STATISTICA, *anno LXII, n. 1, 2002*

A NOTE ON TIIE COVEKGENCE OF BIVAKIATE EXTREME ORDEK STATISTICS

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1. INTRODUCTION

Let $\mathbf{X}_j = (X_{1j}, X_{2j})$, $j = 1, 2, \ldots, n$ be a random sample from a bivariate population X = (X,, X,) with d.f. $F(\mathbf{x}) = F(x_1, x_2)$. Let $X_{i,j:n}$ denote the j-th order statistic of the X_i sample values, $j = 1, 2, \ldots, n$; i = 1, 2. For **a,** b, **x**, **y** E $\mathcal{R}^2$, write ax + b and $\frac{\mathbf{x}}{\mathbf{y}}$ to denote respectively the vectors $(a_1x_1 + b_1,\ a_2x_2 + b_2)$ and $\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right)$ and $\mathbf{x} \leq \mathbf{y}$ to mean x_i **I** y_i, i = 1, 2. Write $\mathbf{Z}_{k,k':n}$, $\mathbf{W}_{k,k':n}$ and $\mathbf{V}_{k,k':n}$ to denote respectively the random vectors $(X_{1,n-k+1:n}, X_{2,n-k'+1:n})$, $(X_{1,k:n}, X_{2,k':n})$ and $(X_{1,k:n}, X_{2,n-k'+1:n})$, where k and k' are any positive integers. Let further $G(\mathbf{x}) = P(X > \mathbf{x})$ and $H_{k,k':n}(\mathbf{x}) = P(\mathbf{Z}_{k,k':n} \leq \mathbf{x})$. Finally, for the i-th ($i = 1, 2$) component of the vector X, let $F_i(x_i)$ and $G_i(x_i)$ be the marginals of $F(\mathbf{x})$ and $G(\mathbf{x})$ respectively, while $H_{k..:n}(x_1)$ and H $_{k':n}(x_2)$ be the marginals of $H_{k,k':n}(\mathbf{x})$. It can be shown that (see Barakat, 1990, 1999)

$$H_{k,k':n}(\mathbf{x}) = \sum_{i=0}^{k-1} \sum_{j=0}^{k'-1} \sum_{r=0 \vee i+j-n}^{i \wedge j} \frac{n!}{(i-r)!\,r!\,(j-r)!\,(n-i-j+1)!} (G_1(x_1) - G(\mathbf{x}))^{i-r}$$

$$\cdot\, G^r(\mathbf{x})(G_2(x_2) - G(\mathbf{x}))^{j-r}(1 - G_1(x_1) - G_2(x_2) + G(\mathbf{x}))^{n-i-j+r}$$

and

$$P(V_{k,k':n} \leq \mathbf{x}) = H_{.,k':n}(x_2) - M_{k,k':n}(\mathbf{x}) \,,$$

where $\max(a, b) = a \vee b$, $\min(a, b) = a \wedge b$ and

$$M_{k,k':n}(\mathbf{x}) = \sum_{i=0}^{k-1} \sum_{j=0}^{k'-1} \sum_{r=0 \vee i+j-n}^{i \wedge j} \frac{n!}{(i-r)!\,r!\,(j-r)!\,(n-i-j+1)!} F^{i-r}(\mathbf{x})$$

$$\cdot\, (F_1(x_1) - F(\mathbf{x}))^r G^{j-r}(\mathbf{x})(1 - G_2(x_2) - F(\mathbf{x}))^{n-i-j+r} \,.$$

Since clearly,

$$n(G_1(u_{1n}-)+G_2(u_{2n}-)) \ge n(1-F(\mathbf{u}_n-)) \ge n(1-F(\mathbf{u}_n)) \ge n(G_1(u_{1n}) \vee G_2(u_{2n})),$$

we have

$$0<\tau_1 \vee \tau_2 \le \liminf_{n\to\infty} n(1-F(\mathbf{u}_n)) \le \limsup_{n\to\infty} n(1-F(\mathbf{u}_n)) \le$$
$$\le \limsup_{n\to\infty} n(1-F(\mathbf{u}_n-)) \le \tau_1+\tau_2<\infty. \tag{2.8}$$

Let us now prove the relation

$$\frac{1-F(\mathbf{x})}{1-F(\mathbf{x}-)} \to 1, \qquad \text{as } \mathbf{x} \to \mathbf{x}^o, \tag{2.9}$$

which is equivalent to

$$\frac{P(\mathbf{x})}{1-F(\mathbf{x}-)} \to 0, \qquad \text{as } \mathbf{x} \to \mathbf{x}^o, \tag{2.10}$$

where $P(\mathbf{x})=F(\mathbf{x})-F(\mathbf{x}-)$. On the other hand (2.7) is equivalent to

$$\frac{P_i(x_i)}{G_i(x_i-)} \to 0, \qquad \text{as } x_i \to x_i^o, \qquad i=1,2; \tag{2.11}$$

where $P_i(x_i)=F_i(x_i)-F_i(x_i-)$, $i=1,2$ and $P(\mathbf{x})=F(\mathbf{x})-F(\mathbf{x}-)$. Suppose, now that (2.10) does not hold. Then, there exist $\varepsilon>0$ and a sequence $\{\mathbf{v}_n\}=\{(v_{1n}, v_{2n})\}$ of vectors such that $\mathbf{v}_n \to \mathbf{x}^o$ and

