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ESTIMATION OF POPULATION RATIO, PRODUCT, AND MEAN USING MULTIAUXILIARY INFORMATION WITH RANDOM NON-RESPONSE

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1. INTRODUCTION

It is well known that in the theory of sampling the precision of estimate is usually increased by the use of some auxiliary variables correlated with the variables under investigation. Ratio, product, regression estimators and their several generalizations have been discussed in literature. These estimators use information in the form of known population means of the auxiliary variables. Srivastava and Jhajj (1981) suggested a family of estimators which use not only the information of the known population mean of the auxiliary variable, but also use the information of its known population variance. The family of estimators of population mean $\bar{Y}_0 = \frac{1}{N}\sum_{j=1}^{N} Y_{0j}$ of a finite population of size N, suggested by Srivastava and Jhajj (1981) is defined by:

$$\bar{y}_t = \bar{y}_0 t(a,b) \tag{1.1}$$

where $\bar{y}_0 = \frac{1}{n}\sum_{j=1}^{n} y_{0j}$, $\bar{x}_1 = \frac{1}{n}\sum_{j=1}^{n} x_{1j}$ are the sample means of size n drawn by simple random sampling without replacement, $s_{x_1}^2 = (n-1)^{-1}\sum_{j=1}^{n}(x_{1j} - \bar{x}_1)^2$ is an unbiased estimator of population variance $S_{x_1}^2$ of the auxiliary character x_1 and $\bar{X}_1$ denotes its known population mean, $a = \bar{x}_1 / \bar{X}_1$, $b = s_{x_1}^2 / S_{x_1}^2$ and $t(a,b)$ is a function of (a,b) such that $t(1,1) = 1$ and satisfies certain regularity conditions as given in Srivastava and Jhajj (1981).

A generalized version of the family (1.1) is given by

$$\bar{y}_T = T(\bar{y}_0, a, b) \tag{1.2}$$

where $T(\bullet)$ is a function of $(\bar{y}_0, a, b)$ such that $T(\bar{Y}_0, 1, 1) = \bar{Y}_0$, $T_1(\bar{Y}_0, 1, 1) = \frac{\partial T(\bullet)}{\partial \bar{y}_0}|_{(\bar{Y}_0,1,1)} = 1$ and satisfies certain regularity conditions. It has been shown that the asymptotic minimum MSE of $\bar{y}_t$ and $\bar{y}_T$ are the same. Using information on $\bar{X}_1$ and $S_{x_1}^2$, Upadhyaya and Singh (1985) suggested a family of estimators of population ratio $R = \bar{Y}_0 / \bar{Y}_1$, $\bar{Y}_1 \neq 0$ as:

$$\hat{R}_h = \hat{R} h(a, b) \tag{1.3}$$

where $\hat{R} = \bar{y}_0 / \bar{y}_1$ $(\bar{y}_1 \neq 0)$ is the conventional estimator of R, $\bar{y}_1$ is the sample mean of the study variable y_1, $h(a,b)$ is a parametric function of (a,b) such that $h(1,1) = 1$ and satisfies certain regularity conditions. A family of estimators wider than $\hat{R}_h$ is defined as:

$$\hat{R}_H = H(\hat{R}, a, b) \tag{1.4}$$

where $H(\bullet)$ is a parametric function of $(\hat{R}, a, b)$ such that, $H_1(R,1,1) = R$, and satisfies certain regularity conditions. It has been shown that the asymptotic minimum MSEs of $\hat{R}_h$ and $\hat{R}_H$ are the same. Quite often information on many supplementary variables are available in the survey which can be utilized to increase the precision of the estimate. Olkin (1958) has considered the use of multi-auxiliary variables, positively correlated with the variables under study to build up a multivariate ratio estimator of the population mean $\bar{Y}_0$. Following Olkin's method of estimation several estimators using multiauxiliary variables have been proposed by various authors; for instance, see Raj (1965), Rao and Mudholkar (1967), Srivastava (1971), Tuteja and Bahl (1991), Agarwal and Panda (1994), Singh and Rani (2005-2006), Tailor and Tailor (2008) etc. In this paper we suggest a family of estimators for population ratio, product and mean when information about population means and variances of $m > 1$ auxiliary variables are available. The properties of the suggested family are also discussed in the presence of random non-response. In this context we refer to Tracy and Osahan (1994), Singh and Joarder (1998), Singh *et al.* (2000, 2007), Dubey and Uprety (2008), Gamrot (2008), and Harel (2008).

2. NOTATIONS

We assume that information on m auxiliary variables $X_1, X_2, ..., X_m$ are available for all the units in the population. Let $U = \{U_1, U_2, ..., U_N\}$ denote the population of N units from which a simple random sample of size n is drawn without replacement. Let Y_{0j}, Y_{1j} and X_{ij} denote the values of the variables Y_0, Y_1 and X_i on the j-th unit of the population, $i=1,2,3...,m;$ $j=1,2,...,N$. Further, let y_{0j}, y_{1j} and x_{ij} denote the value of the sample, $i=1,2,...,m$; $j=1,2,...,n$.

We denote:

$$\bar{y}_0=\frac{1}{n}\sum_{j=1}^{n}y_{0j},\ \bar{y}_1=\frac{1}{n}\sum_{j=1}^{n}y_{1j},\ \bar{x}_i=\frac{1}{n}\sum_{j=1}^{n}x_{ij},\ \bar{Y}_0=\frac{1}{N}\sum_{j=1}^{N}Y_{0j},\ \bar{Y}_1=\frac{1}{N}\sum_{j=1}^{N}Y_{1j},$$

$$\bar{X}_i=\frac{1}{N}\sum_{j=1}^{N}X_{ij},\ s_{y_0}^2=(n-1)^{-1}\sum_{j=1}^{n}(y_{0j}-\bar{y}_0)^2,\ s_{y_1}^2=(n-1)^{-1}\sum_{j=1}^{n}(y_{1j}-\bar{y}_1)^2,$$

$$s_{x_i}^2=(n-1)^{-1}\sum_{j=1}^{n}(x_{ij}-\bar{x}_i)^2,\ s_{y_0y_1}=(n-1)^{-1}\sum_{j=1}^{n}(y_{0j}-\bar{y}_0)(y_{1j}-\bar{y}_1),$$

