line into a number of cells and then comparing the observed cell frequencies with the expected cell frequencies using a chi-square test statistic. The chi-square test is easy to use and applies to almost any parametric family. On the other hand, it also has the drawback that the selection of its partition is arbitrary. Furthermore, for a given partition, the rejection power of the chi-squared test can change dramatically between different alternatives.

Another general approach to goodness of fit test is to use some distance statistics such as the Kolmogorov-Smirnov statistic and Anderson-Darling statistic among others. Compared with chi-square tests, distance tests have the advantage that they do not involve subjective selection of a partition. On the other hand, most distance tests are appropriate only for testing the simple hypothesis whether a set of observations are from some completely specified distribution function. When certain parameters of the distribution must be estimated from the sample, they no longer apply. Lilliefors (1967, 1969, 1973) studied the use of distance statistics to test goodness of fit of the normal, exponential family, and gamma families. In recent years, considerable efforts have been made to extend distance tests to test the composite hypothesis of a general parametric family, using transformations of some generalized empirical processes; see, e.g., Khmaladze (1993) and Sun (1997) among others. However, the resulting tests have not been widely used in practice partly because they are often very complicated and not easy to use.

Many other goodness of fit tests have been proposed in the past for some specific parametric families; see D'Agostino and Stephens (1986) for a comprehensive review of goodness of fit techniques. Among them are some tests of departure from normality using the sample skewness ($\sqrt{b_1}$) and the sample kurtosis (b_2). For a normal distribution, the population skewness and kurtosis are 0 and 3 respectively. Moreover, the exact distributions of $\sqrt{b_1}$ and 17, are known in normal sampling (see Pearson and Hartley, 1970, table 34), which can be used for testing departure from normality. We point out that the ($_1 b_1$, b_2) method is moment-based and applies only to normality testing. Very few results are known for the distributions of $\sqrt{b_1}$ and *h*, in nonnormal sampling.

The main purpose of this article is to introduce a simple moment approach to testing goodness of fit based on some moment structure of a parametric distribution family. This approach has some appealing features For example, the idea is very simple since it only involves comparison of sampling moments with population moments. It is applicable to a variety of discrete or continuous parametric families, while the ($\sqrt{b_1}$, b_2) method is only known for testing normality. The proposed tests only use the standard normal (or chi-square) table and thus are easy use in practice. Compared with chi-square tests, the moment based method does not involve subjective selection of a partition. *It* also demonstrated superior and more stable rejection power than chi-square tests for different alternatives in our limited simulation studies.

In section 2 we describe the general moment-based method. We also illustrate this approach by deriving moment based goodness of tests of some cotntnon parametric families. In section 3 we report results from a simulation study to compare the performances of the moment based tests with some chi-square and distance tests.

2. MOMENT-BASED GOODNESS-OF-FIT TESTS

2.1. *The setup and assumption*

Assume that X,, ..., X, are independent identically distributed random variables from a cumulative distribution function F. Consider the problem of testing the null hypothesis

U,, : F is a member of a parametric family F_θ, 8 E Θ ,

where Θ is a subset of $\mathbb{R}^d$.

Let *in*,= $\int x^i \, dF_\theta(x)$ denote the i-th moment of F_θ. The following basic assumption will be used throughout this paper.

Assumption A. Assume that m,, exists for some positive integer r and that *m*,, ... m_r satisfy the following equation

$$g(m_1, \ldots, m_r) = 0 \qquad \text{for all } \theta \in \Theta$$

for some function g : $\mathbb{R}^r \rightarrow \mathbb{R}$.

In many situations, it is easy to find a function g satisfying Assumption A. For instance, if a parametric distribution is symmetric about its mean, then E(X - *m*,)'= 0. 'This implies that *m*, - $3m_1m_2 + 2m_1^3 \equiv 0$. Thus, one can choose $g(x, y, z) = z - 3xy + 2x^3$.

In general, existence of g satisfying Assumption A is feasible given the fact that m,, m_2, ... all depend on a common finite dimensional parameter $\boldsymbol{\theta}$.

2.2. *The* **test**

Let $\hat{m}_i = \sum_{j=1}^{n} X_j^i / n$ be the sample moment of i-th order, i = I, 2, ...

Theorem 1. Assume that Assumption A holds. Assume further that g is continuously differentiable. Then, under H_0,

$$\sqrt{n}\, g(\hat{m}_1,\ldots,\hat{m}_r) \xrightarrow{d} N(0, V(\theta)),$$

where

$$V(\theta) = \left(\frac{\partial g(m_1,\ldots,m_r)}{\partial m_1}, \ldots, \frac{\partial g(m_1,. \quad m_r)}{\partial m_r} \right) \Sigma \left(\frac{\partial g(m_1,\ldots,m_r)}{\partial m_1}, \ldots, \frac{\partial g(m_1,\ldots,m_r)}{\partial m_r} \right)^T \quad (2.1)$$

and $\Sigma = (\sigma_{ij})_{r\times r}$ with $\sigma_{ij} = m_{i+j} - m_i m_j$.

