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ON THE RELATIONSHIP BETWEEN PARTIAL SUFFICIENCY, INVARIANCE AND CONDITIONAL INDEPENDENCE (*)

J. Montanero, A. G. Nogales, J. A. Oyola, P. Pérez

1. INTRODUCTION AND MATHEMATICAL DEFINITIONS

Conditional independence is a well known and very useful concept, not only in probability theory, but also in the theory of statistical inference, where it can be used as a basic tool to express many of the important concepts of statistics (such as sufficiency, ancillarity, adequacy, etc), given an unified treatment to many areas that are, at first sight, different. We refer the reader to Dawid (1979), where an excellent justification of these statements can be found. A pioneer work in the use of conditional independence in statistical theory is Hall *et al.* (1965) in the study of the relationship between sufficiency and invariance; see also Nogales and Oyola (1996), where the Lemma 3.3 of Hall *et al.* (1965) on conditional independence adopts a more appropriate formulation. Nevertheless, the main result of Hall *et al.* (1965) is the Theorem 3.1, that they called Stein theorem.

We also refer to Florens *et al.* (1990) where an extensive use of conditional independence is made in a Bayesian context. The four references just cited contain many examples about the use of conditional independence in statistical theory, in general, and in the relationship of sufficiency and invariance, in particular.

Recall that two well known data reduction methods in the exact theory of statistical inference are sufficiency – where no information is lost – and invariance – where the loss of information is justified by an argument of symmetry –. While the paper Hall *et al.* (1965) deals with the relationship between sufficiency and invariance, Berk (1972) solves the analogous problem for the almost invariant case; we also refer to Berk *et al.* (1996), where some remarks on their relationship with conditional independence are given.

In this paper, a similar study is given for the concept of partial sufficiency. While sufficiency is usually recognized as the main contribution of Fisher to theoretical statistics, the appealing intuitive concept of partial sufficient statistic (a statistic keeping all the relevant information about a subparameter) is more elusive, as Fisher himself pointed out. The problem of fixing a natural mathematical defi-

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nition of partial sufficiency in a classical setting has been considered in a large number of famous papers, but a satisfactory and definitive solution is not available.

This paper follows the Fraser approach that the interested reader can find in Basu (1978), where a fairly detailed review of the usefulness of partial sufficiency in statistical theory, together with other approaches to this concept (like H-sufficiency of Hájek, Q-sufficiency and L-sufficiency of Raiffa and Schlaifer, or the related concept of Bandorff-cut introduced in Bandorff-Nielsen (1973)), is given. We can also find several illustrative examples and some justifications via Rao-Blackwell type theorems. Godambe (1980) introduces the concept of sufficiency for θ ignoring ϕ, and studies its relationship with Fisher's information; this problem is also considered in Kung-Yee Liang (1983), in Rémon (1984), where partial sufficiency also is named L-sufficiency, and, in a different framework, in Kabaila (1998). See also Zhu and Reid (1994), where a notion of partial sufficiency (named P-sufficiency and including the definitions of Fraser (1956) and Bhapkar (1991)) based on partial information is presented. Finally, the reader is referred to Montanero *et al.* (2003), where the theorem of Stein for partial sufficiency is obtained.

Let us fix the notations to be used throughout the paper. $(\Omega,\mathcal{A},\mathcal{P})$ will be a statistical experiment, *i.e.*, $(\Omega,\mathcal{A})$ is a measurable space and $\mathcal{P}$ a family of probability measures on $(\Omega,\mathcal{A})$.

Usually, we shall suppose the family $\mathcal{P}$ written in the form

$$\mathcal{P} = \{P_{\theta,\phi} : (\theta,\phi) \in \Theta \times \Phi\}, \tag{1}$$

where Θ and Φ are nonempty sets. θ will be considered as the parameter of interest, while ϕ remains as a nuisance parameter. The family $\mathcal{P}$ will be supposed identifiable, in the sense that $P_{\theta,\phi} \neq P_{\theta',\phi'}$ if $(\theta,\phi) \neq (\theta',\phi')$.