$$s_{y_0x_i}=(n-1)^{-1}\sum_{j=1}^{n}(y_{0j}-\bar{y}_0)(x_{ij}-\bar{x}_i),\ s_{y_1x_i}=(n-1)^{-1}\sum_{j=1}^{n}(y_{1j}-\bar{y}_1)(x_{ij}-\bar{x}_i),$$

$$S_{y_0}^2=(N-1)^{-1}\sum_{j=1}^{N}(Y_{0j}-\bar{Y}_0)^2,\ S_{y_1}^2=(N-1)^{-1}\sum_{j=1}^{n}(Y_{1j}-\bar{Y}_1)^2,$$

$$S_{x_i}^2=(N-1)^{-1}\sum_{j=1}^{N}(X_{ij}-\bar{X}_i)^2,\ S_{y_0y_1}=(N-1)^{-1}\sum_{j=1}^{N}(Y_{0j}-\bar{Y}_0)(Y_{1j}-\bar{Y}_1),$$

$$S_{y_0x_i}=(N-1)^{-1}\sum_{j=1}^{N}(Y_{0j}-\bar{Y}_0)(X_{ij}-\bar{X}_i),$$

$$S_{y_1x_i}=(N-1)^{-1}\sum_{j=1}^{N}(Y_{1j}-\bar{Y}_1)(X_{ij}-\bar{X}_i),\ C_{y_0}^2=S_{y_0}^2\big/\bar{Y}_0^2,\ C_{y_1}^2=S_{y_1}^2\big/\bar{Y}_1^2 \text{ and}$$

$$C_{x_i}^2=S_{x_i}^2\big/\bar{X}_i^2,\ i=1,2,...,m.$$

Further let $\rho_{y_0y_1}$, $\rho_{y_0x_i}$, $\rho_{y_1x_i}$ and $\rho_{x_ix_k}$ denote the correlation coefficients between (y_0,y_1), (y_0,x_i), (y_1,x_i) and between the variables (x_i,x_k) respectively, $i\neq k=1,2,....,m$.

Define: $u_i=\bar{x}_i\big/\bar{X}_i$, $i=1,2,....,m$ and $u_i=(s_{x_{i-m}}^2)\big/(S_{x_i-m}^2)$, $i=m+1,m+2,....,2m$

Let u denote the column vector of $2m$ elements $u_1,u_2,...,u_{2m}$. Superscript T over a column vector denotes the corresponding row vector.

Defining:

$$\delta_0=\bar{y}_0\big/\bar{Y}_0-1,\ \delta_1=\bar{y}_1\big/\bar{Y}_1-1,\ \varepsilon_i=u_i-1,\ i=1,2,...,2m; \text{ and}$$

$$\varepsilon^T=(\varepsilon_1,\varepsilon_2,...,\varepsilon_{2m})$$

we have:

$$E(\delta_0)=E(\delta_1)=0\,,\ E(\varepsilon_i)=0\ \ \forall\ i=1,2,...,2m;$$

$$E(\delta_0^2)=\theta C_{y_0}^2\,,\ E(\delta_1^2)=\theta\, C_{y_1}^2\,,\ E(\delta_0\delta_1)=\theta\,\rho_{y_0y_1}C_{y_0}C_{y_1}\,,\ E(\delta_0\varepsilon_i)=\theta\, a_{y_0x_i}\,,$$
$$i=1,2,....,m\,;$$

$$E(\varepsilon_i^2)=\theta C_{x_i}^2\,,\ i=1,2,....,m\,;\ E(\delta_1\varepsilon_i)=\theta\, a_{y_1x_i}\,,\ i=1,2,....,m\,,$$

$$E(\varepsilon_i^2)=\theta\gamma_{x_ix_i}=\theta(\beta_2(x_i)-1)\,, i=m+1,m+2,....,2m\,;$$

$$E\{(\delta_0-\alpha\delta_0)\varepsilon_i\}=\theta\,(a_{y_0x_i}-\alpha a_{y_ix_i})=\theta\, a_i\,, i=1,2,....,m\,;$$

$$E(\varepsilon_i\varepsilon_k)=\theta\,\rho_{x_ix_k}C_{x_i}C_{x_k}=\theta\, a_{x_ix_k}\,,\ i\neq k=1,2,...,m\,;$$

and up to terms of order n^{-1}, we have:

$$E(\delta_0\varepsilon_i)=\theta\,\lambda_{y_0x_i}\,,\ i=m+1,m+2,...,2m\,;\ E(\delta_1\varepsilon_i)=\theta\,\lambda_{y_1x_i}\,,\ i=m+1,m+2,...,2m\,;$$

$$E\{(\delta_0-\alpha\delta_0)\varepsilon_i\}=\theta\,(\lambda_{y_0x_i}-\alpha\lambda_{y_ix_i})=\theta\,\lambda_i\,,\ i=m+1,m+2,...,2m\,;$$

$$E(\varepsilon_i\varepsilon_k)=\theta\, f_{x_ix_k}\,,\ i=1,2,....,m\,,\ k=m+1,m+2,...,2m\,;$$

$$E(\varepsilon_i\varepsilon_k)=\theta\,\gamma_{x_ix_k}\,,\ (i,k)=m+1,m+2,....,2m\,;$$

where $a_{y_0x_i}=\rho_{y_0x_i}C_{y_0}C_{x_i}$; $a_{y_1x_i}=\rho_{y_1x_i}C_{y_1}C_{x_i}$, $\theta=(1/n-1/N)$,

$$\lambda_{y_0x_i}=\sum_{j=1}^{N}(Y_{0j}-\bar{Y}_0)(X_{ij}-\bar{X}_i)^2/((N-1)\bar{Y}_0S_{x_i}^2)\,,$$

$$\lambda_{y_1x_i}=\sum_{j=1}^{N}(Y_{1j}-\bar{Y}_1)(X_{ij}-\bar{X}_i)^2\Big/((N-1)\bar{Y}_1S_{x_i}^2)\,,$$

$$f_{x_ix_k}=\sum_{j=1}^{N}(X_{ij}-\bar{X}_i)(X_{kj}-\bar{X}_k)^2\Big/((N-1)\bar{X}_iS_{x_k}^2)$$

and

$$\gamma_{x_ix_k}=\sum_{j=1}^{N}(X_{ij}-\bar{X}_i)^2(X_{kj}-\bar{X}_k)^2\Big/((N-1)S_{x_i}^2S_{x_k}^2)-1\,.$$