Proof. By the central limit theorem, in conjunction with the Cramér-Wold device, the random vector

$$\sqrt{n}\, \{(\hat{m}_1,\ldots,\hat{m}_r) - (m_1,\ldots,m_r)\}$$

converges in distribution to r-variate normal with mean vector (0, .., 0) and covariance matrix Σ. This, together with the delta method, implies that

$$\sqrt{n}\, g(\hat{m}_1,\ldots,\hat{m}_r) = \sqrt{n}\, \{g(\hat{m}_1,\ldots,\hat{m}_r) - g(m_1,\ldots,m_r)\} \xrightarrow{d} N(0, V(\theta)),$$

where $V(\theta)$ is defined by 12 1)

Let $\hat{\theta} = \theta(X_1, \quad, X_n)$ be a consistent estimate of H under H_0. Assume that m,, .. , m,, are continuous functions of θ. Then. $V(\hat{\theta})$ is a consistent estimate of $V(\theta)$ under H_0 Define

$$T = \sqrt{n}\, g(\hat{m}_1,\ldots, \boldsymbol{h,})/ , V(\theta) \quad (2.2)$$

Then, it follows from Theorem I that under H_0,

$$T \xrightarrow{a} N(0,1)$$

as $n \to \infty$. This leads to the following level α test of H,,: Reject H,, ii $|T| > z_{\alpha/2}$, where $z_{\alpha/2}$ is the upper $\alpha/2$ percentile of the standard normal distribution.

Remark I. A consistent estimate θ of θ under H,, can usually be obtained using either the method of moments or the method of maximum likelihood (see Bickel and Doksum, 1979, chapter 3). It would be difticult to say in general whether one method is preferred to the other. Only one method is used in each of the examples in section 2.3 for the purpose of illustration.

Remark 2. There may exist several functions, say g,, ..., g_k, satisfying Assumption A and, sometimes, one may wish to combine these functions to test goodness of fit. One may conduct several tests based on g_1, ..., g_k, separately. We refer the readers to Mardia et al. (1979, pp. 127-131) for the Roy union-intersection principle and Pallini (1994) for combinations of dependent tests and their optimality. One may also derive a chi-square test using the multidimensional function $g = (g_1, \ldots, g_k)^T$. With g being k-dimensional, Theorem 1 still holds with the limiting distribution being a k-variate normal distribution with mean 0 and covariance matrix $V(\theta)$ given by (2.1). Define $Q = ng^T(\hat{m}_1, \ldots, \hat{m}_r)\, V^{-1}(\hat{\theta})\, g(\hat{m}_1, \ldots, \hat{m}_r)$, where $\hat{8}$ is a consistent estimate of θ under H_0. Then, under H_0 and some regularity conditions, Q has a chi-square limiting distribution with k degrees of freedom. Therefore Q can be used to test H,,in the usual manner.

We note that the proposed test may not have good power to distinguish between certain prametric families such as normal and logistic distributions. A related question is whether or not there exists a test function $g(\cdot)$ which maximizes the rejection power against a given parametric alternative. We have not been able to find a satisfactory solution to this problem so far, and future research is warranted.

2.3. Application to some common distribution families

To illustrate the proposed approach, we derive some moment-based goodness of fit tests for some common parametric distribution families. Although one may obtain more than one g satisfying Assumption A in the examples considered below, for simplicity, we only give a test based on a single function g in each example. Note that to obtain the test statistic T defined by (2.2), we need to find a function g satisfying Assumption A. In addition, we need to express $m_1, \ldots, m_{2r}$ in terms of 8 in order to find $V(\theta)$ defined in (2.1).

2.3.1. Testfor exponential distribution

To test $H_0 : F(x) = 1 - \exp(-x/\theta)$, $x > 0$, we take $r = 2$ and $g(x, y) = y - 2x^2$. Because m, = 8 and m, = $2\theta^2$, we have $g(m_1$, m,) = 0 for all $8 > 0$. After some algebraic calculations, it is shown that $V(\theta) = 4\theta^4$. Estimate 8 by $\hat{\theta} = \hat{m}_1$. Then, the test statistic T defined by (2.2) reduces to

$$T = \frac{\sqrt{n}\,(\hat{m}_2 - 2\hat{m}_1^2)}{2\hat{m}_1^2}$$

This can be further written as

$$T = \frac{\sqrt{n}}{2}\left(\frac{s^2}{\overline{X}^2} - 1\right),$$

where $X = \sum_{i=1}^{n} X_i/n$ and $s^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$ are the sample mean and sample variance respectively.

2.3.2. *Test for normality*

To test the null hypothesis that the sample $X_1, \ldots, X_n$ are from a normal distribution $N(\mu, \sigma^2)$, where μ and σ^2 are unknown, we take $r = 3$ and $g(x, y, z) = z - 3xy + 2x^3$. It can be verified that $g(m_1, m_2, m_3) = 0$ for all $\theta = (\mu, \sigma^2)$. Estimate $\theta = (\mu, \sigma^2)$ by $\hat{\theta} = (\bar{X}, s^2)$ where $\bar{X} = \sum_{i=1}^{n} X_i / n$ and $s^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / n$ are the sample mean and sample variance, respectively. Then the test statistic T is given by

$$T = \frac{\sqrt{n}(\hat{m}_3 - 3\hat{m}_1\hat{m}_2 + 2\hat{m}_1^3)}{\sqrt{V(\hat{\theta})}},$$

where $V(\theta)$ is defined by *(2.1)* with

$$\left(\frac{\partial g(m_1, m_2, m_3)}{\partial m_1}, \frac{\partial g(m_1, m_2, m_3)}{\partial m_2}, \frac{\partial g(m_1, m_2, m_3)}{\partial m_3} \right) = (-3m_2 + 6m_1^2, -3m_1, 1),$$

and

$$m_1 = \mu,$$
$$m_2 = \mu^2 + \sigma^2,$$
$$m_3 = \mu^3 + 3\mu\sigma^2,$$
$$m_4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4,$$
$$m_5 = \mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4,$$
$$m_6 = \mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6.$$

