ON THE CONFIDENCE INTERVALS OF PARAMETRIC FUNCTIONS FOR DISTRIBUTIONS GENERATED BY SYMMETRIC STABLE LAWS

D. Farbod, K. V. Gasparian

1. INTRODUCTION

The “Discrete Distributions Generated by Standard Symmetric Stable Densities” (DSSD in short) are employed as a model for applications in Evolutionary Large-Scale Biomolecular Systems (Astola and Danielian, 2007; Astola *et al.*, 2007, 2008, 2010). It is of interest to consider the statistical analysis of parameters estimators for the DSSD. A major drawback of the DSSD is that neither theirs probability mass functions nor theirs cumulative distribution functions can be expressed in closed form. The main aim of this paper is to discuss asymptotic confidence intervals of the *ML Estimators* for some parametric functions of such distributions.

This paper is formed as follows. Section 2 briefly introduces *Symmetric Stable Laws*. Section 3 considers DSSD and presents the asymptotic properties of the *ML Estimators* for such distributions. The main results of the paper are proposed in Section 4. Section 5 concludes.

2. SYMMETRIC STABLE LAWS

The *Stable Laws,* introduced by Paul Levy in the 1920’s, form a rich class of probability distributions allowing heavy tails, skewness and have many other useful mathematical properties (Farbod and Gasparian, 2008; Nolan, 2010). We define by $S(\alpha,\beta,\gamma,\delta)$ the class of all stable distributions with the following parameters: the index exponent $\alpha\in(0,2]$, the skewness parameter $\beta\in[-1,1]$, the scale parameter $\gamma\in(0,\infty)$, and the location parameter $\delta\in(-\infty,\infty)$. The index exponent α is the most important parameter of the *Stable Laws* and it measures how heavy-tailed the distribution is.

When $\beta=0$, the sub-family $S(\alpha,0,\gamma,\delta)$ is symmetric about δ. A symmetric stable distribution is called *standard* if $\gamma=1$ and $\delta=0$. It is possible without loss of generality to consider only standard symmetric stable distributions (Zolotarev, 1986, p. 60). It is well-known that a standard symmetric stable random variable X is best defined by its characteristic function (Zolotarev, 1986):

$$\varphi_X(t) = \exp(-|t|^{\alpha}), \quad \alpha \in (0,2].$$

In the present paper, we consider the class $S(\alpha,0,1,0)$, $\alpha \neq 1$ with parametric space (compare to DuMouchel, 1973):

$$\Theta_\varepsilon = \{(\alpha,\beta,\gamma,\delta) : 0 < \varepsilon \le \alpha < 1 \ \ or \ \ 1 < \alpha < 2,\ \beta = 0,\ \gamma = 1,\ \delta = 0\},$$

where ε is some small constant, and let D be some subset of Θ_ε whose closure $\bar{D}$ is also contained in Θ_ε.

Then we know the following series expansion for density of the standard symmetric stable distribution from the class $S(\alpha,0,1,0)$, $\alpha \neq 1$ (Matsui and Takemura, 2004; Zolotarev, 1986):

$$s(x;\alpha) = \begin{cases} \dfrac{1}{\pi}\displaystyle\sum_{j=1}^{\infty}\dfrac{\Gamma(j\alpha+1)}{j!}(-1)^{j-1}x^{-\alpha j-1}\sin\left(\dfrac{\pi j\alpha}{2}\right), & if \ \ x > 0, \qquad (a) \\ \dfrac{1}{\pi}\Gamma\left(1+\dfrac{1}{\alpha}\right), & if \ \ x = 0. \qquad (b) \end{cases} \tag{1}$$

If $x < 0$, the density function (1-a) is defined as $s(x;\alpha) = s(-x;\alpha)$. We know that for $0 < \alpha < 1$ the series (1-a) is convergent for each $x \neq 0$, and for $1 < \alpha < 2$ may be justified as an asymptotic expansion as $x \to \infty$. For more on this see Zolotarev (1986). Moreover, for all $x \in \Re$

$$s(x;\alpha) \le s(0;\alpha) = \frac{1}{\pi}\Gamma\left(1+\frac{1}{\alpha}\right).$$

3. DISTRIBUTIONS GENERATED BY SYMMETRIC STABLE LAWS

There are different methods for constructing of parametric families of discrete distributions arising in Large-Scale Biomolecular Systems (see, for example, Kuznetsov, 2003; Danielian and Astola, 2004, 2006). One of them is based on *discretization* of *Stable Densities* (see, for example, Astola *et al.*, 2010). Let us consider the following class of the DSSD constructed from (1).

