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APPROXIMATING THE EXACT VALUE OF AN AMERICAN OPTION

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A mio padre, esempio di vita

1. INTRODUCTION

Many of the options traded in financial markets are of American type, therefore the problem of determining their correct value, according to the noarbitrage principles, cannot be overemphasized. In continuous time models and under standard hypothesis on the market and on the underlying asset, it has been shown (see e.g. Karatzas, 1988; Myneni, 1992) that the arbitrage-free value of the option is the solution of an optimal stopping problem.

It is often impossible to explicitly solve this problem, even in the apparently simpler case of American put options in the Black-Scholes model. Many algorithms have been proposed to get an approximated price, in the Black-Scholes case, the Cox-Ross-Rubinstein (Cox *et al.*, 1979) method is presumably one of the most popular[1].

The Cox-Ross-Rubinstein method is a discrete time approximation of the continuous time market model in the sense that the underlying process is described by a Markov chain that weakly converges to the original diffusion. The discrete market model is complete, hence, once the Equivalent Martingale Measure is determined, the arbitrage-free value of the American option is found by solving a discrete time optimal stopping problem with respect to such a measure. The good news is that the solution of the discrete problem can be computed exactly by using a finite-step algorithm; however, do discrete-time values converge to the continuous-time one when the discretization's interval gets smaller?

The last question, although carefully addressed by the general theory of convergence of stochastic processes (see e.g. Kushner and Dupuis, 1992), has been neglected by the more specific financial literature. As Duffie (Duffie, 1992, page 211) wrote in 1992: "A largely unstudied issue is the convergence of this algorithm to the associated continuous time optional stopping problem characterizing the American security arbitrage free value". In 1994, Amin and Khanna (Amin and Khanna , 1994) analyzed the problem in a financial setting[2]. This paper is a

[1] More recently a novel approach, based on Monte Carlo simulations, and well suited for more complex models, was proposed by Longstaff and Schwartz (Longstaff and Schwartz, 2001).

[2] A more recent contribution to this problem is due to Mulinacci and Pratelli (Mulinacci and Pratelli, 1998).

further attempt in that direction, providing sufficient conditions for convergence, with results that can easily be applied to most of the approximating methods employed in every day practice.

I will consider contingent claims whose payoff is more general than that of the classical American options, so that my results will hold for more sophisticated claims (like exotic options), too. Amin and Khanna make slightly different assumptions: sometimes they are more general, sometimes more restrictive. Since we are following a different approach, we differ in several important points that will be carefully noticed. Differences and similarities between the two approaches will be remarked along the way.

The rest of the paper is as follows. In Section 2 a continuous time market model is introduced. Here it is assumed that the market is constituted by d dividend-paying assets whose prices follow a d dimensional diffusion S_t and a "money market account" B_t with a stochastic interest rate. Such a model is fairly general and contains as a particular case the Black-Scholes one: it is just a little more general than the one proposed by He in He, 1990. The Equivalent Martingale Measure is determined, and, following Karatzas (Karatzas, 1988), the American contingent claim pricing problem is reformulated as an optimal stopping one.

In Section 3 a Markov chain approximation of the process S_t is proposed, along with a corresponding optimal stopping problem in discrete time. It is shown that the Markov chains converge weakly to the diffusion S_t, while the solutions to the discrete-time problems converge to the solution of the continuous-time optimal stopping problem.

In Sections 4 and 5 it is shown that the methods proposed by Cox, Ross and Rubinstein and by He satisfy the hypothesis given in Section 2 and therefore lead to convergent algorithms. This will prove that both methods provide a correct approximation of the exact value of an American option.

2. THE MARKET MODEL

Let us consider a model of a "perfect market" with continuous time trading and no transaction's costs: for a precise definition of these standard concepts see for instance Duffie, 1992. I assume the existence of d assets, whose prices at time t are indicated by S_t^i for $1 \le i \le d$. The d dimensional vector of prices $S_t := (S_t^1, \dots, S_t^d)$ satisfies the following system of stochastic differential equations

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t \tag{1}$$

along with the initial condition $S_0 = S^0$. Here W_t is a d-dimensional Brownian motion with respect to a probability space (Ω, F, F_t, P) and F_t is the filtration generated by it. I also assume that $\mu(x)$ and $\sigma(x)$ are continuous functions from R^d to, respectively, R^d and $R^{d \times d}$, satisfying global Lipschitz and linear growth conditions:

$$\|\mu(x)-\mu(y)\|+\|\sigma(x)-\sigma(y)\|\leq K\|x-y\| \tag{2}$$

$$\|\mu(x)\|^2+\|\sigma(x)\|^2\leq K^2(1+\|x\|^2) \tag{3}$$

for any $x, y \in R^d$, K being a constant (independent of x, y) and $\|\cdot\|$ is a norm in R^d or in $R^{d\times d}$. Moreover, I assume that $\sigma(x)$ is a nonsingular matrix for all x and, for any v, $x \in R^d$, there exists ε, a positive constant such that

$$v^T\sigma(x)\sigma(x)^T v\geq\varepsilon\|v\|^2 \tag{4}$$

where the superscript T means transposition.