2.3.3. *Test for gamma distribution*

To test the null hypothesis that the sample $X_1, \ldots, X_n$ are from a gamma distribution with the following probability density

$$f_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x), \qquad x > 0,$$

where $\theta = (\alpha, \beta)$, $\alpha > 0$ and $\beta > 0$, we take $r = 3$ and $g(x, y, z) = xz + x^2y - 2y^2$. It can be shown that $g(m_1, m_2, m_3) = 0$ for all $\theta = (\alpha, \beta)$, $\alpha > 0$ and $\beta > 0$. Estimate $\theta = (\alpha, \beta)$ by $\hat{\theta} = (\bar{X}^2/s^2, \bar{X}/s^2)$. Then, our test statistic T is given by

$$T = \frac{\sqrt{n}(\hat{m}_1\hat{m}_3 + \hat{m}_1^2\hat{m}_2 - 2\hat{m}_2^2)}{\sqrt{V(\hat{\theta})}},$$

where $V(\theta)$ is defined by *(2.1)* with

$$\left(\frac{\partial g(m_1, m_2, m_3)}{\partial m_1}, \frac{\partial g(m_1, m_2, m_3)}{\partial m_2}, \frac{\partial g(m_1, m_2, m_3)}{\partial m_3} \right) = (m_3 + 2m_1m_2, m_1^2 - 4m_2, m_1),$$

and

$$m_j = \beta^{-j}\alpha(\alpha+1)\ \dots\ (\alpha+j-1)\ , \qquad \text{for } j = 1, \dots, 6\ .$$

2.3.4. *Test for beta distribution*

To test the null hypothesis that the sample $X_1, \dots,$ X, are from a beta distribution with the following probability density

$$f_\theta(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\ , \qquad 0 \le x \le 1\ ,$$

where $8 =$ (a, b), $a > 0$ and $b > 0$ are unknown, we take $r = 3$ and $g(x$, y, $z) = z(2x^2 - x - y) - y(x^2 + xy - 2y)$. It can be verified that $g(m_1$, m,, m,) ≡ 0. Therefore, the test statistic for H_0 is

$$T = \sqrt{n}(m_3(2m_1^2 - m_1 - m_2) - m_2(m_1^2 + m_1m_2 - 2m_2)) / \sqrt{V(\hat\theta)}\ ,$$

where $\hat\theta = (2, b) = (\hat m_1(\hat m_2 - \hat m_1)/(\hat m_1^2 - \hat m_2), \hat a(1 - \hat m_1/\hat m_1)$ and $V(\theta)$ is defined by *(2.1)* with

$$\left(\frac{\partial g(m_1,m_2,m_3)}{\partial m_1}, \frac{\partial g(m_1,m_2,m_3)}{\partial m_2}, \frac{\partial g(m_1,m_2,m_3)}{\partial m_3}\right) =$$

$$= (4m_1m_3 - m, - 2m_1m_2 - m_2^2, 4m_2 - m, - m_1^2 - 2m_1m_2, 2m_1^2 - m, - m,),$$

and

$$m_j = \prod_{i=0}^{j-1} \frac{a+i}{a+b+i}\ , \qquad j = 1, \dots, 6.$$

2.3.5. *Test for uniform distribution*

To test the null hypothesis that the sample X,, ..., X, are from an uniform distribution with the following probability density

$$f_\theta(x) = \frac{1}{b-a}\ , \qquad a < x < b\ ,$$

where $\theta =$ (a, b), $-\infty < a < b < \infty$ are unknown, we take $r = 3$ and g (x, y, z)= $z - 3xy + 2x^3$. *It* can be verified that $g(m_1$, m,, m,) ≡ 0. Therefore, the test statistic for H_0 is

$$T = \frac{\sqrt{n}(\hat m_3 - 3\hat m_1\hat m_2 + 2\hat m_1^3)}{\sqrt{V(\hat\theta)}}\ ,$$

where $\theta = (2, b) =$ (min$(X,)$, max {X,}) and $V(8)$is defined by *(2.1)*with

$$\left(\frac{\partial g(m_1,m_2,m_3)}{\partial m_1}, \frac{\partial g(m_1,m_2,m_3)}{\partial m_2}, \frac{\partial g(m_1,m_2,m_3)}{\partial m_3}\right) = (-3m_2 + 6m_1^2, -3m_1, 1),$$

and

$$m_i = \frac{b^{i+1} - a^{i+1}}{(b-a)(i+1)}, \qquad i = 1,\ldots,6.$$

2.3.6. Test for extreme-value distribution

To test the null hypothesis that the sample $X,, \ldots,$ X, are from an extreme-value distribution with the following probability density

$$f_\theta(x) = \exp\left\{-\exp\left\{-\frac{x-\alpha}{\beta}\right\}\right\} \qquad -\infty < x < \infty,$$

where $\theta = (a,\ p)$, $-\infty < a < \infty$ and $\beta > 0$ are unknown, we take $r = 3$ and $g(x, y, z) = (z - 3xy + 2x^3)^2 - \gamma_1^2(y - x^2)^3$, where $\gamma_1 = E(X_1 - m_1)^3/[E(X_1 - m_1)^2]^{3/2} \equiv 1.29857$ is the coefficient of skewness. It can be verified that $g(m_1,$ m,, m,) $\equiv 0$. Therefore, the test statistic for H_0 is