Putting the above results in matrix notations, we have:

$E(\varepsilon)=0$, $E(\varepsilon\varepsilon^T)=\theta D$, $E\{(\delta_0-\alpha\delta_1)\varepsilon\}=\theta\, d$,

where the vector $d^T=(a^T:\lambda^T)=(a_1,a_2,...,a_m,\lambda_1,\lambda_2,...,\lambda_m)$ and the $2m\times 2m$ matrix:

$$D=\begin{bmatrix} A & \mid F \\ ----- & \\ F^T \mid & \gamma \end{bmatrix}$$

which is assumed to be positive definite. The matrices A, F and γ are of order $m\times m$ matrices $A=[a_{x_ix_k}]$, $F=[f_{x_ix_k}]$, and $\gamma=[\gamma_{x_ix_k}]$.

3. THE SUGGESTED FAMILY OF ESTIMATORS OF $R_{(\alpha)}$

The parameter under investigation is $R_{(\alpha)}=\overline{Y}_0/\overline{Y}_1^{\alpha}$, where α takes values 0, +1, or –1. It reduces to:

(i.) $R_{(1)}=\overline{Y}_0/\overline{Y}_1=R$, for $\alpha=1$ (Ratio of two population means)

(ii) $R_{(-1)}=\overline{Y}_0\overline{Y}_1=P$ for $\alpha=-1$ (Product of two population means)

(iii) $R_{(0)}=\overline{Y}_0$, for $\alpha=0$ (Mean of single population)

The conventional estimator of $R_{(\alpha)}=\overline{Y}_0/\overline{Y}_1^{\alpha}$, is defined by:

$$\hat{R}_{(\alpha)}=\overline{y}_0/\overline{y}_1^{\alpha},\quad \overline{y}_1^{\alpha}\neq 0 \tag{3.1}$$

which reduces to the following set of estimators:

(i) $\hat{R}_{(1)}=\overline{y}_0/\overline{y}_1=\hat{R}$, for $\alpha=1$ (Estimator of ratio of two population means)

(ii) $\hat{R}_{(-1)}=\overline{y}_0\overline{y}_1=\hat{P}$ for $\alpha=-1$ (Estimator of product of two population means)

(iii) $\hat{R}_{(0)}=\overline{y}_0$, for $\alpha=0$ (Estimator of mean of single population)

Let e^T denote the row vector of $2m$ unit elements. Whatever be the sample chosen, let $(\hat{R}_{(\alpha)},u^T)$ assume values in a closed convex subset, Q, of the $2m+1$ dimensional real space containing the point $(R_{(\alpha)},e^T)$.

We define a family of estimators of $R_{(\alpha)}$ as:

$$\hat{R}_{mg}^{(\alpha)} = G(\hat{R}_{(\alpha)}, u_1, u_2, ..., u_{2m}) = G(\hat{R}_{(\alpha)}, u^T) \tag{3.2}$$

where $G(\hat{R}_{(\alpha)}, u^T)$ is a function of $(\hat{R}_{(\alpha)}, u^T)$ such that:

$$G(R_{(\alpha)}, u^T) = R_{(\alpha)} \text{ for all } \hat{R}_{(\alpha)}, \tag{3.3}$$

and such that it satisfies the following conditions:

(1) The function $G(\hat{R}_{(\alpha)}, u^T)$ is continuous and bounded in Q.

(2) The first and second order partial derivatives of the function $G(\hat{R}_{(\alpha)}, u^T)$ exist and are continuous and bounded in Q.

To obtain the minimum mean squared error (MSE) of $\hat{R}_{mg}^{(\alpha)}$, we expand the function $G(\hat{R}_{(\alpha)}, u^T)$ about the point $G(R_{(\alpha)}, e^T)$ in second order Taylor's series. We obtain

$$\begin{aligned}\hat{R}_{mg}^{(\alpha)} &= G(R_{(\alpha)}, e^T) + (\hat{R}_{(\alpha)} - R_{(\alpha)}) \frac{\partial G(\bullet)}{\partial \hat{R}_{(\alpha)}}\Big|_{(R_{(\alpha)}, e^T)} + (u-e)^T G^{(1)}(R_{(\alpha)}, e^T) \\ &+ \frac{1}{2}\Bigg\{ (\hat{R}_{(\alpha)} - R_{(\alpha)})^2 \frac{\partial^2 G(\bullet)}{\partial \hat{R}_{(\alpha)}^2}\Big|_{(\hat{R}_{(\alpha)}^*, u^{*T})} + 2(\hat{R}_{(\alpha)} - R_{(\alpha)})(u-e)^T \frac{\partial G(\bullet)}{\partial \hat{R}_{(\alpha)}}\Big|_{(\hat{R}_{(\alpha)}^*, u^{*T})} \\ &+ (u-e)^T G^{(2)}(\hat{R}_{(\alpha)}^*, u^{*T})(u-e) \Bigg\}\end{aligned} \tag{3.4}$$

where $\hat{R}_{(\alpha)}^* = R_{(\alpha)} + \eta(\hat{R}_{(\alpha)} - R_{(\alpha)})$, $u^* = e + \eta(u-e)$, $0 < \eta < 1$; $G^{(1)}(\bullet)$ denote the $2m$ elements column vector of the first partial derivatives of $G(\bullet)$ and $G_{(\bullet)}^{(2)}$ denotes the $2m \times 2m$ matrix of second order partial derivatives of $G(\bullet)$ with respect to u. Substituting for $\bar{y}_0, \bar{y}_1$ and u in terms of δ_0, δ_1 and $\in$ using (3.3), we have:

$$\begin{aligned}\hat{R}_{mg}^{(\alpha)} &= R_{(\alpha)} + R_{(\alpha)}((1+\delta_0)(1+\delta_1)^{-\alpha} - 1) \frac{\partial G(\bullet)}{\partial \hat{R}_{(\alpha)}}\Big|_{(R_{(\alpha)}, e^T)} + \varepsilon^T G^{(1)}(R_{(\alpha)}, e^T) \\ &+ \frac{1}{2}\Bigg\{ R_{(\alpha)}^2((1+\delta_0)(1+\delta_1)^{-\alpha} - 1) \frac{\partial^2 G(\bullet)}{\partial \hat{R}_{(\alpha)}^2}\Big|_{(\hat{R}_{(\alpha)}^*, u^{*T})}\end{aligned}$$

$$+2R_{(\alpha)}((1+\delta_0)(1+\delta_1)^{-\alpha}-1)\varepsilon^T \frac{\partial G^{(1)}(\bullet)}{\partial \hat{R}_{(\alpha)}}|_{(\hat{R}^*_{(\alpha)},u^{*T})} +\varepsilon^T G^{(2)}(\hat{R}^*_{(\alpha)},u^{*T})\varepsilon \quad \Bigg\} \tag{3.5}$$