It is well known (see e.g. Karatzas and Shreve, 1987; theorem 5.2.9) that such hypotheses are sufficient for strong existence and uniqueness of the price process S_t, for $0\leq t\leq\infty$, that is, for a given probability space (Ω,F,F_t,P) and Brownian motion W_t, there is a unique process S_t satisfying (1).

Let B_t be the d+1-th security, called the *Money Market Account* indicating the value at time t of one currency unit deposited in a bank account at time 0 with interest rate $r(\cdot)$ that is a function of time and possibly, of the vector of asset prices S_t. The money market account B_t is the solution of

$$dB_t=B_t r(S_t)dt \tag{5}$$

where $r(x)$ is a strictly positive continuous and bounded function from R^d to R. Let us set B_0=1 as initial condition.

To complete the setting let us introduce the function $\delta(x) \in R^d$ to denote the *continuous dividend yield*, i.e. $\delta_i(S_t)S_t^i dt$ is the dividend paid by stock i in the interval $(t,t+dt)$. It is assumed that, for all x and i:

$$0\leq\delta_i(x)\leq 1$$

An *American Contingent Claim* (ACC) is a security that can be exercised at any time t between 0 and its expiration time T and whose payoff at a time τ is

$$P(\tau):=\int_0^\tau k(S_u)du+g(S_\tau) \tag{6}$$

where $k(x)$ and $g(x)$ are positive, continuous functions form R^d to R. Moreover, they are assumed to satisfy a uniform integrability condition, i.e., for some α > 1

$$E\left(\sup_{0\leq t\leq T}P(t)\right)^\alpha<\infty\,. \tag{7}$$

The uniform integrability condition (7) is necessary to obtain a finite fair price for the ACC (see Karatzas, 1988).

This definition of ACC includes, as particular cases, American put and American call options. This general definition, however, will allow us to consider some exotic options (e.g. options whose payoff depends on the path of the underlying) as well. In practice, the possibility of early exercise is more important for exotic options than for others “vanilla” options where the difference from the prices of the corresponding Europeans is often negligible.

To determine che artitrage-free price for the ACC we need to construct an *Equivalent Martingale Measure* for our market model. Following the lines of Girsanov’s Theorem, let’s define the R^d-valued process

$$\theta(x) := \sigma(x)^{-1}(\mu(x) + (\delta(x) - r(x))x) . \tag{8}$$

The real valued process

$$Z_t := \exp\left\{-\int_0^t \theta(S_u)^T dW_u - \frac{1}{2}\int_0^t \|\theta(S_u)\|^2 du\right\} \tag{9}$$

is a P-martingale. For any finite $T>0$ we can define a probability measure on the sigma-algebra F_T as

$$Q(A) := E(Z_T I_A) \qquad\qquad A \in F_T .$$

Then P and Q are mutually absolutely continuous and

$$\tilde{W}_t = W_t + \int_o^t \theta(S_u)du$$

is a d-dimensional Brownian motion on (Ω, F, F_t, Q). In the financial literature, Q is usually called *Equivalent Martingale Measure* because it is equivalent to the original measure P and the discounted (ex-dividend) price processes $S_t^i e^{-\delta_i(S_t) - r(S_t)}$ are martingales with respect to Q.

The stochastic dynamic of S_t under the measure Q is given by

$$dS_t = b(S_t)dt + \sigma(S_t)d\tilde{W}_t . \tag{10}$$

where

$$b(S_t) := S_t(r(S_t) - \delta(S_t)) .$$

Let $V(t,x)$ be the no-arbitrage price at time t of the ACC when $S_t=x$. Then Karatzas (1988), theorem 5.4, shows that

$$V(0,x) = \sup_{\tau \in \mathrm{T}[0,T]} E_x^Q \overline{P}(\tau) \tag{11}$$

where $\mathrm{T}[0,T]$ is the set of stopping times with values in $[0,T]$, E_x^Q is the operator of expectation with respect to Q when $S_0=x$ and $\overline{P}(\tau)$ is the claim's payoff discounted at time 0, i.e.

$$\overline{P}(\tau) := \int_0^\tau \frac{k(S_u)}{B_u} du + \frac{g(S_\tau)}{B_\tau} \tag{12}$$

To find the arbitrage-free price of an ACC one has to solve the optimal stopping problem (11). This, in general, does not have a closed form solution, although there are important exceptions, like, for instance, the case of an American call option written on a non-dividend-paying underlying asset. In fact in this case it can be shown that early exercise is never optimal and the problem is easily solved. For most of the other cases one has to resort to some kind of approximation.

3. A DISCRETE TIME APPROXIMATION'S METHOD

To determine an approximation for the value of the ACC expiring in T we will proceed by discretizing the time interval $[0,T]$ into small intervals of length h.