Given, $\phi \in \Phi$ we shall write $\mathcal{P}_\phi = \{P_{\theta,\phi} : \theta \in \Theta\}$; $\mathcal{N}$ (resp., $\mathcal{N}_\phi$) will denote the family of the $\mathcal{P}$-null (resp., $\mathcal{P}_\phi$-null) events. For two statistics, f and g, we shall write $f \sim g$ (resp., $\overset{\phi}{\sim}$) if $\{f \neq g\}$ belongs to $\mathcal{N}$ (resp., $\mathcal{N}_\phi$). In this case, f and g are said to be $\mathcal{P}$-equivalent (resp., $\mathcal{P}_\phi$-equivalent). For a sub-σ-field $\mathcal{B} \subset \mathcal{A}$, $[\mathcal{B}]$ (resp., $[\mathcal{B}]^+$) will denote the class of the $\mathcal{B}$-measurable (resp., $\mathcal{B}$-measurable and non negative) functions, and we shall write $\overline{\mathcal{B}}$ for the completion of $\mathcal{B}$ with the $\mathcal{P}$-null sets. Given two sub-σ-fields $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{A}$, we say that both are equivalent (we write $\mathcal{C} \sim \mathcal{D}$) when $\overline{\mathcal{C}} = \overline{\mathcal{D}}$.

Recall that a sub-σ-field $\mathcal{B}$ of $\mathcal{A}$ is said to be sufficient when, for all $A \in \mathcal{A}$, $\bigcap_{P\in P} P(A|\mathcal{B}) \neq \emptyset$, where $P(A|\mathcal{B})$ denotes the conditional probability of A given $\mathcal{B}$, i.e., the class of all real functions $f \in [\mathcal{B}]$ such that $E_P(I_A \cdot f) = P(A \cap B)$, for all $B \in \mathcal{B}$. Writing the family $\mathcal{P}$ as in (1), $\mathcal{B}$ is said to be θ-oriented when the restriction $P^{\mathcal{B}}_{\theta,\phi}$ of the probability $P_{\theta,\phi}$ to $\mathcal{B}$ does not depend on ϕ. $\mathcal{B}$ is said to be specific θ-sufficient if it is sufficient for the statistical experiments $(\Omega,\mathcal{A},\mathcal{P}_\phi)$, $\phi\in\Phi$. $\mathcal{B}$ is said to be partially θ-sufficient (in the sense of Fraser, 1956) if it is θ-oriented and specific θ-sufficient.

A transformation on a set Ω is a bijective map from Ω onto itself. We say that a group G of bimeasurable transformations on $(\Omega,\mathcal{A})$ leaves invariant the statistical experiment $(\Omega,\mathcal{A},\mathcal{P}_\phi)$, when, for all $g\in G$ and $P\in\mathcal{P}$, the probability distribution P^g of g with respect to P lies in $\mathcal{P}$. Thus, for every transformation g, there exists a bijective map $\overline{g}:\mathcal{P}\to\mathcal{P}$ such that $P^g=\overline{g}(P)$, for all $P\in\mathcal{P}$. A statistic $f:(\Omega,\mathcal{A},\mathcal{P})\to(\Omega',\mathcal{A}')$ is said to be G-invariant if $f\circ g=f$, for all $g\in G$; f is said to be almost G-invariant (resp., ϕ-almost G-invariant for a given $\phi\in\Phi$) if $f\sim f\circ g$ (resp., $f\overset{\phi}{\sim} f\circ g$), for all $g\in G$. An event $A\in\mathcal{A}$ is said to be G-invariant, almost G-invariant or ϕ-almost G-invariant when so is its indicator function I_A. In the next, $\mathcal{A}_I$, $\mathcal{A}_A$ and $\mathcal{A}^\phi_A$ will denote, respectively, the σ-fields of the G-invariant almost G-invariant or ϕ-almost G-invariant events. Obviously, every G-invariant, almost G-invariant or ϕ-almost G-invariant statistic is $\mathcal{A}_I$-measurable, $\mathcal{A}_A$-measurable or $\mathcal{A}^\phi_A$-measurable, resp.; we can find in Florens *et al.* (1990) some conditions of regularity under which the converse implications are also true. A sub-σ-field $\mathcal{B}$ is said to be stable (resp., essentially stable) when $g\mathcal{B}=\mathcal{B}$ (resp., $g\mathcal{B}\sim\mathcal{B}$), for all $g\in G$.