Taking expectation in (3.5) and noting that the second partial derivatives are bounded, we have:

Theorem 3.1.

$$E(\hat{R}^{(\alpha)}_{mg}) = R_{(\alpha)} + O(n^{-1})$$

From Theorem 3.1, it follows that the bias of the suggested family is of order n^{-1} and hence its contribution to the MSE of $\hat{R}^{(\alpha)}_{mg}$ will be of order n^{-2}.

We now prove the following theorem:
Theorem 3.2. Up to terms of order n^{-1}, the MSE of $\hat{R}^{(\alpha)}_{mg}$ is minimized for

$$G^{(1)}(R_{(\alpha)}, e^T) = -R_{(\alpha)} D^{-1} d \tag{3.6}$$

and the minimum MSE is given by

$$MSE(\hat{R}^{(\alpha)}_{mg}) = MSE(\hat{R}_{(\alpha)}) - \theta\, R^2_{(\alpha)} d^T D^{-1} d \tag{3.7}$$

where

$$MSE(\hat{R}_{(\alpha)}) = \theta\, R^2_{(\alpha)} [C^2_{y_0} + \alpha\{\alpha C^2_{y_1} - 2\rho_{y_0 y_1} C_{y_0} C_{y_1}\}] \tag{3.8}$$

is the MSE of $\hat{R}_{(\alpha)}$ to the first degree of approximation.

Proof. From (3.5), we have up to terms of order n^{-1},

$$MSE(\hat{R}^{(\alpha)}_{mg}) = E[\hat{R}^{(\alpha)}_{mg} - R_{(\alpha)}]^2$$
$$= MSE(\hat{R}_{(\alpha)}) + \theta\,[2R_{(\alpha)} d^T G^{(1)}(R_{(\alpha)}, e^T) + (G^{(1)}(R_{(\alpha)}, e^T))^T D (G^{(1)}(R_{(\alpha)}, e^T))] \tag{3.9}$$

which is minimized for

$$G^{(1)}(R_{(\alpha)}, e^T) = -R_{(\alpha)} D^{-1} d \tag{3.10}$$

Substitution of (3.10) in (3.9) yields the minimum MSE of $\hat{R}^{(\alpha)}_{mg}$ as given in (3.7). Hence the theorem.

Theorem 3.3. Up to terms of order n^{-1},

$$MSE(\hat{R}_{mg}^{(\alpha)}) \geq MSE(\hat{R}_{(\alpha)}) - \theta R_{(\alpha)}^2 d^T D^{-1} d$$

with equality holding if

$$G^{(1)}(R_{(\alpha)}, e^T) = -R_{(\alpha)} D^{-1} d .$$

Any parametric function $G(\hat{R}_{(\alpha)}, u^T)$ satisfying the conditions (1) and (2) can generate an asymptotically acceptable estimator. The family of estimators is large.

Some examples are:

(1) $\hat{R}_{mg}^{(1)} = \hat{R}_{(\alpha)} \exp(\phi^T \log u)$ (2) $\hat{R}_{mg}^{(2)} = \hat{R}_{(\alpha)}(1 + \phi^T(u - e))$

(3) $\hat{R}_{mg}^{(3)} = \hat{R}_{(\alpha)} \exp(\phi^T(u - e))$ (4) $\hat{R}_{mg}^{(4)} = \hat{R}_{(\alpha)} + \phi^T(u - e)$

(5) $\hat{R}_{mg}^{(5)} = \hat{R}_{(\alpha)} \sum_{i=1}^{2m} w_i \exp((\phi_i / w_i) \log u_i)$ (6) $\hat{R}_{mg}^{(6)} = \hat{R}_{(\alpha)} \prod_{i=1}^{2m} \mu_i^{\phi_i}$

(7) $\hat{R}_{mg}^{(7)} = \hat{R}_{(\alpha)}^2 / \{\hat{R}_{(\alpha)} + \phi^T(u - e)\}$, where $\phi^T = (\phi_1, \phi_2, ..., \phi_{2m})$ is a vector of $2m$ constants.

The optimum values of these constants which minimizes the MSE of $\hat{R}_{mg}^{(\alpha)}$ are obtained from the conditions (3.6). The MSE of any estimator of the family (3.2) is obtained from (3.9). From (3.7), the minimum MSE is not larger than the MSE of the conventional estimator $\hat{R}_{(\alpha)}$, since $d^T D^{-1} d > 0$.

Remark 3.1. The suggested family of estimators $\hat{R}_{mg}^{(\alpha)}$ reduces to:

(1) Srivastava and Jhajj (1983) estimator of $\overline{Y}_0 : \hat{R}_s^{(1)} = G(\overline{y}_0, u^T)$, for $\alpha = 0$

(2) the generalized version of Upadhyaya and Singh (1985) estimator of population ratio $R : \hat{R}_s^{(2)} = G(\hat{R}, u^T)$, for $\alpha = 1$

(3) the product estimator: $\hat{R}_s^{(3)} = G(\hat{P}, u^T)$ for $\alpha = -1$.