We indicate with $\{S_n^h\}_{n=0}^\infty$ the Markov chain approximation to the diffusion S_t. S_n^h is a d-dimensional process defined on a probability space (Ω^h, Ψ^h, Q^h); its transition probability at step n is indicated by Q_n^h, while $\{F_n^h\}_{n=0}^\infty$ is the filtration generated by it.

Let $b^h(x)$; $a^h(x)$ be measurable functions on R^d such that, for any compact set $K \in R^d$,

$$b^h(x) = b(x) + o(1), \quad h \to 0 \tag{13}$$

$$a^h(x) = \sigma(x)\sigma(x)^T + o(1), \quad h \to 0 \tag{14}$$

uniformly for $x \in K$.

The process $\{S_n^h\}_{n<\infty}$ must satisfy the following condition:

$$E_n(S_{n+1}^h - S_n^h) = b^h(S_n^h)h \tag{15}$$

$$Var_n(S_{n+1}^h - S_n^h) = a^h(S_n^h)h\,, \tag{16}$$

where E_n and Var_n are the mean and variance operators with respect to Q_n^h. Equations (15) and (16) state that the local mean and variance of the Markov chain and of the diffusion process are close to each other, for this reason in the literature (see e.g. Kushner and Dupuis, 1992) they are sometimes referred to as *local consistency conditions*. Amin and Khanna (1994) impose conditions that are very close to these, only a bit more restrictive because they try to approximate a more general diffusion process.

To insure convergence to continuous sample paths one has to impose that

$$\lim_{h\to 0}\sup_{n,\omega}\left\|S_{n+1}^h - S_n^h\right\| = 0 \tag{17}$$

where ω is an event in Ω. Note that conditions (15), (16), (17) do not uniquely determine a Markov chain. See Kushner and Dupuis (1992) for a construction of some chains satisfying them.

The approximation B_n^h to the process for the money market account B_t is given by the iterative formula

$$B_{n+1}^h = (1 + r(S_n^h))B_n^h h \tag{18}$$

along with the initial condition $B_n^h := 1$.

For $N := [T/h]$, let $\mathrm{T}^h[0,N]$ be the set of F_n^h-stopping times assuming values between 0 and N. The optimal stopping problem for the Markov chain S_n^h that is a natural translation of problem (11) is

$$V^h(0,x) := \sup_{v\in \mathrm{T}^h[0,N]} E_0(\overline{P}^h(v)) \tag{19}$$

where $x = S_0^h$ and

$$\overline{P}^h(v) := \sum_{n=0}^{v-1} \frac{k(S_n^h)h}{B_n^h} + \frac{g(S_v^h)}{B_v^h} \tag{20}$$

is the discrete time version of (12).

Remark 3.1 Is is not necessarily true that $V^h(0,x)$ is the arbitrage-free price of an American option in the discrete time market model described by S_n^h and B_n^h. In fact this holds only if the process $\{\overline{P}^h(n)\}_{n=0}^{\infty}$ is a F_n^h-martingale. To get such a nice property one should follow the construction of the discrete model as given by He (1990); we will see later that it also satisfies our conditions.

It is well known from the theory of optimal stopping problems in discrete time (for a nice exposition in a financial setting see chapter 2 of Lamberton and Lapeyre, 1992) that the exact solution can be determined by a finite iterative method solving

$$\begin{cases} V^h(n,x) = \max\left\{ g(x), \dfrac{E_n V^h(n+1, S^h_{n+1}) + k(x)h}{(1+r(x))h} \right\} \\ V^h(N,x) = g(x) \end{cases} \tag{21}$$

In words (21) states that the value at time n of an ACC is given by the best expected result between two opportunities: immediately exercise or hold. Moreover it provides an easy algorithm for computing $V^h(0,x)$ by moving backward (in time) along the chain.

In order to prove convergence of $V^h(0,x)$ to $V(0,x)$ we first need to show that the discrete process is a nice approximation to the diffusion, to this end we will construct a continuos time interpolation of the Markov chain S^h_n. To do it we will use *random jump times* τ^h_n as in Kushner and Dupuis (1992).

Let us set $\tau^h_0 := 0$ and let $\{\tau^h_n\}_{n<\infty}$ be a real valued Markov process independent of $\{S^h_n\}_{n=0}^{\infty}$, such that

$$\Delta\tau^h_n := \tau^h_{n-1} - \tau^h_n \qquad n=1,2,...$$

have an exponential distribution with mean h, that is

$$Q^h(\Delta\tau^h_n < t \mid \tau^h_n) = 1 - e^{t/h} \qquad n=1,2,...$$

Let N_t be the number of jumps from time 0 to time t, that is a Poisson process with parameter $1/h$. We define the continuous time approximating process as

$$S^h(t) := S^h_{N_t} \tag{22}$$

$S^h(\cdot)$ is a process whose sample paths are Right Continuous with Left Limits (RCLL): in fact its trajectories are constant between jump times and take the values of the discrete time process $\{S^h_n\}_{n<\infty}$.