For $0 < \varepsilon \le \alpha < 1$ or $1 < \alpha < 2$, and $\beta = 0$ we have (see, for example, Farbod, 2011):

$$p(x;\alpha) = c_\alpha^{-1} s(x;\alpha) \qquad x = 0,1,2,..., \tag{2}$$

where $c_\alpha = \sum_{y=0}^{\infty} s(y;\alpha)$.

The main goal of the present paper is to consider some functions of parameters involved (2), and then to obtain the asymptotic confidence intervals of the *ML Estimators* for this functions.

3.1. *ML Estimator*

Let $X^n = (X_1,...,X_n)$, with realization $x^n = (x_1,...,x_n)$, be a sample from (2). Let us assume the following regularity conditions (compare to Borovkov, 1998):

1. $\overline{D} \subset \Theta_\varepsilon$ is a compact set of $\Re$;
2. $p(x;\alpha_1) \neq p(x;\alpha_2)$ for all $\alpha_1 \neq \alpha_2$ where $\alpha_1, \alpha_2 \in \Theta_\varepsilon$;
3. Probability distributions P_α with probability mass functions as in (2) have a common support, i.e. the set

Support $\mathrm{P}_\alpha = \{x : p(x;\alpha) > 0\}$,

does not depend on $\alpha \in \Theta_\varepsilon$;

4. The function $l(x;\alpha) \equiv \ln p(x;\alpha)$ is twice continuously differentiable with respect to α for all $x = 0,1,2,...$ In addition,

$$\left| \frac{\partial^2 l(x;\alpha)}{\partial \alpha^2} \right| \leq M(x),$$

where $E_\alpha[M(X_1)] < \infty$;

5. For all $\alpha \in \Theta_\varepsilon$ the Fisher's information quality $I(\alpha)$ contained in observation X_1, satisfies to the following condition

$$0 < I(\alpha) = E_\alpha \left[\frac{\partial l(X_1;\alpha)}{\partial \alpha} \right]^2 = -E_\alpha \left[\frac{\partial^2 l(X_1;\alpha)}{\partial \alpha^2} \right] < \infty,$$

and is continuous in α.

It was proved (Farbod, 2007) that the regularity conditions mentioned above are satisfied by the model (2). Under this conditions, we have the following well-known Theorem (Borovkov, 1998; Lehmann, 1983) about asymptotic behavior of the *ML Estimators*.

Theorem 3.1: Suppose that the regularity conditions 1-5 are fulfilled. Then the likelihood equation

$$\frac{\partial \ln L(x^n;\alpha)}{\partial \alpha} = 0,$$

with $L(X^n;\alpha)=\prod_{i=1}^n p(X_i;\alpha)$ has a unique solution $\hat{\alpha}_n=\hat{\alpha}_n(X^n)$ in D. This solution is a ML Estimator for α and has the following properties:

(I) Consistency, i.e.

$$\hat{\alpha}_n \xrightarrow{\mathrm{P}_\alpha} \alpha,\ (\mathrm{P}_\alpha(|\hat{\alpha}_n-\alpha|>\varepsilon)\to 0 \quad \forall\varepsilon>0 \ as\ n\to\infty)\cdot$$

(II) Asymptotic normality, i.e.

$$\hat{u}_n \equiv \sqrt{n}(\hat{\alpha}_n-\alpha)\xrightarrow{d}\xi\in N(0,\sigma^2(\alpha)). \tag{3}$$

(III) Asymptotic efficiency: the asymptotic variance $\sigma^2(\alpha)$ in (3) is $\sigma^2(\alpha)=I^{-1}(\alpha)$, such that

$$\hat{u}_n \equiv \sqrt{n}(\hat{\alpha}_n-\alpha)\xrightarrow{d}\xi\in N(0,I^{-1}(\alpha)).$$