$$T = \frac{\sqrt{n}((\hat{m}_3 - 3\hat{m}_1\hat{m}_2 + 2\hat{m}_1^3)^2 - \gamma_1^2(\hat{m}_2 - \hat{m}_1^2)^3)}{\sqrt{V(\hat{\theta})}},$$

where 0 = (&,B)= $(m, - \gamma_1\hat{\beta}, \sqrt{6(\hat{m}_2} - A:)$[K) and V($) is defined by *(2.1)*with

$$\frac{\partial g(m_1, m_2, m_3)}{\partial m_1} = 6(2m_1^2 - m_2)(m_3 - 3m_1m_2 + 2m_1^3) + 6\gamma_1^2 m_1(m_2 - m_1^2)^2,$$

$$\frac{\partial g(m_1, m_2, m_3)}{\partial m_2} = 6m_1(m_3 - 3m_1m_2 + 2m_1^3) + 3\gamma_1^2(m_2 - m_1^2)^2,$$

$$\frac{\partial g(m_1, m_2, m_3)}{\partial m_3} = 2(m_3 - 3m_1m_2 + 2m_1^3),$$

and

$$m_1 = \alpha + \gamma_1\beta,$$

$$m_2 = (\alpha + \gamma_1\beta)^2 + \frac{\pi^2\beta^2}{6},$$

$$m_3 = m_1\kappa_3 + m_2\kappa_2 + 2m_1m_2,$$

$$m_4 = m_1\kappa_4 + 2m_2\kappa_3 + m_3\kappa_2 + 2m_1^2 + 2m_1m_3,$$

$$m_5 = m_1\kappa_5 + 2m_2\kappa_4 + 3m_3\kappa_3 + m_4\kappa_2 + 6m_2m_3 + 2m_1m_4,$$

$$m_6 = m_1\kappa_6 + 2m_2\kappa_5 + 5m_3\kappa_4 + 3m_4\kappa_3 + 6m_3^2 + 8m_2m_4 + 2m_1m_5,$$

$$\kappa_1 = \alpha + \gamma_1\beta,$$

$$\kappa_2 = \frac{\pi^2\beta^2}{6},$$

$$\kappa_r = (-\beta)^r\phi^{(r-1)}(1) \quad \text{for } r \geq 2.$$

Here $\phi(\cdot)$ is a digamma function (see Mood *et al.*, 1974, p. 543).

2.3.7. *Test for logistic distribution*

To test the null hypothesis that the sample $X_1, \ldots,$ X, are from a logistic distribution with the following cumulative distribution function

$$F_\theta(x) = [1 + \exp\{-(x - \alpha)/\beta\}]^{-1} \qquad -\infty < x < \infty,$$

where $\theta = (a, \beta)$, $-\infty < a < \infty$, and $\beta > 0$ are unknown, we take $r = 3$ and $g(x, y, z) = z - 3xy + 2x^3$. It can be verified that $g(m_1,$ m,, m,) $\equiv 0$. Therefore, the test statistic for H_0 is

$$T = \frac{\sqrt{n}(\hat{m}_3 - 3\hat{m}_1\hat{m}_2 + 2\hat{m}_1^3)}{\sqrt{V(\hat{\theta})}},$$

where $\hat{\theta} = (\hat{\alpha}, \hat{\beta}) = (m_1, \pi^{-1}\sqrt{3(\hat{m}_2 - \hat{m}_1^2)})$ and $V(\theta)$ is defined by *(2.1)* with

$$\left(\frac{\partial g(m_1, m_2, m_3)}{\partial m_1}, \frac{\partial g(m_1, m_2, m_3)}{\partial m_2}, \frac{\partial g(m_1, m_2, m_3)}{\partial m_3}\right) = (-3m_2 + 6m_1^2, -3m_1, 1),$$

and

$$\begin{aligned}
m_1 &= \alpha, \\
m_2 &= \alpha^2 + \frac{\pi^2\beta^2}{3}, \\
m_3 &= 3m_1m_2 - 2m_1^3, \\
m_4 &= m_1\kappa_4 + 2m_2\kappa_3 + m_3\kappa_2 + 2m_1^2 + 2m_1m_3, \\
m, &= m_1\kappa_5 + 2m_2\kappa_4 + 3m_3\kappa_3 + m_4\kappa_2 + 6m_2m_3 + 2m_1m_4, \\
m_6 &= m_1\kappa_6 + 2m_2\kappa_5 + 5m_3\kappa_4 + 3m_4\kappa_3 + 6m_3^2 + 8m_2m_4 + 2m_1m_5, \\
\kappa_{2r-1} &= 0, \quad r = 1, 2, \ldots; \quad \kappa_{2r} = 6(2^{2r} - 1)\beta^{2r}B_{2r}, \quad r = 1, 2, \ldots.
\end{aligned}$$

Here B,, is a Bernoulli number (sec Kendall and Stuart, *1958*, p. *80)*.