We also define a continuous time version of the money market account process by setting

$$B^h(t) := B^h_{N_t} \tag{23}$$

Remark 3.2 Stochastic processes with jumps have been used several times in the Mathematical Finance literature: perhaps the closest model, in spirit, with the one we have just introduced is Dengler and Jarrow's (1993). They show that a process with two independent and exponentially distributed jump times with constant jump amplitudes, weakly converge to the Black-Scholes diffusion.

Here we use $S^h(\cdot)$ only for proving convergence: it doesn't play any role in the algorithm.

Of course the first step in proving convergence is to show that the process $S^h(\cdot)$ is weakly convergent to the original process S_t with respect to the equivalent martingale measure Q: the main tool we are going to use here is a version of the Martingale Central Limit Theorem as stated by He (1990), Lemma 1), that is just a translation of theorem 7.4.1 of Ethier and Kurtz (1986).

Theorem 3.1 The RCLL process $S^h(\cdot)$ converges weakly to the process S_t with respect to the equivalent martingale measure Q.

Proof. Let $t \in [0,T]$, suppose $N_t = \bar{n}$, $S^h(t) = S^h_{\bar{n}} = x$. We denote by Ψ^h_t the sigma algebra generated by $S^h(u)$, for $0 \le u \le t$. Let E^h_t be the expected value conditioned to Ψ^h_t.
By independence we have:

$$E^h_t(S^h(t+\theta) - S^h(t)) = Q^h(\text{jump on}[t,t+\theta] \mid \Psi^h_t) E_{\bar{n}}(S^h_{\bar{n}+1} - S^h_{\bar{n}}) + o(\theta), \quad \theta \to 0 \tag{24}$$

From the definition of τ^h_n

$$Q^h(\text{jump on}[t,t+\theta] \mid \Psi^h_t) = 1 - e^{-\theta/h} = \theta/h + o(\theta), \quad \theta \to 0 \tag{25}$$

By substitution of equations (15) and (25) into equation (24) we get

$$E^h_t(S^h(t+\theta) - S^h(t)) = \theta b^h(x) + o(\theta) \quad \theta \to 0 \tag{26}$$

Hence,

$$\lim_{\theta \to 0} E^h_t \frac{S^h(t+\theta) - S^h(t)}{\theta} = b^h(x). \tag{27}$$

Therefore $b^h(S^h(t))$ is the "predictable compensator" of $S^h(t)$, that is

$$M^h_t := S^h(t) - \int_0^t b^h(S^h(u))du \tag{28}$$

is a Ψ^h_t-martingale (see e.g. Ethier and Kurtz, 1986, Proposition 4.1.7).

To compute the predictable compensator of the process $M^h_t M^{hT}_t$ we use the same argument as above,

$$A_t^b(M_t^b M_t^{bT}) := \lim_{\theta\to 0} E_t^b \frac{(M_{t+\theta}^b M_{t+\theta}^{b\ T}) - (M_t^b M_t^{bT})}{\theta}$$

$$:= \lim_{\theta\to 0} E_t^b \frac{(M_{t+\theta}^b - M_t^b)(M_{t+\theta}^b - M_t^b)^T}{\theta} \tag{29}$$

where (29) follows from the fact that M_t^b is a martingale.
From the definition of M_t^b we get

$$M_{t+\theta}^b - M_t^b = S^b(t+\theta) - S^b(t) - \int_t^{t+\theta} b^b(S^b(u))du\,. \tag{30}$$

For $S^b(t) = x$ we have (a.s. and uniformly in x)

$$M_{t+\theta}^b - M_t^b = S^b(t+\theta) - S^b(t) - \int_t^{t+\theta} b^b(S^b(u))du\,; \tag{31}$$

hence, from (27)

$$\int_t^{t+\theta} b^b(S^b(u))du = E_b^t S^b(t+\theta) - S^b(t) + o(\theta), \quad \theta \to 0\,, \tag{32}$$

where (32) also holds a.s. and uniformly in x. Substituting (32) into (30) we get

$$M_{t+\theta}^b - M_t^b = S^b(t+\theta) - E_b^t S^b(t+\theta) - S^b(t) + o(\theta), \quad \theta \to 0\,,$$

hence from (29) follows

$$A_t^b(M_t^b M_t^{bT}) := \lim_{\theta\to 0} E_t^b \frac{[S^b(t+\theta) - E_t^b S^b(t+\theta)]\,[S^b(t+\theta) - E_t^b S^b(t+\theta)]^T}{\theta}$$

$$:= a^b(x) \tag{33}$$

where (33) follows from the same argument used to derive (27). Therefore the process

$$M_t^b M_t^{bT} - \int_0^t a^b(S^b(u))du \tag{34}$$

is a martingale.