(IV) Convergence of moments:

$$E_\alpha[\hat{u}_n^k]\to E_\alpha[\xi^k] \quad \forall k\geq 1. \tag{4}$$

From (4), when $k=1$, the property of asymptotic unbiasedness also holds, that is

$$E_\alpha[\hat{\alpha}_n]=\alpha+o(n^{-\frac{1}{2}}).$$

(V) If $h(t)$ is a some differentiable function on $\Re$ such that $h'(\alpha)\neq 0$, then:

$$\sqrt{n}(h(\hat{\alpha}_n)-h(\alpha))\xrightarrow{d}\xi\in N\left(0,\frac{[h'(\alpha)]^2}{I(\alpha)}\right). \tag{5}$$

Remark 3.1: Substituting $k=2$ into (4) we have

$$E_\alpha(\hat{\alpha}_n-\alpha)^2=\frac{1+o(1)}{nI(\alpha)}.$$

Also, (5) can be represented as

$$E_\alpha(h(\hat{\alpha}_n)-h(\alpha))^2=\frac{[h'(\alpha)]^2}{n\,I(\alpha)}\cdot(1+o(1)).$$

Remark 3.2: The estimator $\hat{\alpha}_n$ is asymptotically optimal (see, for instance, Lehmann, 1983), that is for all asymptotically normal α^* estimators which satisfy condition (II) with asymptotic variance $(\sigma^*(\alpha))^2$, the following inequality holds:

$$I^{-1}(\alpha)=\lim_{n\to\infty}E_\alpha[n(\hat{\alpha}_n-\alpha)^2]\leq\lim_{n\to\infty}E_\alpha[n(\alpha_n^*-\alpha)^2]=(\sigma^*(\alpha))^2.$$

4. THE CONFIDENCE INTERVALS OF PARAMETRIC FUNCTIONS

We consider the questions about estimators and their properties for some "useful" functions of the index α. Before that, let us prove the following Lemma:

Lemma 4.1: The function $I(\alpha)$ of the model (2) admits the following representation

$$I(\alpha) = Var_\alpha\{U(s(X_1;\alpha))\}, \qquad \text{(Var means Variance)},$$

where $U(s(X_1;\alpha)) = (\ln s(X_1;\alpha))' = \dfrac{s'(X_1;\alpha)}{s(X_1;\alpha)}$, $\left(s'(X_1;\alpha) = \dfrac{\partial s(X_1;\alpha)}{\partial\alpha}\right)$, is a contribution function of X_1 for the stable law (1), and $0 < I(\alpha) < \infty$.

Proof. It is readily seen that

$$U(p(x;\alpha)) = (\ln p(x;\alpha))' = \frac{s'(x;\alpha)}{s(x;\alpha)} - \frac{(c_\alpha)'}{c_\alpha}, \qquad (c_\alpha)' = \frac{\partial(c_\alpha)}{\partial\alpha},$$

such that keeping in mind the representation

$$E_\alpha[U(s(X_1;\alpha))] = \sum_{x=0}^{\infty}\frac{s'(x;\alpha)}{c_\alpha} = (c_\alpha)^{-1}\cdot(c_\alpha)'.$$

we have (compare to Farbod, 2011)

$$I(\alpha) = E_\alpha[U^2(p(X_1;\alpha))] = E_\alpha[U^2(s(X_1;\alpha))] - \left[\frac{(c_\alpha)'}{c_\alpha}\right]^2 = Var_\alpha\{U(s(X_1;\alpha))\}.$$

It is obvious $I(\alpha) > 0.$ To prove $I(\alpha) = Var_\alpha\{U(s(X_1;\alpha))\} < \infty,$ it suffices to show that

$$E_\alpha[U^2(s(X_1;\alpha))] = \frac{1}{c_\alpha}\cdot\sum_{x=0}^{\infty}\frac{(s'(x;\alpha))^2}{s(x;\alpha)} < \infty,$$

which is met obviously. The Lemma 4.1 is proved.