2.3.8. *Test for Laplace distribution*

To test the null hypothesis that the sample X,, ..., X, are from a Laplace distribution with the following probability density function

$$f_\theta(x) = \frac{1}{2\beta}\exp\left\{-\frac{|x - \alpha|}{\beta}\right\}, \qquad -\infty < x < \infty,$$

where $\theta = (a, P)$, $-\infty < \alpha < \infty$, and $\beta > 0$ are unknown, we take $r = 3$ and $g(x, y, z) = z - 3xy + 2x^3$. It can be verified that $g(m_1,$ m,, $m_3) \equiv 0$. Therefore, the test statistic for H,, is

$$T = \frac{\sqrt{n}(\hat{m}_3 - 3\hat{m}_1\hat{m}_2 + 2\hat{m}_1^3)}{\sqrt{V(\hat{\theta})}},$$

where $\hat{\theta} = (\hat{\alpha}, \hat{\beta}) = (h, \sqrt{(\hat{m}_2 - \hat{m}_1^2)/2})$ and $V(\theta)$ is defined by *(2.1)*with

$$\left(\frac{\partial g(m_1, m_2, m_3)}{\partial m_1}, \frac{\partial g(m_1, m_2, m_3)}{\partial m_2}, \frac{\partial g(m_1, m_2, m_3)}{\partial m_3} \right) = (-3m_2 + 6m_1^2, -3m_1, 1),$$

and

$$m_1 = \alpha,$$
$$m_2 = \alpha^2 + 2\beta^2,$$
$$m_3 = \alpha^3 + 6\alpha\beta^2,$$
$$m_4 = \alpha^4 + 12\alpha^2\beta^2 + 12\beta^4,$$
$$m_5 = \alpha^5 + 20\alpha^3\beta^2 + 60\alpha\beta^4,$$
$$m_6 = \alpha^6 + 30\alpha^4\beta^2 + 180\alpha^2\beta^4 + 120\beta^6$$

2.3.9. *Tests for Poisson distribution*

To test the null hypothesis that the sample X,, ..., X, are from a Poisson distribution with the following probability frequency function

$$f_\theta(x) = e^{-\theta}\theta^x/x!\,, \qquad x = 0, 1, 2, \dots,$$

where $\theta > 0$ is unknown, we take $r = 2$ and $g(x, y) = y - x - x^2$. For Poisson distribution, $m, = 8$ and $m_2 = 8 + \theta^2$. Hence, $g(m_1, m_2) \equiv 0$. After some algebraic calculations, it is seen that $V(\theta) = 20^2$. Let $\theta = \hat{m}_1$. Therefore, the test statistic for H_0 is given by

$$T = \sqrt{n}\, g(\hat{m}_1, \dots, \hat{m}_r)/\sqrt{V(\hat{\theta})} = \sqrt{n}\,(\hat{m}_2 - \hat{m}_1 - \hat{m}_1^2)/\sqrt{2}\,\hat{m}_1\,,$$

which can be further written as

$$T = \sqrt{\frac{n}{2}} \left(\frac{s^2}{\bar{X}} - 1 \right),$$

where X and s^2 denote the sample mean and the sample variance.

It is worth noting that the above statistic T can be written as $T = (\chi^2 - n)/\sqrt{2n}$ where $\chi^2 = ns^2/\bar{X}$ is the well known Fisher index of dispersion statistic (see Fisher, 1970, p. 58).

2.3.10. *Test for binomial distribution*

To test the null hypothesis that the sample X,, ..., X, are from a binomial (m, θ) distribution where $0 < \theta < 1$ is unknown, we take $r = 2$ and $g(x, y) = y - x - ((m - 1)/m)x^2$. It is easy to verify that $g(m_1$, m,) $\equiv 0$ for binomial distribution. Therefore, the test statistic for H,, is

$$T = \frac{\sqrt{n}\left(\hat{m}_2 - \hat{m}_1 - \frac{m-1}{m}\hat{m}_1^2\right)}{\sqrt{V(\hat{\theta})}},$$

where $\hat{\theta} = \bar{X}/m$ and $V(\theta)$ is defined by *(2.1)*with

$$\left(\frac{\partial g(m_1, m_2)}{\partial m_1}, \frac{\partial g(m_1, m_2)}{\partial m_2}\right) = \left(-1 - \frac{2(m-1)}{m} m_1, 1\right),$$

and

$$m_1 = m\theta,$$

$$m_2 = m\theta(m\theta - \theta + 1),$$

$$m_3 = m\theta(1 - 2\theta) + (m\theta)^2(1 - 3\theta),$$

$$m_4 = m\theta(1 - \theta) - 6m\theta^2(1 - \theta)^2 + (m\theta)^2(7 - 12\theta + 5\theta^2) - 2(m\theta)^3(3\theta + 1) - 3(m\theta)^4 .$$

2.3.11. *Test for negative binomial distribution*

To test the null hypothesis that the sample X_1, , X,, are from a negative binomial (m, θ) distribution where $0 < \theta < 1$ is unknown, we take $\imath = 2$ and $g(x, y) = y \quad x - ((m + 1)/m)x^2$ *It* is easv to verify that $g(m_1, m_2) \equiv 0$ for negative binomial distribution Theretore, the test statistic for H_0 is

$$T = \frac{\sqrt{n}\left(\hat{m}_2 - \hat{m}_1 - \frac{m+1}{m}\hat{m}_1^2\right)}{\sqrt{V(\hat{\theta})}},$$

where $\hat{\theta} = m/(m + X)$ and $V(\theta)$ is defined bv (2.1) with

$$\left(\frac{\partial g(m_1, m_2)}{\partial m_1}, \frac{\partial g(m_1, m_2)}{\partial m_2}\right) = \left(-1 - \frac{2(m+1)}{m} m_1, 1\right),$$

and

$$m_1 = n(1 - \theta)/\theta,$$

$$m_2 = \frac{m(1 - \theta) + m^2(1 - \theta)^2}{\theta^2}$$

$$m_3 = \frac{m(2 - \theta)(1 - \theta)}{\theta^3} + 3m_1m_2 - 2m_1^2,$$

$$m_4 = \frac{-\theta)}{\theta^4}[1 + 4(1 - H) + (1 - \theta)^2 + 3m(1 - \theta)] + 4m_1m_3 - 6m_1^2 - 2m_1^2m_2 + 3m_1^4 .$$