To show convergence we refer to the Lemma 1 in He (1990): point (a) is obviously true, while (b) follows from (28) and (34). Point (c) is verified because the jumps of M_t^b are the same as those of $S^b(t)$ and therefore their amplitude uniformly goes to zero as b decreases because of (17). As for the last point, let τ_t^b be the first exit time of process $S^b(t)$ from the sphere of radius r. Then we need to show that, for any r>0,

$$\lim_{h\to 0}\sup_{t\le\tau_r^h\wedge T}\left|\int_0^t b_i^h(S^h(u))-b_i(S^h(u))du\right|=0,\qquad 1\le i\le n \tag{35}$$

$$\lim_{h\to 0}\sup_{t\le\tau_r^h\wedge T}\left|\int_0^t a_{ij}^h(S^h(u))-a_{ij}(S^h(u))du\right|=0,\qquad 1\le i,j\le n \tag{36}$$

in probability. Equations (35) and (36) hold for the uniform convergence over compact sets of $b^h(x)$, $a^h(x)$ to, respectively, $b(x)$ and $a(x)$.

Let $I^h(t,T)$ be the set of stopping times $\tau_n^h \wedge T$ where τ_n^h are jump times, with $n \ge N_t$. We define an optimal stopping problem for $S^h(t)$ by restricting the set of feasible stopping times to $I^h(t,T)$, for $x=S^h(t)$.

$$V^{S^h}(t,x)=\sup_{\tau\in I^h(t,T)} E_t^h\overline{P}(\tau), \tag{37}$$

where $\overline{P}$ is defined by (12), with $S^h(u)$ and $B^h(u)$ substituting S_u and B_u.

Now we want to show that solving the optimal stopping problem for the discrete time process S_n^h is almost the same as solving (37). To do that, we observe that, from the principle of optimality (see Kushner and Dupuis, 1992, pg. 44),

$$V^{S^h}(0,x)=\frac{1}{1+r(x)\delta}\max\{V^{S^h}(\delta,x),E_0^hV^{S^h}(\delta,S^h(\delta))+k(x)\delta\}+o(1)\quad \delta\to 0 \tag{38}$$

Using the hypothesis of independence, we can compute an approximation of the expected value on the right hand side

$$E_0^hV^{S^h}(\delta,S^h(\delta))=\left(1-\frac{\delta}{h}\right)V^{S^h}(\delta,x)+\frac{\delta}{h}E^hV^{S^h}(\delta,S_1^h)+o(\delta),\quad \delta\to 0\,.$$

Substituting it into (38) and setting $\delta=h$, we get

$$V^{S^h}(0,x)=\frac{1}{1+r(x)h}\max\{V^{S^h}(h,x),E^hV^{S^h}(h,S_1^h)+k(x)h\}+o(1)\quad h\to 0 \tag{39}$$

The optimality equation satisfied by V^h is

$$V^h(0,x)=\frac{1}{1+r(x)h}\max\{V^h(h,x),E^hV^h(h,S_1^h)+k(x)h\}\,.$$

Therefore we have proved that

$$V^{S^h}(0,x)=V^h(0,x)+o(1),\quad h\to 0 \tag{40}$$

Amin and Khanna (1994) do not have to care about such an issue, since all their arguments are related to the continuous-time interpolated value function (the correspondent of our $V^{S^h}(t,x)$).
Let ρ^h be the optimal stopping time for the problem (37), that is

$$V^{S^h}(0,x) = E_t^h \overline{P}(\rho^h)\,; \tag{41}$$

since the family ρ^h is *tight* (it is bounded), from the Prohorov's Theorem, there exists a subsequence ρ^{h_n} that is weakly convergent to some random variable ρ as h_n goes to zero. Note that ρ could depend on the particular subsequence chosen.

We now want to show that ρ is a solution to the original stopping problem. Let us define a process $w^h(t)$ as

$$w^h(t) := \int_0^t \sigma(S^h(u))^{-1} dM_u^h\,. \tag{42}$$

Note that the integral can be computed path by path in the sense of Riemann-Stieltjes: in fact M^h is a process with path of finite variation. Moreover, the jump's amplitude of $w^h(t)$ goes to zero as h goes to zero.