4. ESTIMATORS BASED ON ESTIMATED OPTIMUM VALUES OF CONSTANTS

Following the same approach as adopted by Srivastava and Jhajj (1983), we suggest a family of estimators of $R_{(\alpha)}$ (based on estimated optimum values of constants) as:

$$\hat{R}_{mg}^{*(\alpha)} = G^*(\hat{R}_{(\alpha)}, u^T, \hat{\psi}^T) \tag{4.1}$$

where

$$\hat{\psi}^T = \hat{D}^{-1}\hat{d} \tag{4.2}$$

is a consistent estimator of $\psi^T = D^{-1}d$ with:

$$\hat{D} = \begin{bmatrix} \hat{A}, & \hat{F} \\ \hat{F}, & \hat{\gamma} \end{bmatrix},\ \hat{d}^T = (\hat{a}^T : \hat{\lambda}^T) = (\hat{a}_1, \hat{a}_2, ..., \hat{a}_m, \hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_m),\ \hat{A} = [\hat{a}_{x_i x_k}]_{m\times m},$$

$$\hat{F} = [\hat{f}_{x_i x_k}]_{m\times m},\ \hat{\gamma} = [\hat{\gamma}_{x_i x_k}]_{m\times m},\ \hat{a}_i = (\hat{a}_{y_0 x_i} - \alpha\, \hat{a}_{y_1 x_i}),\ \hat{\lambda}_i = (\hat{\lambda}_{y_0 x_i} - \alpha\hat{\lambda}_{y_1 x_i}),$$

$$\hat{a}_{y_0 x_i} = \hat{\rho}_{y_0 x_i}\hat{C}_{y_0}\hat{C}_{x_i},\ i = 1,2,...,m;\ \hat{a}_{y_1 x_i} = \hat{\rho}_{y_1 x_i}\hat{C}_{y_1}\hat{C}_{x_i},\ i = 1,2,...,m;$$

$$\hat{\rho}_{y_0 x_i} = s_{y_0 x_i}/(s_{y_0}s_{x_i}),\ i = 1,2,...,m;\ \hat{\rho}_{y_1 x_i} = s_{y_1 x_i}/(s_{y_1}s_{x_i}),\ i = 1,2,...,m;$$

$$\hat{C}_{y_0} = s_{y_0}/\overline{y}_0,\ \hat{C}_{y_1} = s_{y_1}/\overline{y}_1,\ s_{y_0 x_i} = (n-1)^{-1}\sum_{j=1}^{n}(y_{0j} - \overline{y}_0)(x_{ij} - \overline{x}_i),$$

$$s_{y_1 x_i} = (n-1)^{-1}\sum_{j=1}^{n}(y_{1j} - \overline{y}_1)(x_{ij} - \overline{x}_i),\ s_{y_0}^2 = (n-1)^{-1}\sum_{j=1}^{n}(y_{0j} - \overline{y}_0)^2,$$

$$s_{y_1}^2 = (n-1)^{-1}\sum_{j=1}^{n}(y_{1j} - \overline{y}_1)^2,\ s_{x_i}^2 = (n-1)^{-1}\sum_{j=1}^{n}(x_{ij} - \overline{x}_i)^2,\ i = 1,2,...,m;$$

$$\hat{\lambda}_i = (\hat{\lambda}_{y_0 x_i} - \alpha\hat{\lambda}_{y_1 x_i}),\ i = m+1, m+2, ..., 2m;\ \hat{\lambda}_{y_0 x_i} = \frac{\sum_{j=1}^{n}(y_{0j} - \overline{y}_0)(x_{ij} - \overline{x}_i)^2}{(n-1)\overline{y}_0 s_{x_i}^2},$$

$$\hat{\lambda}_{y_1 x_i} = \frac{\sum_{j=1}^{n}(y_{1j} - \overline{y}_1)(x_{ij} - \overline{x}_i)^2}{(n-1)\overline{y}_1 s_{x_i}^2},\ \hat{a}_{x_i x_k} = \hat{\rho}_{x_i x_k}\hat{C}_{x_i}\hat{C}_{x_k},\ i \neq k = 1,2,...,m;$$

$$\hat{\rho}_{x_i x_k} = \frac{s_{x_i x_k}}{s_{x_i}s_{x_k}};\ s_{x_i x_k} = (n-1)^{-1}\sum_{j=1}^{n}(x_{ij} - \overline{x}_i)(x_{kj} - \overline{x}_k);$$

$$\hat{f}_{x_i x_k} = \frac{\sum_{j=1}^{n}(x_{ij} - \overline{x}_i)(x_{kj} - \overline{x}_k)^2}{(n-1)\overline{x}_i^2 s_{x_k}^2},\ \hat{\gamma}_{x_i x_k} = \frac{\sum_{j=1}^{n}(x_{ij} - \overline{x}_i)^2(x_{kj} - \overline{x}_k)^2}{(n-1)s_{x_i}^2 s_{x_k}^2} - 1,$$

$G^*(\hat{R}_{(\alpha)}, u^T, \hat{\psi}^T)$ is a parametric function of $(\hat{R}_{(\alpha)}, u^T, \hat{\psi}^T)$ such that:

$$G^*(R_{(\alpha)}, e^T, \psi^T) = R_{(\alpha)} \text{ for all } R_{(\alpha)} \tag{4.3}$$

which implies

$$\frac{\partial G^*(\bullet)}{\partial \hat{R}_{(\alpha)}}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = 1 \tag{4.4}$$

$$\frac{\partial G^*(\bullet)}{\partial u}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = -R_{(\alpha)}\psi \tag{4.5}$$

and

$$\frac{\partial G^*(\bullet)}{\partial \hat{\psi}}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = 0. \tag{4.6}$$

Expanding the function $G^*(\hat{R}_{(\alpha)}, u^T, \hat{\psi}^T)$ about the point $(R_{(\alpha)}, e^T, \psi^T)$ in Taylor's series and using (4.3) to (4.6), we have

$$\begin{aligned}\hat{R}_{mg}^{*(\alpha)} &= G^*(R_{(\alpha)}, e^T, \psi^T) + (\hat{R}_{(\alpha)} - R_{(\alpha)})\frac{\partial G^*(\bullet)}{\partial \hat{R}_{(\alpha)}}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} + (u-e)^T \frac{\partial G^*(\bullet)}{\partial u}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} \\ &\quad + (\hat{\psi} - \psi)^T \frac{\partial G^*(\bullet)}{\partial \hat{\psi}}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} + \text{second order terms} \\ &= R_{(\alpha)} + R_{(\alpha)}(\delta_0 - \alpha\delta_1) + \in^T(-R_{(\alpha)}\psi) + \text{second order terms}\end{aligned} \tag{4.7}$$

Since $\hat{\psi}$ is a consistent estimator of ψ, the expectation of the second order terms in (4.7) will be $O(n^{-1})$ and hence

$$E(\hat{R}_{mg}^{*(\alpha)}) = R_{(\alpha)} + O(n^{-1})$$

The mean squared error of $\hat{R}_{mg}^{(\alpha)}$ up to terms of order n^{-1}, from (4.7) is

$$MSE(\hat{R}_{mg}^{*(\alpha)}) = E[\hat{R}_{mg}^{*(\alpha)} - R_{(\alpha)}]^2 = MSE(\hat{R}_{(\alpha)}) - \theta\, R_{(\alpha)}^2 d^T D^{-1} d$$

which is same as (3.7). We thus have proved the following theorem.