3. A SIMULATION STUDY

We conducted a Monte Carlo study to compare the performance of the proposed test for exponential family with some existing goodness-of-fit tests. Specifically, we consider the moment based test (T5) derived in Section 2.3.1, the Lilliefors test (T4), and three chi-squared tests (T1, T2, and T3) with partitions (6,13,20), (4, 8, 12, 16, 20), and (3, 6, 9, 15, 18, 20), respectively. Each entry in the tables is based on 1,000 Monte Carlo samples and represents the simulated rejection probability multiplied by 1,000.

Table 1 reports results of simulations to estimate the size (probability of type I error) of goodness-of-fit tests for exponential family for various combinations of nominal levels (α), sample sizes (n), and underlying exponential (θ) distributions. It is seen that the simulated sizes of Lilliefors's test are verv closc to the nominal levels. 'She simulated sizcs of the moment-based test are reasonabl\ closc to the nominal levels when $\alpha = 0.1$ and 0 05, but may require a larger sample size when $\alpha = 0.01$. The three chi-square tests T1-T3 did not perform as well as T4 and T5.

Table 2 reports simulated rejection power of the five aforementioned goodness-of-fit tests for exponential familv for various combinations of nominal levels (α), sample sizcs (n) and Weibull (a, b) alternatives: $F(x) = 1 - \exp(-x^a/b)$

Table 3 reports simulated rejection power ot the five tests for exponential family lor various combinations of nominal levels (α), sample sizes (n) and lognormal alternatives

TABLE 1

Simulated size (multiplied by 1,000) of goodness-of-fit tests for exponential family

		$\alpha = 0.10$					$\alpha = 0.05$					$\alpha = 0.01$				
		T1	T2	T3	T4	T5	T1	T2	T3	T4	T5	T1	T2	T3	T4	T5
0.5	20	182	198	252	100	117	115	128	188	57	81	25	58	102	9	38
	30	162	161	214	100	127	81	95	134	78	84	12	36	58	14	41
	40	161	170	204	113	119	88	97	122	52	69	10	34	50	8	33
	50	154	149	178	113	113	89	77	106	60	73	15	22	36	10	22
1.0	20	191	212	261	100	116	115	145	182	50		22	58	90	13	38
	30	190	190	254	118	126	105	122	188	85	76	24	35	80	1?	31
	40	158	159	185	93	104	78	86	108	49	66	13	20	36	8	28
	50	139	126	160	93	89	73	66	85	42	49	11	10	28	7	19
1.5	20	177	207	280	103	136	114	139	206	46	88	21	54	97	11	47
	30	160	145	219	97	93	78	87	149	69	68	15	26	65	8	33
	40	155	152	171	85	94	85	87	111	41	57	12	24	46	7	22
	50	164	154	175	95	88	84	90	108	51	55	20	28	34	8	18
2.0	20	182	206	265	103	115	113	139	198	50	77	21	49	92	10	38
	30	152	166	221	108	119	87	100	156	81	80	17	39	72	11	34
	40	154	157	178	103	101	81	89	108	55	58	19	25	32	7	26
	50	136	156	173	96	120	74	91	122	44	72	12	22	46	7	23

Note: The tests T1, Γ2, and T3 are chi-square tests with partitions (6, 13, 20), (4, 8, 12, 16, 20), and (3, 6, 9, 15, 18, 20), respectively. T4 represents the Lilliefors test. T5 represents the moment based test derived in Section 2.3 1. For each selected θ, the data were generated trom tlic exponential distribution $F(x) = 1 - \exp(-x/\theta)$. 1,000 Monte Carlo samples were used to obtain each entry.

TABLE 2

Simulated rejection power (multiplied by 1,000) of goodness-of-fit tests for exponential family under Weibull alternatives