Let us also recall the concept of conditional independence. Given three sub-σ-fields $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\subset\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be conditionally independent given $\mathcal{A}_3$ (we write $\mathcal{A}_1\perp\mathcal{A}_2\mid\mathcal{A}_3$) when, for every $A_1\in\mathcal{A}_1$, $A_2\in\mathcal{A}_2$ and $P\in\mathcal{P}$, $P(A_1\cap A_2\mid\mathcal{A}_3)=P(A_1\mid\mathcal{A}_3)\cdot P(A_2\mid\mathcal{A}_3)$; it is well known that this is equivalent to the fact that, for all $A_1\in\mathcal{A}_1$, and $P\in\mathcal{P}$, $P(A_1\mid\mathcal{A}_3)\cap P(A_1\mid\mathcal{A}_2\vee\mathcal{A}_3)\neq\emptyset$, where $\mathcal{A}_2\vee\mathcal{A}_3$ is the least σ-field containing $\mathcal{A}_2$ and $\mathcal{A}_3$.

The relationship between invariance and sufficiency is studied in Hall et al. (1965), whose main theorem, attributed to Stein, is the following: "Given a statistical experiment which remains invariant under the action of a group G and a sufficient and essentially G-stable σ-field $\mathcal{B}$ such that $\mathcal{B}\cap\mathcal{A}_A\sim\mathcal{B}\cap\mathcal{A}_I$, the σ-field $\mathcal{B}\cap\mathcal{A}_I$ is sufficient for $\mathcal{A}_I$". The main part of their paper deals with the four propositions below:

(SI1) For all $A\in\mathcal{A}_I$, there exists an invariant version in $\cap_{P\in P}P(A\mid\mathcal{B})$.

(SI2) $\mathcal{B}\perp\mathcal{A}_I\mid\mathcal{B}\cap\mathcal{A}_I$.

(SI3) $\mathcal{B} \cap \mathcal{A}_I$ is sufficient for $\mathcal{A}_I$.

(SI4) For all $A \in \mathcal{A}_I$, there exists an almost-invariant version in $\cap_{P\in\mathcal{A}} P(A|\mathcal{B})$.

Although it is asserted in Hall *et al.* (1965) that (SI2) implies (SI3), it is shown in Nogales and Oyola (1996) that it is not true, and that the relationship between these propositions are the following: (SI1) $\Leftrightarrow$ (SI2) + (SI3) and (SI2) $\Rightarrow$ (SI4).

If the principle of invariance is understood as a reduction to the σ-field $\mathcal{A}_A$ of the almost invariant events, the following result of Berk (1972) (see the discussion after Lemma 3; see also Nogales and Oyola (1996, p. 907, Remark 1(v))) solves the main problem considered in Hall *et al.* (1965):

"If $\mathcal{B}$ is sufficient and essentially stable, $\mathcal{B} \cap \mathcal{A}_A$ is sufficient for $\mathcal{A}_A$".

2. PARTIAL SUFFICIENCY AND INVARIANCE

In this section we are interested in the relationship between partial sufficiency and invariance. First, we consider the following propositions, the partial sufficiency analogue of propositions (SI1), (SI2), (SI3) and (SI4). $\mathcal{B}$ will be a partially sufficient σ-field.

(PSI1) For all $\phi \in \Phi$ and $A \in \mathcal{A}_I$, there exists an invariant version in $\cap_{\theta\in\Theta} P_{\theta,\phi}(A|\mathcal{B})$.

(PSI2) $\mathcal{B} \perp \mathcal{A}_I \mid \mathcal{B} \cap \mathcal{A}_I$.

(PSI3) $\mathcal{B} \cap \mathcal{A}_I$ is partially θ-sufficient for $\mathcal{A}_I$.

(PSI4) For all $\phi \in \Phi$ and $A \in \mathcal{A}_I$, there exists a ϕ-almost-invariant version in $\cap_{\theta\in\Theta} P_{\theta,\phi}(A|\mathcal{B})$.

Equivalent formulations of propositions (PSI2) and (PSI4) are:

(PSI2) $\forall(\theta,\phi) \in \Theta\times\Phi$, $\forall A \in \mathcal{A}_I$, $P_{\theta,\phi}(A|\mathcal{B}\cap\mathcal{A}_I) \subset P_{\theta,\phi}(A|\mathcal{B})$.