Theorem 4.1. If the optimum values of constants in (3.6) is replaced by their consistent estimators and conditions (4.5) and (4.6) hold, the resulting estimator $\hat{R}_{mg}^{*(\alpha)}$ has the same mean squared error up to terms of order n^{-1}, as that of optimum $\hat{R}_{mg}^{(\alpha)}$.

5. RANDOM NON-RESPONSE AND SOME EXPECTED VALUES

If $z\ (z = 0,1,2,...,(n-2))$ denotes the number of sampling units on which information could not be obtained due to random non-response, then the remaining $(n-z)$ units in the sample can be treated as SRSWOR sample from U. Since we are considering the problem of unbiased estimation of variance of the estimators of general parameter $R_{(\alpha)}$, therefore, we are assuming that z should be less than $(n-1)$, that is, $0 \le z \le (n-2)$. We assume that if p denotes the probability of non-response among the $(n-2)$ possible values of non-responses, then z has the following discrete distribution given by

$$P(z) = \frac{(n-z)}{nq+2p}\, {}^{n-2}C_{z}\, p^{z} q^{n-2-z} \tag{5.1}$$

where $q = (1-p)$, $z = 0,1,2,...,(n-2)$ and ${}^{n-2}C_{z}$ denote the total number of ways of obtaining z non-responses out of the $(n-2)$ total possible responses, for instance, see Singh and Joarder (1998).

Let us define:

$$\delta_0^* = \bar{y}_0^* / \bar{Y}_0 - 1,\ \delta_1^* = \bar{y}_1^* / \bar{Y}_1 - 1,\ \epsilon_i^* = u_i^* - 1, \text{ and } \epsilon_i = u_i - 1,$$

where

$$u_i^* = \begin{cases} \bar{x}_i^* / \bar{X}_i, & i = 1,2,....,m \\ s_{x_{i-m}}^{*2} / S_{x_{i-m}}^2, & i = m+1, m+2,...,2m \end{cases},\quad u_i = \begin{cases} \bar{x}_i / \bar{X}_i, & i = 1,2,....,m \\ s_{x_{i-m}}^2 / S_{x_{i-m}}^2, & i = m+1, m+2,...,2m \end{cases}$$

$\epsilon^T = (\epsilon_1, \epsilon_2, ..., \epsilon_{2m})$, $\epsilon^{*T} = (\epsilon_1^*, \epsilon_2^*, ..., \epsilon_{2m}^*)$ where $s_{x_i}^{*2} = (n-z-1)^{-1} \sum_{i=1}^{n-z} (x_{ij} - \bar{x}_i^*)^2$ is conditionally unbiased estimator of $S_{x_i}^2$ $(i = 1,2,...,m)$ and $\bar{y}_0^* = (n-z)^{-1} \sum_{j=1}^{n-z} y_{0j}$, $\bar{y}_1^* = (n-z)^{-1} \sum_{j=1}^{n-z} y_{1j}$, $\bar{x}_i^* = (n-z)^{-1} \sum_{j=1}^{n-z} x_{ij}$, $i = 1,2,...,m.$

Thus under the probability model (5.1), we have the following results:

$$E(\delta_0^*) = E(\delta_1^*) = 0,\ E(\epsilon_i) = E(\epsilon_i^*) = 0 \text{ for all } i = 1,2,...,2m;$$

$$E(\delta_0^{*2}) = \theta^* C_{y_0}^2,\ E(\delta_1^{*2}) = \theta^* C_{y_1}^2,\ E(\delta_0^* \delta_1^*) = \theta^* \rho_{y_0 y_1} C_{y_0} C_{y_1},$$

$$E(\in_i^{*2}) = \begin{cases} \theta^* C_{x_i}^2, \; i = 1, 2, ..., m \\ \theta^* \gamma_{x_i x_i} = \theta^* (\beta_2(x_i) - 1), \\ \qquad i = m+1, m+2, ..., 2m \end{cases}$$

$$E(\delta_0^* \in_i^*) = \begin{cases} \theta^* a_{y_0 x_i}, \; i = 1, 2, ..., m \\ \theta^* \lambda_{y_0 x_i}, \; i = m+1, m+2, ..., 2m. \end{cases}$$

$$E(\in_i^* \in_k^*) = \begin{cases} \theta^* f_{x_i x_k}, \\ \qquad i = 1, 2, ..., m; k = m+1, m+2, ..., 2m \\ \theta^* \gamma_{x_i x_k}, \\ \qquad i, k = m+1, m+2, ..., 2m. \end{cases}$$

$$E(\delta_1^* \in_i^*) = \begin{cases} \theta^* a_{y_1 x_i}, \; i = 1, 2, ..., m \\ \theta^* \lambda_{y_1 x_i}, \; i = m+1, m+2, ..., 2m. \end{cases}$$

$$E((\delta_0^* - \alpha \delta_1^*) \in_i^*) = \begin{cases} \theta^* (a_{y_0 x_i} - \alpha \, a_{y_1 x_i}) = \theta^* a_i, \\ \qquad i = 1, 2, ..., m \\ \theta^* (\lambda_{y_0 x_i} - \alpha \, \lambda_{y_1 x_i}) = \theta^* \lambda_i, \\ \qquad i = m+1, m+2, ., 2m. \end{cases}$$

$$E((\delta_0^* - \alpha \delta_1^*) \in_i) = \begin{cases} \theta \, a_i, \; i = 1, 2, ..., m \\ \theta \, \lambda_i, \; i = m+1, m+2, ..., 2m. \end{cases}$$