		$\alpha = 0.10$					$\alpha = 0.05$					$\alpha = 0.01$				
		T1	T2	T3	T4	T5	T1	T2	T3	T4	T5	T1	T2	T3	T4	T5
0.5	20	322	387	484	545	770	214	274	400	370	697	81	144	219	147	543
	30	427	485	513	694	883	292	363	391	631	830	109	186	201	282	700
	40	498	559	618	793	943	359	452	499	673	911	152	239	290	385	809
	50	647	680	683	898	971	489	548	556	801	954	189	299	314	547	877
1.0	20	341	399	464	540	811	224	299	366	393	746	87	149	215	158	592
	30	455	544	542	720	911	334	417	430	671	858	139	202	246	287	719
	40	571	596	638	835	946	410	452	510	704	910	150	245	315	426	807
	50	660	696	677	906	978	516	558	552	827	961	223	332	342	561	876
1.5	20	347	431	543	549	812	239	330	433	405	738	86	173	252	182	585
	30	439	522	541	725	898	308	407	438	663	851	113	199	237	289	733
	40	540	585	657	812	949	397	465	542	713	910	154	251	309	430	792
	50	671	695	712	914	966	517	594	582	822	951	202	318	340	600	882
2.0	20	337	426	497	538	793	231	325	409	402	726	104	162	255	156	566
	30	441	487	548	694	892	315	368	427	641	836	122	186	229	269	707
	40	547	597	613	833	943	391	483	514	706	903	152	236	301	429	799
	50	640	670	680	877	973	495	531	563	789	952	205	284	324	530	873
0.5	20	691	747	839	938	991	544	653	745	865	989	275	418	528	607	973
	30	842	904	910	994	1000	729	818	837	989	1000	409	580	602	858	999
	40	906	961	968	998	1000	820	922	938	991	1000	487	755	799	942	999
	50	981	994	993	1000	1000	942	975	974	999	1000	673	884	902	992	1000
1.0	20	680	752	842	941	997	521	640	752	855	993	251	413	526	590	974
	30	821	891	903	985	1000	701	820	828	982	1000	397	590	619	844	997
	40	940	962	970	996	1000	840	921	935	991	1000	510	745	807	951	1000
	50	983	991	986	1000	1000	942	971	971	1000	1000	681	875	888	991	1000
1.5	20	662	757	812	919	993	509	636	725	843	980	264	426	528	596	947
	30	838	898	910	997	1000	731	812	846	988	999	395	597	646	860	994
	40	936	961	973	997	1000	853	919	938	995	1000	511	768	823	948	1000
	50	982	989	987	1000	1000	937	965	966	999	1000	650	853	886	986	1000
2.0	20	682	749	825	933	998	539	630	718	851	994	253	406	515	580	964
	30	840	897	900	987	1000	714	826	833	978	999	403	608	640	844	994
	40	938	968	975	999	1000	849	921	953	994	1000	529	754	807	956	1000
	50	975	988	988	1000	1000	928	971	978	998	1000	674	850	890	985	1000

Note: T1, T2, and T3 are the same chi-square tests as those in Table 1. T4 represents the Lilliefors test. T5 represents the moment based test derived in Section 2.3.1. For each given (a, b), the data were generated from the Weibull distribution $F(x) = 1 - \exp(-x^a/b)$. 1,000 Monte Carlo samples were used to obtain each entry.

Table 4 reports simulated rejection power of the five tests for exponential family for various combinations of nominal levels (α), sample sizes (n) and truncated normal alternatives.

It is observed from tables 2, 3 and 4 that the proposed moment based test (T5) demonstrared better rejection power than the chi-square tests (T1-T3) and the Lilliefors test (T4) in almost every situation considered.

Under lognormal alternatives, all five tests performed well with the proposed

TABLE 3

Simulated rejection power (multiplied by 1,000) of goodness-of-fit tests for exponential family under lognormal alternatives

		$\alpha = 0.10$					$\alpha = 0.05$					$\alpha = 0.01$				
		T1	T2	T3	T4	T5	T1	T2	T3	T4	T5	T1	T2	T3	T4	T5
0.4	20	975	995	996	1000	999	926	977	992	1000	996	712	885	935	996	991
	30	998	1000	1000	L000	1000	989	1000	999	1000	1000	877	980	998	1000	999
	40	1000	1000	1000	1000	998	1000	1000	1000	1000	998	973	1000	1000	1000	998
	50	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	995	1000	1000	1000	1000
0.5	20	857	914	930	996	971	738	832	883	986	961	444	622	717	888	935
	30	948	982	985	1000	993	882	954	971	1000	989	632	852	890	994	971
	40	991	999	999	1000	994	958	996	999	1000	991	793	960	969	1000	985
	50	997	1000	1000	1000	997	994	1000	1000	1000	996	903	990	999	1000	992
0.6	20	676	739	787	939	873	514	629	697	857	823	264	388	482	584	740
	30	812	883	895	993	901	660	783	822	983	878	351	541	612	854	808
	40	906	960	967	999	937	799	910	917	998	911	525	721	768	961	853
	50	934	983	985	1000	945	871	960	965	1000	925	608	833	880	990	888
0.4	20	969	991	996	1000	999	919	971	989	1000	999	697	894	934	997	996
	30	997	1000	1000	1000	999	984	999	998	1000	999	876	989	992	1000	999
	40	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	968	1000	1000	1000	1000
	50	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	998	1000	1000	1000	1000
0.5	20	845	912	934	995	970	738	842	882	984	958	452	620	712	876	924
	30	947	985	990	1000	994	858	958	963	1000	989	597	831	861	991	973
	40	988	998	1000	1000	988	961	990	997	1000	988	782	956	969	999	976
	50	999	1000	1000	1000	1000	995	1000	1000	1000	999	911	995	996	1000	997
0.6	20	680	746	794	943	856	527	624	699	873	819	255	385	464	569	728
	30	774	890	897	990	871	638	811	816	982	836	367	535	599	834	768
	40	890	955	963	1000	922	787	914	928	997	896	499	730	755	958	845
	50	[illegible]	[illegible]	[illegible]	1000	952	878	960	970	1000	930	662	830	874	990	882

Note: T1, '1'2, and T3 are the same chi-square tests as those in Table I. T4 represents the Lilliefors test. T5 represents the moment based test derived in Section 2.3.I. For each given (μ, σ), ttic data were generated from the lognormal distribution with mean μ and standard deviation σ. 1,000 Monte Carlo samples were used to obtain each entry.

moment based test (T5) (see table 3) being slightly better. Under Weibull and truncated normal alternatives, however, substantially increased reject power is achieved by the proposed moment based test (T5) compared with the other four tests (T1-T4) (see tables 2 and 4). This is true especially for small samples. For example, it is seen from table 4 that the simulated rejection probabilities for the five tests T1, T2, T3, T4, and T5 are 0.196, 0.278, 0.329, 0.422, and 0.909, respectively, when $\mu = 1$, $\sigma = 1$, $n = 30$, and $\alpha = 0.01$.