(PSI4) $\forall\phi \in \Phi$, $\forall A \in \mathcal{A}_I$, $\cap_{\theta\in\Theta} P_{\theta,\phi}(A|\mathcal{B}) \subset [\mathcal{A}_A^\phi]^+$.

It is our aim to study the relationship between these four propositions. We shall need the following lemma, whose proof can be found in Montanero *et al.* (2003).

Lemma 1. If $\mathcal{B}$ is a σ-field θ-oriented and essentially G-stable, then $\mathcal{B} \cap \mathcal{A}_A^\phi = \mathcal{B} \cap \mathcal{A}_A$, for all $\phi \in \Phi$.

The following proposition is a direct application of the result of Berk cited above to the statistical experiment $(\Omega, \mathcal{A}, \mathcal{P}_\phi)$.

Proposition 2. Let $\mathcal{B}$ be a specific θ-sufficient and essentially stable σ-field. If G leaves invariant every family $\mathcal{P}_\phi$, $\phi \in \Phi$, then (PSI4) holds.

Now, we are ready to obtain the main result of the paper. See Nogales and Oyola (1996) for a similar result for sufficiency.

Theorem 3. If $\mathcal{B}$ is partial sufficient and G leaves invariant every family $\mathcal{P}_\phi$, $\phi \in \Phi$, then

(i) (PSI1) $\Leftrightarrow$ (PSI2) + (PSI3)
(ii) (PSI2) $\Rightarrow$ (PSI4).

Proof. (i) Given $\phi \in \Phi$ and $A \in \mathcal{A}_I$, let be p_A^ϕ an invariant statistic in $\cap_{\theta\in\Theta} P_{\theta,\phi}(A \mid \mathcal{B})$. (PSI2) follows immediately from this. Moreover, $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A \mid \mathcal{B} \cap \mathcal{A}_I)$, and (PSI3) also holds.

For the converse, choose $\phi \in \Phi$ and $A \in \mathcal{A}_I$. By (PSI3), there exists a statistic $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A \mid \mathcal{B} \cap \mathcal{A}_I)$. Then, $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A \mid \mathcal{B})$ follows from (PSI2). This gives the proof as p_A^ϕ is invariant.

(ii) Let $\phi \in \Phi$ and $A \in \mathcal{A}_I$. Since $\mathcal{B}$ is specific θ-sufficient, there exists $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A \mid \mathcal{B})$. Given $g \in G$ and $\theta \in \Theta$ let $f_A^{\theta,\phi}$ and $f_A^{\bar{g}(\theta,\phi)}$ be versions of $P_{\theta,\phi}(A \mid \mathcal{B} \cap \mathcal{A}_I)$ and $P_{\theta,\phi}^g(A \mid \mathcal{B} \cap \mathcal{A}_I)$, respectively. Then, we have that

$$\begin{aligned} P_{\theta,\phi}(\{p_A^\phi \circ g \neq p_A^\phi\}) \leq\ & P_{\theta,\phi}(\{p_A^\phi \circ g \neq f_A^{\bar{g}(\theta,\phi)} \circ g\}) \\ & + P_{\theta,\phi}(\{f_A^{\bar{g}(\theta,\phi)} \circ g \neq f_A^{\bar{g}(\theta,\phi)}\}) \\ & + P_{\theta,\phi}(\{f_A^{\bar{g}(\theta,\phi)} \neq f_A^{\theta,\phi}\}) \\ & + P_{\theta,\phi}(\{f_A^{\theta,\phi} \neq p_A^\phi\}) \end{aligned} \tag{2}$$

By hypothesis, G leaves invariant the family $\mathcal{P}_\phi$. So, there exists $\bar{g}_1(\theta) \in \Theta$ such that $\bar{g}_1(\theta,\phi) = (\bar{g}_1(\theta),\phi)$. Then, by (PSI2) we have that $f_A^{\theta,\phi} \in P_{\theta,\phi}(A \mid \mathcal{B})$ and $f_A^{\bar{g}(\theta,\phi)} \in P_{\bar{g}_1(\theta),\phi}(A \mid \mathcal{B})$. So,

$$P_{\theta,\phi}(\{p_A^\phi \circ g \neq f_A^{\bar{g}(\theta,\phi)} \circ g\}) = P_{\bar{g}_1(\theta),\phi}(\{p_A^\phi \neq f_A^{\bar{g}(\theta,\phi)}\}) = 0$$