Let us now find the asymptotic confidence intervals for parametric function $h(\alpha)$ of α:

Theorem 4.1: If the regularity conditions 1-5 are satisfied for the model (2) and $h(t)$ is some differentiable function on Θ_ε such that $h'(\alpha) \neq 0$ (α is a true value of parameter), then the η-level ($0<\eta<1$) asymptotic confidence interval for $h(\alpha)$ is

$$h(\hat{\alpha}_n) \mp \frac{h'(\hat{\alpha}_n)}{\sqrt{n\, I(\hat{\alpha}_n)}} \cdot z_{\frac{\eta}{2}},$$

where $z_{\frac{\eta}{2}}$ is $\frac{\eta}{2}$-level critical point for standard normal law.

Proof. From property (V) of Theorem 3.1 one can obtain that if $n \to \infty$

$$\frac{\sqrt{n\, I(\alpha)}}{h'(\alpha)} \cdot (h(\hat{\alpha}_n) - h(\alpha)) \xrightarrow{d} N(0,1).$$

Then, by the Continuity Theorems (Borovkov, 1998) we have

$$\frac{\sqrt{n\, I(\hat{\alpha}_n)}}{h'(\hat{\alpha}_n)} \cdot (h(\hat{\alpha}_n) - h(\alpha)) \xrightarrow{d} N(0,1),$$

such that

$$P_\alpha\left(h(\hat{\alpha}_n) - \frac{h'(\hat{\alpha}_n)}{\sqrt{n\, I(\hat{\alpha}_n)}} \cdot z_{\frac{\eta}{2}} \le h(\alpha) \le h(\hat{\alpha}_n) + \frac{h'(\hat{\alpha}_n)}{\sqrt{n\, I(\hat{\alpha}_n)}} \cdot z_{\frac{\eta}{2}} \right) \to 1 - \eta,$$

when $n \to \infty$. The proof is complete.

We consider some useful parametric functions:

Example 1. Let $h(\alpha) = I(\alpha)$. Then by Theorem 4.1 we have the following $\eta-$level asymptotic confidence interval for $I(\alpha)$:

$$P_\alpha\left(I(\alpha) \in (I(\hat{\alpha}_n) \mp \frac{I'(\hat{\alpha}_n)}{\sqrt{n\, I(\hat{\alpha}_n)}} \cdot z_{\frac{\eta}{2}}) \right) \to 1 - \eta,$$

where (see Lemma 4.1)

$$I'(\alpha) = 2\, Cov_\alpha[U(s(X_1;\alpha)), U'(s(X_1;\alpha))]. \qquad \text{(Cov means Covariance).}$$

Example 2. Let $h_x(\alpha) = p(x;\alpha) = \dfrac{s(x;\alpha)}{c_\alpha}$, where $x \in N \cup \{0\}$ is fixed. Again using Theorem 4.1 we obtain:

$$P_\alpha\left(p(x;\alpha) \in (p(x;\hat{\alpha}_n) \mp \frac{p'(x;\hat{\alpha}_n)}{\sqrt{n\, I(\hat{\alpha}_n)}} \cdot z_{\frac{\eta}{2}}) \right) \to 1 - \eta,$$

where

$$p'(x;\alpha)=\frac{\partial p(x;\alpha)}{\partial\alpha}$$

$$=\frac{1}{c_\alpha}\left[s'(x;\alpha)-\frac{(c_\alpha)'}{c_\alpha}\,s(x;\alpha)\right]$$

$$=U(s(x;\alpha))\cdot p(x;\alpha)-p(x;\alpha)\cdot E_\alpha[U(s(X_1;\alpha))]$$

$$=p(x;\alpha)\cdot\{U(s(x;\alpha))-E_\alpha[U(s(X_1;\alpha))]\}.$$

Example 3. Set now $h_t(\alpha)=\overline{F}_\alpha(t)\equiv 1-F_\alpha(t)=\sum_{x=t}^{\infty}p(x;\alpha)$, where $t\in(0,\infty)$. From Theorem 4.1 we have:

$$P_\alpha\left(\overline{F}_\alpha(t)\in(\overline{F}_{\hat\alpha_n}(t)\mp\frac{\overline{F}'_{\hat\alpha_n}(t)}{\sqrt{n\,I(\hat\alpha_n)}}\cdot z_{\frac{\eta}{2}})\right)\to 1-\eta,$$