From (34) it follows that the quadratic variation process for M_h^t (see Protter, 1990, pg. 65, Corollary 2) is given by

$$[M^h, M^h]_t = \int_0^t a^h(S_u^h)du\,,$$

therefore,

$$\begin{aligned}
[w^h, w^h]_t &= \int_0^t \sigma(S^h(u))^{-1} a^h(S^h(u))(\sigma(S^h(u))^T)^{-1} du \\
&= \int_0^t \sigma(S^h(u))^{-1}[a(S^h(u)) + o(1)I](\sigma(S^h(u))^T)^{-1} du \quad h \to 0 \\
&= tI + o(1)I, \quad h \to 0
\end{aligned} \tag{43}$$

where I is the n-dimensional identity matrix, and (43) holds for the hypothesis of uniform convergence of $a^h(\cdot)$ to $a(\cdot)$ over compact sets.
Since, from (43), for any Ψ_t^h-stopping time $\tilde{v}_h$,

$$E_{\tilde{v}_h}^h \left\| w^h(\tilde{v}_h + \varepsilon) - w^h(\tilde{v}_h) \right\|^2 = O(\varepsilon)\,, \qquad \varepsilon \to 0\,,$$

then Theorem 9.2.1 of Kushner and Dupuis (1992) states that $w^h(\cdot)$ is tight. Let $(w^{h_n}(\cdot), \rho^{h_n})$ denote a weakly convergent subsequence of $(w^h(\cdot), \rho^h)$ and let (w_t, ρ) denote its limit. From (43) and the continuity of its sample paths, it follows that w_t is a Brownian Motion. Since $(\rho^h \leq s)$ is independent of $(w_t^h - w_s^h)$ for any $t \geq s \geq 0$, it follows that $(\rho \leq s)$ is independent of $(w_t - w_s)$. Hence we can say that ρ is *nonanticipative* with respect to w_t. Then (see Chapter 8 of Kushner, 1977) there exists a probability space $(\overline{\Omega}, \overline{F}, \overline{F}_t, \overline{Q})$, with a Brownian Motion $\overline{w}_t$ and a random variable $\overline{\rho}$ such that $(\overline{w}_t, \overline{\rho})$ has the same distribution as (w_t, ρ). Moreover, the filtration $\overline{F}_t$ is generated by $\overline{w}_t$, hence $\overline{\rho}$, being nonanticipative with respect to $\overline{w}_t$, is a $\overline{F}_t$-stopping time. From (42) we get

$$dM_t^h = \sigma(S^h(t))dw_t^h,$$

hence, substituting into (28), we get the following differential form for $S^h(t)$

$$dS^h(t) = b^h(S^h(u))du + \sigma(S^h(u))dw_u^h. \tag{44}$$

Let $\overline{S}_t$ be the solution of

$$d\overline{S}_t = b(\overline{S}_u)du + \sigma(\overline{S}_u)d\overline{w}_u$$

with $\overline{S}_0 = S_0$. From the weak uniqueness of (1), $\overline{S}_t$ has the same distribution as S_t. Therefore, for an alternative proof of Theorem 3.1, we could now use Theorem 4.3 of Duffie and Protter (1992), to conclude that it is the weak limit of the sequence $S^h(t)$. Observe that, unlike ρ, $\overline{S}_t$ is not sequence-dependent.

Remark 3.3 The optimal stopping time problem for the process $\overline{S}_t$ with the probability space $(\overline{\Omega}, \overline{F}, \overline{F}_t, \overline{Q})$, is equivalent, from the market modeling point of view, to the original problem (11). In fact what is fundamental for the construction of the Equivalent Martingale Measure and therefore for the formulation of (11) is that the filtration F_t, that represents the amount of information available at time t, is generated by W_t; in this way all the information is carried by the price process S_t. Other particular features of the probability space are not important for the model, hence the equivalence of the two problems.

We can now state the main result

Theorem 3.2 Let

$$P^h(n) := \sum_{i=0}^{n} k(S_i^h)h + g(S_n^h)$$

be uniformly integrable (with respect to h), for all n, $0 \leq n \leq N$, then $V^h(0,x)$, the optimal value for the discrete problem, converges to $V(0,x)$ as h goes to zero.

Proof. Let ρ^h be given by (41), then

$$\limsup_{h\to 0} V^h(0,x) = \limsup_{h\to 0} E\overline{P}^h(\rho^h) \tag{45}$$

$$\leq \overline{E}\overline{P}^h(\rho) \tag{46}$$

$$\leq V(0,x); \tag{47}$$

where $x = S_0^h = \overline{S}_0$ and $\overline{E}$ is the expectation with respect to $\overline{Q}$. The lim sup in (45) is necessary because of the sequence dependency of ρ, (46) follows from the uniform integrability of $P^h(\cdot)$ (and hence of $\overline{P}^h(\cdot)$) and from the weak convergence to ρ. Equation (47) is a consequence of Remark 3.3 and the suboptimality of ρ.

To show the reverse inequality we have to use the fact that ρ^h are optimal stopping times for $S^h(t)$. Using a result by Shiryayev (1973) (see also Kushner and Dupuis, 1992), for any $\varepsilon>0$, we can choose δ such that if we restrict the stopping times for the continuous time problem to take only the values $\{n\delta, n\delta \leq T\}$ and we call $V_\delta(0,x)$ the optimal value that we get, then

$$V_\delta(0,x) \geq V(0,x) - \varepsilon .$$

Moreover if ρ_ε is the optimal stopping time under such a restriction, its probability law is determined by $Q(\rho_\varepsilon=0)$ and by

$$Q(\rho_\varepsilon = n\delta \mid W_s, s \leq n\delta, \rho_\varepsilon > (n-1)\delta) = F_n(W_{p\theta}, p\theta \leq n\delta) \tag{48}$$

where F_n is a continuous function, p is an integer number and $\theta \leq \delta$.