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TABLE 3

Simulated rejection power (multiplied by 1,000) of goodness-of-fit tests for exponential family under truncated normal alternatives

		$\alpha = 0.10$					$\alpha = 0.05$					$\alpha = 0.01$				
		T1	T2	T3	T4	T5	T1	T2	T3	T4	T5	T1	T2	T3	T4	T5
-1	20	162	211	289	144	313	103	147	217	83	234	27	60	115	22	128
	30	159	187	253	172	344	101	122	173	132	251	25	50	81	22	125
	40	169	205	241	213	417	92	133	147	129	319	24	47	56	45	178
	50	174	197	236	226	423	93	124	159	146	327	27	47	60	45	183
0	20	195	259	384	283	556	126	193	306	188	469	46	93	162	47	294
	30	237	274	343	359	653	142	192	257	314	570	49	84	136	95	379
	40	270	307	357	453	760	162	209	257	318	682	55	79	113	116	491
	50	316	335	346	520	830	196	229	244	382	743	60	94	120	150	550
1	20	392	508	608	661	925	291	384	502	525	885	122	210	298	245	768
	30	548	626	645	841	978	430	495	541	787	957	196	278	329	422	909
	40	678	728	774	916	992	533	616	696	850	988	225	361	459	611	958
	50	769	823	813	966	1000	638	715	716	918	997	291	448	483	731	988
-1	20	169	260	329	188	358	103	176	254	105	292	30	77	130	33	181
	30	173	212	276	235	441	90	134	198	180	340	21	65	84	38	200
	40	169	209	228	246	493	91	138	163	143	386	20	47	75	43	207
	50	207	216	238	304	550	124	138	172	207	447	30	51	67	65	263
0	20	196	271	363	279	565	126	196	273	174	461	42	102	136	62	309
	30	208	267	301	355	669	128	179	212	302	570	43	84	103	95	382
	40	289	330	366	480	757	175	223	252	331	662	52	88	118	143	474
	50	308	324	364	529	814	199	229	242	379	737	51	77	107	151	551
1	20	271	397	508	516	831	173	308	402	371	749	77	152	240	151	587
	30	390	473	476	638	903	273	349	381	587	847	104	161	206	239	720
	40	492	553	599	796	973	352	428	478	649	943	133	217	275	375	852
	50	603	627	643	860	990	450	506	517	764	979	145	270	305	486	922
-1	20	155	222	297	184	383	96	158	214	120	295	27	67	116	27	167
	30	156	206	261	231	461	85	149	182	179	351	30	60	78	35	227
	40	207	240	257	315	567	114	156	172	213	473	35	61	71	56	290
	50	219	223	272	336	624	131	158	188	225	513	31	50	77	77	326
0	20	205	260	367	311	581	121	201	284	200	484	49	102	155	60	328
	30	236	297	323	387	654	154	208	235	333	572	51	85	126	96	400
	40	266	305	354	458	758	154	209	244	298	657	54	75	122	112	455
	50	302	310	347	503	824	190	210	253	376	739	49	94	123	157	561
1	20	258	322	446	427	731	159	242	348	294	660	44	132	204	97	495
	30	371	439	464	602	883	249	316	359	543	814	88	139	180	187	663
	40	408	445	499	680	927	267	329	385	552	887	86	154	209	267	764
	50	516	543	577	787	963	359	423	446	670	929	122	199	220	374	835

Note: T1, T2, and 'S3 arc the same chi-square tests as those in Table 1. T4 represents the Lilliefors test. T5 represents the moment based test derived in Section 2.3.1. For each given (μ, σ), the data were generated from the truncated normal distribution by truncating the normal distribution with mean μ and standard deviation σ at 0. 1,000 Monte Carlo samples were used to obtain each entry.

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RIASSUNTO

Una nota sui test di bontà dell'adattamento basati sui momenti

Lo scopo del presente articolo è di introdurre un metodo generale, basato sui momenti, per derivare formalmente un test di bonth dell'adattamento di una famiglia parametrica. Si mostra come, in generale, un test approssimato gaussiano o chi-quadrato possa essere derivato analizzando la struttura dei momenti di una famiglia parametrica, sotto l'ipotesi che i momenti fino ad un dato ordine esistano. L'idea è semplice e i test ottenuti possono essere implementati facilmente. Al fine di illustrare l'uso della metodologia proposta sono derivati, per alcune famiglie parametriche discrete e continue note, i test di bontà dell'adattamento basati sui momenti. I test proposti sono inoltre confrontati con l'usuale test chi-quadrato di Pearson e Fisher e con alcuni test di distanza mediante uno studio di simulazione.

SUMMARY

A note on goodness of fit test using moments

The purpose of this article is to introduce a general moment-based approach to derive formal goodness of fit tests of a parametric family. We show that, in general, an approximate normal test or a chi-squared test can be derived by exploring the moment structure of a parametric family, when moments up to certain order exist. The idea is simple and the resulting tests are easy to implement. To illustrate the use of this approach, we derive moment-based goodness of fit tests for some common discrete and continuous parametric families. We also compare the proposed tests with the well known Pearson-Fisher chi-square test and some distance tests in a simulation study.