where

$$\overline{F}'_\alpha(t)=\sum_{x=t}^{\infty}p'(x;\alpha)$$

$$=\sum_{x=t}^{\infty}\frac{s'(x;\alpha)}{s(x;\alpha)}\,p(x;\alpha)-\sum_{x=t}^{\infty}\frac{(c_\alpha)'}{c_\alpha}\,p(x;\alpha)$$

$$=E_\alpha[U(s(X_1;\alpha))]\cdot 1_{X_1\ge t}-E_\alpha[U(s(X_1;\alpha))]\cdot\overline{F}_\alpha(t)$$

$$=(1_{X_1\ge t}-\overline{F}_\alpha(t))\cdot E_\alpha[U(s(X_1;\alpha))].$$

(Here $1_{X_1\ge t}=\begin{cases}1 & \text{if } X_1\ge t,\\ 0 & \text{if } X_1<t.\end{cases}$).

Example 4. Assuming $h(\alpha)=c_\alpha=\sum_{y=0}^{\infty}s(y;\alpha)$. By Theorem 4.1 we have:

$$P_\alpha\left(c_{\hat\alpha_n}-\frac{(c_{\hat\alpha_n})'}{\sqrt{n\,I(\hat\alpha_n)}}\cdot z_{\frac{\eta}{2}}<c_\alpha<c_{\hat\alpha_n}+\frac{(c_{\hat\alpha_n})'}{\sqrt{n\,I(\hat\alpha_n)}}\cdot z_{\frac{\eta}{2}}\right)\to 1-\eta,$$

where

$$(c_\alpha)' = c_\alpha \cdot \sum_{x=0}^{\infty} \frac{s'(x;\alpha)}{s(x;\alpha)} p(x;\alpha) = c_\alpha \cdot E_\alpha[U(s(X_1;\alpha))].$$

Example 5. Let us have $h(\alpha) = \frac{(c_\alpha)'}{c_\alpha}$. Using Theorem 4.1 we have:

$$P_\alpha\left((\ln c_{\hat{\alpha}_n})' - \frac{(\ln c_{\hat{\alpha}_n})''}{\sqrt{n\, I(\hat{\alpha}_n)}} \cdot z_{\frac{\eta}{2}} < (\ln c_{\alpha_n})' < (\ln c_{\hat{\alpha}_n})' + \frac{(\ln c_{\hat{\alpha}_n})''}{\sqrt{n\, I(\hat{\alpha}_n)}} \cdot z_{\frac{\eta}{2}} \right) \to 1 - \eta,$$

where

$$(\ln c_\alpha)'' = \left(\frac{(c_\alpha)'}{c_\alpha} \right)' = E_\alpha[U'(s(X_1;\alpha))].$$

5. CONCLUSIONS

In this paper, the asymptotic confidence interval for the parametric function $h(\alpha)$ has been obtained (Theorem 4.1). We also considered some useful parametric functions of the index parameter α (Examples 1-5). With the help of Theorem 3.1, Lemma 4.1 and Theorem 4.1 the asymptotic confidence intervals for such parametric functions have been proposed.

This theoretical results may be applied for obtaining the respective statistical inferences in *Bioinformatics*, and in other theories where the *Stable Laws* are used, for example, in *Financial Mathematics* and *Economics*.

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Department of Mathematics,
Quchan Institute of Engineering and Technology,
Quchan, Razavi Khorasan Province, Iran
DAVOOD FARBOD

Chair of Probability Theory and Mathematical Statistics,
Yerevan State University, Yerevan, Armenia
KAREN V. GASPARIAN

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SUMMARY

On the confidence intervals of parametric functions for Distributions Generated by Symmetric Stable Laws

In this paper we consider "Discrete Distributions Generated by Standard Symmetric Stable Densities" (DSSD in short) arising in Bioinformatics (Astola and Danielian, 2007). Using well-known asymptotic properties of the maximum likelihood (ML) estimators we obtain the respective asymptotic confidence intervals for some useful parametric functions of the DSSD.