$$E(\delta_0^* \in_i) = \begin{cases} \theta \, a_{y_0 x_i}, \; i = 1, 2, ..., m \\ \theta \, \lambda_{y_0 x_i}, \; i = m+1, m+2, ..., 2m. \end{cases}$$

$$E(\delta_1^* \in_i) = \begin{cases} \theta \, a_{y_1 x_i}, \; i = 1, 2, ..., m \\ \theta \, \lambda_{y_1 x_i}, \; i = m+1, m+2, ..., 2m. \end{cases}$$

$$E(\in_i^* \in_i) = \begin{cases} \theta \, C_{x_i}^2, \; i = 1, 2, ..., m \\ \theta \, \gamma_{x_i x_i} = \theta(\beta_2(x_i) - 1), \\ \qquad i = m+1, m+2, ..., 2m. \end{cases}$$

$$E(\in_i^* \in_k) = \begin{cases} \theta\, f_{x_i x_k}, i = 1,2,...,m; \\ \qquad\qquad k = m+1, m+2,...,2m \\ \theta\, \gamma_{x_i x_k}, i,k = m+1, m+2,...,2m \end{cases}$$

and

$$E(\in_i^* \in_k^*) = \theta\, a_{x_i x_k}, i \neq k = 1,2,...,m ,$$

where

$$\theta^* = \left[\frac{1}{nq+2p} - \frac{1}{N}\right].$$

Putting the results in matrix notations, we have

$E(\in^*) = 0$, $E(\in^* \in^{*T}) = \theta^* D$, $E\{(\delta_0^* - \alpha\,\delta_1^*) \in^*\} = \theta^* d$, $E(\in \in^{*T}) = \theta\, D$, and $E\{(\delta_0^* - \alpha\,\delta_1^*) \in\} = \theta\, d$.

It may be noted that if $p = 0$ that is if there is no non-response, the above expected values coincide with the usual results. In the following sections we consider three different strategies as follows:

6. SUGGESTED STRATEGY-I

We are considering the situation when random non-response exists on study variables Y_0, Y_1 and auxiliary variables $X_1, X_2, ..., X_m$. It is assumed that the population means $\bar{X}_i$ and variances $S_{x_i}^2, (i = 1,2,...,m)$ of the auxiliary variables $X_1, X_2, ..., X_m$ are known. Thus we define a family of estimators of $R_{(\alpha)}$ as:

$$d_1 = J(\hat{R}_{(\alpha)}^*, u^{*T}), \tag{6.1}$$

where $\hat{R}_{(\alpha)}^* = \bar{y}_0^* / \bar{y}_1^{*\alpha}$ is conventional estimator of $R_{(\alpha)}$, $J(\hat{R}_{(\alpha)}^*, u^{*T})$ is a parametric function of $(\hat{R}_{(\alpha)}^*, u^{*T})$ such that

$$J(R_{(\alpha)}, e^T) = R_{(\alpha)} \text{ for all } \hat{R}_{(\alpha)} \tag{6.2}$$

which implies that:

$$\frac{\partial J(\bullet)}{\partial \hat{R}^{*}_{(\alpha)}}\Big|_{(R_{(\alpha)},e^{T})}=1 \tag{6.3}$$

and satisfies certain conditions as given for $\hat{R}^{(\alpha)}_{mg}$. To terms of order n^{-1}, it can be easily shown that:

$$E(d_1)=R_{(\alpha)}+O(n^{-1}) \tag{6.4}$$

and

$$MSE(d_1)=MSE(\hat{R}^{*}_{\alpha})+\theta^{*}[2R_{(\alpha)}d^{T}J^{(1)}(R_{(\alpha)},e^{T})+(J^{(1)}(R_{(\alpha)},e^{T})^{T}D(J^{(1)}(R_{(\alpha)},e^{T}))) \tag{6.5}$$

where

$$J^{(1)}{}_{(R_{(\alpha)},e^{T})}=\frac{\partial J(\bullet)}{\partial u^{*}}\Big|_{(R_{(\alpha)}.e^{T})}$$

and

$$MSE(\hat{R}^{*}_{(\alpha)})=\theta^{*}R^{2}_{(\alpha)}[C^{2}_{y_0}+\alpha\{\alpha C^{2}_{y_1}-2\rho_{y_0y_1}C_{y_0}C_{y_1}\}] \tag{6.6}$$

is the MSE of $\hat{R}^{*}_{(\alpha)}$ to terms of order n^{-1}. The $\mathrm{MSE}(d_1)$ at (6.5) is minimized for

$$J^{(1)}(R_{(\alpha)},e^{T})=-R_{(\alpha)}D^{-1}d \tag{6.7}$$

Thus the resulting (minimum) MSE of d_1 is given by

$$\min.MSE(d_1)=MSE(\hat{R}^{*}_{(\alpha)})-\theta^{*}R^{2}_{(\alpha)}d^{T}D^{-1}d \tag{6.8}$$

which clearly indicates that the proposed estimator is more efficient than the conventional estimator $\hat{R}^{*}_{(\alpha)}$. Thus we proved the following theorem:

Theorem 6.1. To the first degree of approximation

$$MSE(d_1)\geq MSE(\hat{R}^{*}_{(\alpha)})-\theta^{*}R^{2}_{(\alpha)}d^{T}D^{-1}d$$

with equality holding if

$$J^{(1)}(R_{(\alpha)},e^{T})=-R_{(\alpha)}D^{-1}d\,.$$

The following estimators: (1) $d_1^{(1)} = \hat{R}^*_{(\alpha)} \exp(\phi^T \log u^*)$, (2) $d_1^{(2)} = \hat{R}^*_{(\alpha)}(1+\phi^T(u^* - e))$ etc. may be identified as particular members of the suggested family d_1. The MSEs of these estimators can be easily obtained from (6.5). Proceeding in a similar manner as section 4, we state the following theorem.