$$P(\mathbf{v}_n) \ge 2\varepsilon(1-F(\mathbf{v}_n-)). \tag{2.12}$$

On the other hand, for sufficiently large n, by (2.11) we have,

$$P_i(v_{in})<\varepsilon G_i(v_{in}-), \qquad i=1,2,$$

which yields

$$P_1(v_{1n})+P_2(v_{2n})<\varepsilon(G_1(v_{1n}-)+G_2(v_{2n}-)). \tag{2.13}$$

Now, by using the obvious relation $P_1(x_1)+P_2(x_2)-P(\mathbf{x})=G(\mathbf{x}-)-G(\mathbf{x})\ge 0$, we get

$$P_1(x_1)+P_2(x_2) \ge P(\mathbf{x}), \qquad \forall \mathbf{x}. \tag{2.14}$$

Therefore, by (2.12), (2.14) and by using the relation $\frac{1}{2}(F_1+F_2) \ge \sqrt{F_1F_2} \ge F$, we get

$$P_1(v_{1n})+P_2(v_{2n}) \ge P(\mathbf{v}_n) \ge 2\varepsilon\left(1-\frac{F_1(v_{1n}-)+F_2(v_{2n}-)}{2}\right)=\varepsilon(G_1(v_{1n}-)+G_2(v_{2n}-)),$$

for all sufficiently large n, which contradicts (2.13), *i.e.* (2.9) is proved.

Now, if

$$\liminf_{n\to\infty} n(1-F(\mathbf{u}_n)) = \limsup_{n\to\infty} n(1-F(\mathbf{u}_n-)), \tag{2.15}$$

(2.8) clearly guarantees the existence of a real number $0 < \tau < \infty$, $0 < \tau_1 \vee \tau_2 \leq \tau \leq \tau_1 + \tau_2 < \infty$, such that $n(1 - F(\mathbf{u}_n)) \to \tau$, which, in view of the relation $1 - F(\mathbf{u}_n) = G_1(u_{1n}) + G_2(u_{2n}) - G(\mathbf{u}_n)$, leads immediately to (2.3) (since clearly $\mathbf{u}_n \to \mathbf{x}^o$). On the other hand, if (2.15) does not hold, then

$$0 < \liminf_{n\to\infty} n(1 - F(\mathbf{u}_n)) < \limsup_{n\to\infty} n(1 - F(\mathbf{u}_n -)) < \infty,$$

by which we can select a subsequence n_k such that

$$\lim_{k\to\infty} \frac{1 - F(\mathbf{u}_{n_k})}{1 - F(\mathbf{u}_{n_k} -)} \neq 1,$$

which contradicts (2.9).

Remark 2.1. Although the statement of Theorem 2.1 will not be changed if all d.f.'s ($F(\mathbf{x})$, $F_1(x_1)$ and $F_2(x_2)$ are assumed to be left continuous, the relation (2.7) should become $\frac{G_i(x_i)}{G_i(x_i +)} \to 1$, as $x_i \to x_i^o$, $i = 1, 2$. Similar obvious changes will be done for (2.9), (2.19), etc...

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Assume that the marginals $H_{k,n}(u_{1n})$ and $H_{,k',n}(u_{2n})$ converge to a nondegenerate limits (i.e., positive and finite numbers). Then, in view of Lemma 2.1, (2.1) and (2.2) are satisfied. Therefore, in view of Lemma 2.2, (2.7) is also satisfied and the first part of the theorem is now followed by applying the second part of Lemma 2.2. To prove the second part it suffices to use Geffroy condition (see Geffroy, 1958) which is a sufficient and necessary condition for the independence of the limit marginals (see Barakat, 1998, 2000). However, in view of the relation $G(\mathbf{x}) = G_1(x_1) + G_2(x_2) - (1 - F(\mathbf{x}))$, Geffroy condition is equivalent to

$$\frac{1 - F(\mathbf{x})}{G_1(x_1) + G_2(x_2)} \to 1, \qquad \text{as } \mathbf{x} \to \mathbf{x}^o \tag{2.16}$$

Now the second part of theorem follows immediately by using (2.16) and the obvious relation

$$1 \geq \frac{1 - F(\mathbf{x})}{G_1(x_1) + G_2(x_2)} \geq \frac{1 - F_1(x_1) F_2(x_2)}{G_1(x_1) + G_2(x_2)} = 1 - \left(\frac{1}{G_1(x_1)} + \frac{1}{G_2(x_2)} \right)^{-1}.$$

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ACKNOWLEDGEMENT

The author would like to thank the anonymous reviewers for several helpful comments.

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RIASSUNTO

Una nota sulla convergenza di statistiche d'ordine estreme bivariate

In questa nota si mostra che per ogni vettore bivariato di statistiche d'ordine estreme, esiste almeno una successione di vettori reali per cui la funzione di distribuzione (d f) converge ad un limite finito se e solo re le marginali hanno limiti finiti Inoltre tale limite si può scrivere come il prodotto delle marginali se la d f bivariata da cui sono calcolate le statistiche d'ordine è quella di una variabile casuale dipendente *negative quadmnte*

SUMMARY

A note on the convergence of bivariate extreme order statistics

In this note an interesting fact is proved that, for any vector of bivariate extreme order statistics there exists (at least) a sequence of vectors of real numbers for which the distribution function (d.f.) of this vector converges to a nondegenerate limit if and only if its marginals converge to nondegenerate limits. Moreover, the limit splits into the product of the limit: marginals if the bivariate d.f., irom which the order statistics are drawn, is of negative quadrant dependent random variables (r.v., s).