To construct an analogous of ρ_ε for the approximating processes we proceed as follows: let τ_k^h be the jump times of $S^h(t)$; let's define

$$\sigma_n^h = \min_k \{\tau_k^h : \tau_k^h \geq n\delta\}$$

in words, σ_n^h is the first jump time after $n\delta$. We can now define, for any h, the stopping time ρ_ε^h that takes value on the jump times only and whose probability law is given by

$$P(\rho_\varepsilon^h = 0) = Q(\rho_\varepsilon = 0)$$

and

$$P(\rho_\varepsilon^h = \sigma_k^h \mid w^h(s), s \leq \sigma_k^h, \rho_\varepsilon^h > \sigma_{k-1}^h) = F_n(w^h(p\theta), p\theta \leq \sigma_k^h)$$

where $w^h(s)$ is defined by (42).

Since ρ_ε^h are Ψ^h-stopping times that are feasible for problem (37), we get, from (40),

$$\begin{aligned} V^h(0,x) &= V^{S^h}(0,x) + o(1), \quad h \to 0 \\ &\geq E\overline{P}^h(\rho_\varepsilon^h) + o(1), \quad h \to 0 \end{aligned}$$

We observe that $\sigma_{k+1}^h - \sigma_k^h$ converges to δ in probability. From this and the fact that the probability law of ρ_ε^h is a continuous function of $w^h(t)$, that weakly converges to w_t, it follows that $(\rho_\varepsilon^h, w^h(t))$ is weakly convergent to $(\rho_\varepsilon, w(t))$.
Hence we have

$$\begin{aligned} \lim_{h\to 0} V^h(0,x) &\geq \lim_{h\to 0} E\overline{P}^h(\rho_\varepsilon^h) \\ &= E^Q\overline{P}(\rho_\varepsilon) && (49) \\ &\geq V(0,x) - \varepsilon && (50) \end{aligned}$$

where (49) follows from the weak convergence of $(\rho_\varepsilon^h, w^h(t))$ to $(\rho_\varepsilon, w(t))$ and from the uniform integrability of $\overline{P}^h(\cdot)$. The result then follows from (47) and (50).

4. THE COX ROSS RUBINSTEIN APPROXIMATION

In 1979, Cox Ross and Rubinstein (Cox *et al.*, 1979) constructed a discrete time approximation of the Black Scholes model

$$\begin{aligned} dS_t &= S_t\mu dt + S_t\sigma dW_t \\ dB_t &= rB_t dt \end{aligned}$$

Their model is still the most used approximation's technique to price American Options in the Black and Scholes setting. It is given by a binomial process S_n^h, where $n = 1,...,N$ and $h{:=}T/N$ is the time interval, defined by recursion in the following way:

$$S_0^h := S_0 \tag{51}$$

$$S_{n+1}^h := \begin{cases} uS_n^h & \text{with probability} \quad p \\ dS_n^h & \text{with probability} \quad 1\text{-}p \end{cases} \tag{52}$$

where $u := \exp(\sigma\sqrt{h}), d := \exp(-\sigma\sqrt{h})$ and $p \in (0,1)$. The approximation for B_t is given by

$$B_0^h := 1 \tag{53}$$

$$B_{n+1}^h := RB_n^h \tag{54}$$

where $R := e^{rh}$.

The transition probability for the discrete model in the risk-neutral setting is given by

$$Q_n^h(S_{n+1}^h = uS_n^h) := \frac{\mathrm{R} - d}{u - d}$$

while the dynamic of the stock price with respect to the Equivalent Martingale Measure in the continuous setting is given by

$$dS_t = S_t r dt + S_t \sigma d\tilde{W}_t \,.$$

To check conditions (15) and (16) we have to compute

$$\begin{aligned} E_n(S_{n+1}^h - S_n^h) &= S_n^h(\mathrm{R} - 1) \\ &= S_n^h(e^{rh} - 1) \\ &= S_n^h rh + o(h), \qquad h \to 0 \end{aligned}$$

and

$$\begin{aligned} Var_n(S_{n+1}^h - S_n^h) &= E_n(S_{n+1}^h)^2 - 2E_n S_{n+1}^h S_n^h + (S_n^h)^2 - [S_n^h(\mathrm{R} - 1)]^2 \\ &= E_n(S_{n+1}^h)^2 - (S_n^h)^2 \mathrm{R}^2 \\ &= (S_n^h)^2 [1 + 2rh + \sigma^2 h - e^{2rh}] + o(h), \quad h \to 0 \\ &= (S_n^h \sigma)^2 h + o(h), \quad h \to 0\,. \end{aligned}$$

Therefore local consistency conditions (15) and (16) hold. Moreover, since the jump size uniformly goes to zero as h goes to zero, we can conclude from Theorem 3.2 that when the Cox-Ross-Rubinstein method is applied to pricing an ACC and the uniform integrability condition for the option's payoff in the discrete parametrization is fulfilled, then the approximation is convergent to the continuous time value.