Analogously,

$$P_{\theta,\phi}(\{f_A^{\theta,\phi} \neq p_A^\phi\}) = 0$$

The second term of the sum in the right-hand side of (2) is also null, because $f_A^{\bar{g}(\theta,\phi)}$ is $\mathcal{A}_I$ -measurable. Finally, if $B \in \mathcal{B} \cap \mathcal{A}_I$ then

$$\begin{aligned}\int_B I_A \, dP_{\theta,\phi} &= \int_{gB} I_A \circ g^{-1} \; dP^g_{\theta,\phi} = \int_B I_A \; dP^g_{\theta,\phi} \\ &= \int_B f_A^{\bar{g}(\theta,\phi)} \; dP^g_{\theta,\phi} = \int_{g^{-1}(B)} f_A^{\bar{g}(\theta,\phi)} \circ g \; dP_{\theta,\phi} \\ &= \int_B f_A^{\bar{g}(\theta,\phi)} \; dP_{\theta,\phi}\end{aligned} \tag{3}$$

Thus, $f_A^{\bar{g}(\theta,\phi)} \in P_{\theta,\phi}(A \mid \mathcal{B} \cap \mathcal{A}_I)$, and the third term is 0. So, we have shown that p_A^ϕ is ϕ-almost-invariant.

Remark 4. As a consequence of the previous result we get the version for σ-fields of the Stein theorem for partial sufficiency, that the reader can find in Montanero *et al.* (2003). Namely, if G leaves invariant every family $\mathcal{P}_\phi$ and $\mathcal{B}$ is a partially θ-sufficient and essentially stable σ-field such that $\mathcal{B} \cap \mathcal{A}_A \sim \mathcal{B} \cap \mathcal{A}_I$, then the propositions (PSI1), (PSI2), (PSI3) and (PSI4) hold.

The paper Nogales and Oyola (1996) contains an example showing that (SI2) does not imply (SI3), as it is assured in Hall *et al.* (1965). The following proposition shows how this example can be adapted to prove that (PSI2) does not imply (PSI3).

Proposition 5. There exist a statistical experiment $(\Omega, \mathcal{A}, \{P_{\theta,\phi} : (\theta,\phi) \in \Theta \times \Phi\})$, a group of bimeasurable transformations leaving invariant every family $\mathcal{P}_\phi$ and a partially θ-sufficient σ-field $\mathcal{B}$ satisfying (PSI2) but not (PSI3).

Proof. Let $(\Omega_1, \mathcal{A}_1, \{P_\theta : \theta \in \Theta\})$ be a statistical experiment, G_1 be a group of bimeasurable transformations leaving it invariant and $\mathcal{B} \subset \mathcal{A}_1$ be a sufficient σ-field such that (SI2) holds and (SI3) does not hold (see Nogales and Oyola (1996) for such an example).

Let us consider the statistical experiment

$$(\Omega_1 \times \mathbb{N}, \mathcal{A}_1 \times \mathsf{P}(\mathbb{N}), \{P_\theta \times \varepsilon_n : \; (\theta, n) \in \Theta \times \mathbb{N}\})$$

and the group $G = \{(g_1, id_{\mathbb{N}}) : \; g_1 \in G_1\}$ where ε_n is the probability distribution degenerated at n and $id_{\mathbb{N}}$ denotes the identity map on $\mathbb{N}$. It is clear that G leaves invariant each family $\{P_\theta \times \varepsilon_n : \; \theta \in \Theta\}, n \in \mathbb{N}$. We shall show the σ-field $\mathcal{B} \times \{\emptyset, \mathbb{N}\}$ is partially θ-sufficient and (PSI2) holds for it, but (PSI3) doesn't.

First, we note that $\mathcal{A}_I = \mathcal{A}_I^1 \times \mathsf{P}(\mathbb{N})$, $\mathcal{A}_I^1$ and $\mathcal{A}_I$ being the σ-fields of the G_1-invariant and G-invariant events, respectively. Indeed, given $n \in \mathbb{N}$, $A \in \mathcal{A}_I$ and

$g_1 \in G_1$, if $g := (g_1, id_{\mathbb{N}})$ and $A_n := \{\omega \in \Omega_1 : (\omega, n) \in A\}$, then $(gA)_n = g_1 A_n$; so

$$\bigcup_{n\in\mathbb{N}} g_1 A_n \times \{n\} = \bigcup_{n\in\mathbb{N}} (gA)_n \times \{n\} = gA = A = \bigcup_{n\in\mathbb{N}} A_n \times \{n\},$$

which proves that $g_1(A_n) = A_n$, for all $n \in \mathbb{N}$, and hence, $A \in \mathcal{A}_I^1 \times \mathsf{P}(\mathbb{N})$. The inclusion $\mathcal{A}_I^1 \times \mathsf{P}(\mathbb{N}) \subset \mathcal{A}_I$ is obvious.

Now, fix $(\theta, n) \in \Theta \times \mathbb{N}$, $A_1 \in \mathcal{A}_I^1$ and $N \in \mathsf{P}(\mathbb{N})$. From (SI2), the existence of a $\mathcal{A}_I$-measurable statistic $p_{A_1}^{\theta}$ in $P_\theta(A|\mathcal{B})$ follows. Since the map

$$p_{A_1\times N}^{\theta,n} : (\omega, n') \in \Omega_1 \times \mathbb{N} \mapsto \varepsilon_n(N) \cdot p_{A_1}^{\theta}(\omega)$$

is a $\mathcal{A}_I$-measurable version of $(P_\theta \times \varepsilon_n)(A_1 \times N \,|\, \mathcal{B} \times \{\emptyset, \mathbb{N}\})$, the theorem of Dynkin proves that, for every $A \in \mathcal{A}_I = \mathcal{A}_I^1 \times \mathsf{P}(\mathbb{N})$, there exists a $\mathcal{A}_I$-measurable version of $(P_\theta \times \varepsilon_n)(A \,|\, \mathcal{B} \times \{\emptyset, \mathbb{N}\})$,. Thus (PSI2) holds.

An analogous argument applied to a version $p_{A_1} \in \cap_{\theta\in\Theta} P_\theta(A_1 \,|\, \mathcal{B})$ shows that $p_{A_1\times N}^{n}(\omega, n') := \varepsilon_n(N) \cdot p_{A_1}^{\theta}(\omega)$ belongs to $\cap_{\theta\in\Theta}(P_\theta \times \varepsilon_n)(A_1 \times N \,|\, \mathcal{B} \times \{\emptyset, \mathbb{N}\})$; hence, $\mathcal{B} \times \{\emptyset, \mathbb{N}\}$ is specific θ-sufficient. Moreover, being also θ-oriented, it is partially θ-sufficient.

Finally, let us see that (PSI3) does not hold: otherwise, given $A_I = \mathcal{A}_I^1$ and $n \in \mathbb{N}$, there would exist

$$q_{A_1}^{n} \in \bigcap_{\theta\in\Theta} (P_\theta \times \varepsilon_n)(A_1 \times \mathbb{N} \,|\, (\mathcal{B} \times \{\emptyset, \mathbb{N}\}) \cap \mathcal{A}_I).$$

Then, being $(\mathcal{B} \times \{\emptyset, \mathbb{N}\}) \cap \mathcal{A}_I = (\mathcal{B} \cap \mathcal{A}_I^1) \times \{\emptyset, \mathbb{N}\}$, there would exist a $(\mathcal{B} \cap \mathcal{A}_I^1)$-measurable statistic $q_{A_1}^{n}$ such that $p_{A_1}^{n}(\omega) = p_{A_1}^{n}(\omega, n')$, for all $\omega \in \Omega_1$ and all $n' \in \mathbb{N}$. It would follow that $p_{A_1}^{n} \in \cap_{\theta\in\Theta} P_\theta(A \,|\, \mathcal{B} \cap \mathcal{A}_I^1)$, which leads to a contradiction, since (SI3) does not hold.