5. THE HE MODEL

In a paper of 1990, H. He (He, 1990) proposed a discretization's technique for general continuous time model based on the idea of preserving the properties of completeness for the market model (see Remark 3.1). He considered a security market consisting of N risky stocks S_t and one bond B_t, whose dynamic of prices is given by

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t \tag{55}$$

$$dB_t = B_t r(S_t)dt\,. \tag{56}$$

Here the probability space and the functions $\mu(x)$, $\sigma(x)$, $r(x)$, have the same properties as in Section 2.

The discrete approximation is defined on the time interval [0,1] with n equally spaced time steps and it is determined by

$$S^h_{k+1} = S^h_k + h\mu(S^h_k) + \sqrt{h}\sigma(S^h_k)\tilde{\varepsilon}^k \tag{57}$$

$$B^h_{k+1} = B^h_k(1 + hr(S^h_k)) \tag{58}$$

where $h{:=}1/n$ and $\tilde{\varepsilon}^k$ are independent and identically distributed N dimensional random vectors, with mean 0 and variance I (the identity matrix). He showed that the discrete model is complete and determined the Equivalent Martingale Measures Q^h.

It is not hard to show that such a discrete approximation is locally consistent to the continuous parameter process when both are considered with respect to the Equivalent Martingale Measures. In fact it follows from (10) that the dynamic of the continuous process under the new measure Q is given by

$$dS_t = r(S_t)S_t dt + \sigma(S_t)d\tilde{W}_t$$

$$dB_t = B_t r(S_t)dt\,.$$

For the discrete process, we have

$$\begin{aligned} E_k S^h_{k+1} - S^h_k &= (1 + hr(S^h_k))S^h_k - S^h_k \\ &= hr(S^h_k)S^h_k\,, \end{aligned}$$

that is condition (15). Moreover

$$\begin{aligned} Var_k(S^h_{k+1} - S^h_k) &= Var_k(\sqrt{h}\sigma(S^h_k)\tilde{\varepsilon}^k) \\ &= h\sigma(S^h_k)\sigma(S^h_k)^T Var_k\tilde{\varepsilon}^k\,, \end{aligned}$$

but

$$Var_k \tilde{\varepsilon}^k = E^{Q_h} \tilde{\varepsilon}^k \tilde{\varepsilon}^{kT}$$

$$+h(\sigma(S_k^h)^{-1}(\mu(S_k^h) - r(S_k^h)S_k^h))(\sigma(S_k^h)^{-1}(\mu(S_k^h) - r(S_k^h)S_k^h))^T$$

Hence, by substitution, we get

$$\begin{aligned} Var_k(S_{k+1}^h - S_k^h) &= h\sigma(S_k^h)\sigma(S_k^h)^T E^{Q_h} \tilde{\varepsilon}^k \tilde{\varepsilon}^{kT} + o(h), \quad h \to 0 \\ &= h\sigma(S_k^h)\sigma(S_k^h)^T (I + O(h^{1/2})) + o(h), \quad h \to 0 \\ &= h\sigma(S_k^h)\sigma(S_k^h)^T + o(h), \quad h \to 0 \end{aligned}$$

where convergence is uniform if $\mu(x)$, $\sigma(x)$ and $r(x)$ satisfy hypothesis of Section 2. Hence condition (16) also holds.

This proves that He's method gives a correct approximation of the continuous time value when applied to pricing ACCs.

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RIASSUNTO

Approssimare il valore esatto di una option *americana*

Una *option* americana è un diritto derivato che può essere esercitato in qualunque momento prima della scadenza. Sotto ipotesi standard si può dimostrare che il suo prezzo *arbitrage-free* è la soluzione di un problema di tempo ottimale di stop. Solitamente la soluzione del problema di stop non ha una forma chiusa. Al fine di pervenire a soluzioni approssimate sono stati proposti modelli discreti. In questo lavoro si formulano alcune condizioni sul processo discreto per assicurare la convergenza dell'approssimazione al valore esatto e si mostra come applicare tali condizioni per controllare la correttezza di alcuni dei più popolari schemi di discretizzazione.

SUMMARY

Approximating the exact value of an American option

An American option is a derivative security that can be exercised at any time before expiration. Under standard hypotheses it can be shown that its arbitrage-free price is the solution of an optimal stopping problem. Usually, if the underlying asset follows a diffusion, the stopping time problem does not have a closed form solution. Therefore, discrete time models have been proposed to determine an approximated solution. I formulate some conditions on the discrete process to insure convergence of the approximations to the exact value. I also show how to apply such conditions to check the correctness of some of the most popular discretization schemes.