Remark 6. Under the equality $\mathcal{A}_A = \overline{\mathcal{A}}_I$, the propositions (PSI1), (PSI2), (PSI3) and (PSI4) are equivalent. In Lehmann (1986), section 6.5, sufficient conditions are given to get this equality. To illustrate this situation, let us recall an example from Montanero *et al.* (2003). Consider the statistical experiment

$$\left((\mathbb{R}^n)^2, (\mathcal{R}^n)^2, \left\{N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\theta^2 & \psi\\ \psi & \xi^2\end{pmatrix}\right)^n : \theta,\xi>0, \psi\in\mathbb{R}\right\}\right)$$

corresponding to a n sized sample of a bivariate normal distribution with mean 0 and unknown covariance matrix. Setting $\beta = \psi/\theta^2$ and $\sigma^2 = \xi^2 - \psi^2/\theta^2$, define $\phi = (\beta, \sigma^2)$. Let $G = \{g_\Lambda : \Lambda \in O_n\}$, where O_n is the group of orthogonal matrices of order n and $g_\Lambda(x,y) := (\Lambda x, \Lambda y)$, for $(x,y)\in(\mathbb{R}^n)^2$ and $\Lambda\in O_n$. It can be easily checked that G leaves invariant each family $\mathcal{P}_\phi$ and that $\mathcal{R}^n \times \{\varnothing, \mathbb{R}^n\}$ is a G-stable, partially θ-sufficient σ-field. Since all the regularity conditions are satisfied, the propositions (PSI1), (PSI2), (PSI3) and (PSI4) hold.

3. THE ALMOST-INVARIANT CASE

Replacing invariance by almost invariance, the four propositions (PSI1), (PSI2), (PSI3) and (PSI4) become:

(PSA1) For all $\phi\in\Phi$ and $A\in\mathcal{A}_A$, there exists an almost-invariant statistic $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A\,|\,\mathcal{B})$.

(PSA2) $\mathcal{B} \perp \mathcal{A}_A \mid \mathcal{B}\cap\mathcal{A}_A$.

(PSA3) $\mathcal{B}\cap\mathcal{A}_A$ is partially θ-sufficient for $\mathcal{A}_A$.

(PSA4) For all $\phi\in\Phi$ and $A\in\mathcal{A}_A$, there exists an ϕ-almost-invariant statistic $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A\,|\,\mathcal{B})$.

The next theorem states the relationship between them.

Theorem 7. If $\mathcal{B}$ is partially θ-sufficient and G leaves invariant every family $\mathcal{P}_\phi$, then:

(i) (PSA1) $\Leftrightarrow$ (PSA2) + (PSA3).

(ii) (PSA2) $\Rightarrow$ (PSA4).

Proof. (i) Given $\phi\in\Phi$ and $A\in\mathcal{A}_A$, let be p_A^ϕ an almost-invariant statistic in $\cap_{\theta\in\Theta} P_{\theta,\phi}(A\,|\,\mathcal{B})$. (PSA2) follows immediately from this. Moreover, $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A\,|\,\mathcal{B}\cap\mathcal{A}_A)$, and (PSA3) also holds. For the converse, choose $\phi\in\Phi$ and $A\in\mathcal{A}_A$. By (PSA3), there exists a statistic $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A\,|\,\mathcal{B}\cap\mathcal{A}_A)$. From (PSA2), we have $p_A^\phi \in \cap_{\theta\in\Theta} P_{\theta,\phi}(A\,|\,\mathcal{B})$. This gives the proof of (i) since p_A^ϕ is almost-invariant.

(ii) Let $\phi \in \Phi$ and $A \in \mathcal{A}_A$. $\mathcal{B}$ being specific θ-sufficient, there exists $p_A^\phi \in \cap_{\theta \in \Theta} P_{\theta,\phi}(A \mid \mathcal{B})$. Given $g \in G$ and $\theta \in \theta$, let $f_{\mathrm{A}}^{\theta,\phi} \in P_{\theta,\phi}(A \mid \mathcal{B} \cap \mathcal{A}_A)$ and $f_A^{\bar{g}(\theta,\phi)} \in P_{\theta,\phi}^{g}(A \mid \mathcal{B} \cap \mathcal{A}_A)$. As in (2), it can be shown that $P_{\theta,\phi}(\{p_A^\phi \circ g \neq p_A^\phi\}) = 0$, and this shows that p_A^ϕ is ϕ-almost invariant, which gives the proof.

Remark 8. As the previous theorem is the almost-invariant analogue of Theorem 3, we can derive from it an analogue to Remark 4; namely, if G leaves invariant every family $\mathcal{P}_\phi$ and $\mathcal{B}$ is a partially θ-sufficient, and essentially stable σ-field, then the four propositions (PSA1), (PSA2), (PSA3) and (PSA4) hold.

Departmento de Matemáticas
Universidad de Extremadura

JESÚS MONTANERO
AGUSTÍN G. NOGALES
JOSÉ A. OYOLA
PALOMA PÉREZ

REFERENCES

BANDORFF-NIELSEN, O. (1973), *On M-ancillarity*, "Biometrika", 72 (3), pp. 447-455.

BASU, D. (1978), *On the elimination of nuisance parameters*, "Journal of the American Statistical Asociation", 72 (358), pp. 355-366.

BERK, R.H. (1972), *A note on Sufficiency and Invariance*, "Annals of Mathematical Statistics", 39, pp. 647-650.

BERK, R.H., A.G. NOGALES, J.A. OYOLA (1996), *Some counterexamples concerning sufficiency and invariance*, "Annals of Statistics", 24 (2), pp. 902-905.

BHAPKAR, V.P. (1991), *Loss of information in the presence of nuisance parameters and partial sufficiency*, "Journal of Statistical Planning and Inference", 28, pp. 185-203.

DAWID, A.P. (1979), *Conditional independence in statistical theory*, "The Journal of the Royal Statistical Society", Series B 41, pp. 1-31.

FLORENS, J.P., M., MOUCHART, J.M., ROLIN (1990), *Elements of Bayesian Statistics*, Marcel Dekker Inc., New York.

FRASER, D.A.S. (1956), *Sufficient statistics with nuisances parameters*, "Annals of Mathematical Statistics", 27, pp. 838-842.

GHOSH, J.K. (1988), *Statistical Information and Likelihood*. *A collection of critical essays by Dr. D. Basu*, "Lecture Notes in Statistics", 45, Springer Verlag, New York.

GODAMBE, V. (1980), *On sufficiency and ancillarity in the presence of a nuisance parameters*, "Biometrika", 67 (1), pp. 155-162.

HALL, W.J., R.A. WIJSMAN and J.K. GHOSH (1965), *The relationship between sufficiency and invariance with applications in sequential analysis*, "Annals of Mathematical Statistics", 36, 575-614.

KABAILA, P. (1998), *A note on sufficiency and information loss*, "Statistics & Probability Letters", 37, pp. 111-114.

KUNG-YEE LIANG (1983), *On information and ancillarity in the presence of a nuisances parameter*, "Biometrika", 70 (3), pp. 607-612.

MONTANERO, J., NOGALES, A.G., OYOLA, J.A., P. PÉREZ (2003), *On partial sufficiency and invariance*, "Statistica Sinica", 13 (4), pp. 1269-1280.

NOGALES, A.G., J.A. OYOLA (1996), *Some remarks on sufficiency, invariance and conditional independence,* "The Ann. Statist." 24 (2), pp. 906-909.
RÉMON, M. (1984), *On a concept of partial sufficiency: L-sufficiency,* "International Statistical Review", 52 (2), pp. 127-135.
ZHU, Y., REID, N. (1994), *Information, ancillarity, and sufficiency in the presence of nuisance parameters*, "Canadian Journal of Statistics", 22 (1), pp. 111-123.

RIASSUNTO

Sulla relazione fra parziale sufficienza, invarianza e independenza condizionale

Molte proposizioni che appaiono in modo naturale nella letteratura sulla sufficienza e invarianza, includendo la dipendenza condizionale delle sigma-algebra invarianti e sufficienti data la loro intersezione, sono adattate per la sufficienza parziale (nel senso di Fraser) ed é studiata la relazione tra esse.

SUMMARY

On the relationship between partial sufficiency, invariance and conditional independence

Several propositions that appear in a natural way in the literature on sufficiency and invariance, including the conditional independence of the invariant and the sufficient σ-fields given its intersection, are adapted for partial sufficiency (in the sense of Fraser), and the relationship between them is studied.