Theorem 6.2. The family of estimators:

$$d_1^* = J^*(\hat{R}^*_{(\alpha)}, u^{*T}, \hat{\psi}^{*T})$$

of $R_{(\alpha)}$ based on estimated optimum values of constants has MSE to the first degree of approximation equal to that of minimum MSE of d_1 that is

$$MSE(d_1^*) = \min.MSE(d_1) \tag{6.9}$$

where $\min.MSE(d_1)$ is cited in (6.8), $J^*(\hat{R}^*_{(\alpha)}, u^{*T}, \hat{\psi}^{*T})$ is a function of $(\hat{R}^*_{(\alpha)}, u^{*T}, \hat{\psi}^{*T})$ such that

$$J^*(R_{(\alpha)}, e^T, \psi^T) = R_{(\alpha)} \text{ for all } \hat{R}^*_{(\alpha)}$$

which implies that

$$\frac{\partial J^*(\bullet)}{\partial \hat{R}^*_{(\alpha)}}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = 1, \quad \frac{\partial J^*(\bullet)}{\partial u^*}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = -R_{(\alpha)}\psi \text{ and } \frac{\partial J^*(\bullet)}{\partial \hat{\psi}^*}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = 0$$

and $\hat{\psi}^* = \hat{D}^{*-1}\hat{d}^*$ is the estimator of $\psi = D^{-1}d$, $\hat{D}^* = \begin{bmatrix} \hat{A}^* & | & \hat{F}^* \\ - - - & - & - - - \\ \hat{F}^{*T} & | & \hat{\gamma}^* \end{bmatrix}$,

$\hat{d}^{*T} = (\hat{a}^{*T} : \hat{\lambda}^{*T}) = (\hat{a}_1^*, \hat{a}_2^*, ..., \hat{a}_m^*, \hat{\lambda}_1^*, \hat{\lambda}_2^*, ..., \hat{\lambda}_m^*)$, $\hat{A}^* = [\hat{a}^*_{x_i x_k}]_{m\times m}$, $\hat{F}^* = [\hat{f}^*_{x_i x_k}]_{m\times m}$,

$\hat{\gamma}^* = [\hat{\gamma}^*_{x_i x_k}]_{m\times m}$, $\hat{a}_i^* = \hat{a}_{y_0 x_i} - \alpha\, \hat{a}^*_{y_1 x_i}$, $\hat{\lambda}_i^* = \hat{\lambda}^*_{y_0 x_i} - \alpha \hat{\lambda}^*_{y_1 x_i}$, $\hat{a}^*_{y_0 x_i} = \hat{\rho}^*_{y_0 x_i} \hat{C}^*_{y_0} \hat{C}^*_{x_i}$,

$\hat{a}^*_{y_1 x_i} = \hat{\rho}^*_{y_1 x_i} \hat{C}^*_{y_1} \hat{C}^*_{x_i}$, $\hat{C}^*_{y_0} = \dfrac{s^*_{y_0}}{\overline{y}_0^*}$, $\hat{C}^*_{y_1} = \dfrac{s^*_{y_1}}{\overline{y}_1^*}$,

$s^*_{y_0 y_1} = (n - z - 1)^{-1} \sum_{j=1}^{n-r} (y_{0j} - \overline{y}_0^*)(y_{1j} - \overline{y}_1^*)$, $s^{*2}_{y_0} = (n - z - 1)^{-1} \sum_{j=1}^{n-z} (y_{0j} - \overline{y}_0^*)^2$,

$s^{*2}_{y_1} = (n - z - 1)^{-1} \sum_{j=1}^{n-z} (y_{1j} - \overline{y}_1^*)^2$, $\hat{\rho}^*_{y_0 x_i} = s^*_{y_0 x_i} \big/ (s^*_{y_0} s^*_{x_i})$, $\hat{\rho}^*_{y_1 x_i} = s^*_{y_1 x_i} \big/ (s^*_{y_1} s^*_{x_i})$,

$$\hat{\rho}^*_{x_i x_k} = s^*_{x_i x_k} \Big/ (s^*_{x_i} s^*_{x_k}),\ \hat{\lambda}_{y_0 x_i} = (n-z-1)^{-1} \sum_{j=1}^{n-z} (y_{0j} - \bar{y}^*_0)(x_{ij} - \bar{x}^*_i)^2 \Big/ (\bar{y}^*_0 s^{*2}_{x_i}),$$

$$\hat{\lambda}_{y_1 x_i} = (n-z-1)^{-1} \sum_{j=1}^{n-z} (y_{1j} - \bar{y}^*_1)(x_{ij} - \bar{x}^*_i)^2 \Big/ (\bar{y}^*_1 s^{*2}_{x_i}),\ \hat{a}^*_{x_i x_k} = \hat{\rho}^*_{x_i x_k} \hat{C}^*_{x_i} \hat{C}^*_{x_k},$$

$$\hat{C}^*_{x_i} = s^*_{x_i} \Big/ \bar{x}^*_i,\ \hat{C}^*_{x_k} = s^*_{x_k} \Big/ \bar{x}^*_k,$$

$$\hat{f}^*_{x_i x_k} = \sum_{j=1}^{n-z} (x_{ij} - \bar{x}^*_i)(x_{kj} - \bar{x}^*_k)^2 \Big/ ((n-z-1)\bar{x}^{*2}_i s^{*2}_{x_k})$$

and $\hat{\gamma}^*_{x_i x_k} = \sum_{j=1}^{n-z} (x_{ij} - \bar{x}^*_i)^2 (x_{kj} - \bar{x}^*_k)^2 \Big/ ((n-z-1) s^{*2}_{x_i} s^{*2}_{x_k}) - 1$.

6.1. *Special cases*

For the numerical comparison of the various estimators, we consider a situation when only two auxiliary variables are available. Let d_1 be an estimator making use of the population means $\bar{X}_1$ and $\bar{X}_2$ as well as known population variances $S^2_{x_1}$ and $S^2_{x_2}$. It can be easily seen that the percent relative efficiency (RE) of the estimator d_1 with respect to the control estimator $\hat{R}^*_{(\alpha)}$ is give by:

$$\text{RE}(d_1, \hat{R}^*_{(\alpha)}) = \left[A + \frac{(a_1\Delta_1 + a_2\Delta_2 + \lambda_1\Delta_3 + \lambda_2\Delta_4)}{\Delta} \right]^{-1} \times 100\% \tag{6.1.1}$$

Let $d_1^{(1)}$ be an estimator making use of the population means $\bar{X}_1$ and $\bar{X}_2$, then the percent relative efficiency (RE) of $d_1^{(1)}$ with respect to the control estimator $\hat{R}^*_{(\alpha)}$ is given by:

$$\text{RE}(d_1^{(1)}, \hat{R}^*_{(\alpha)}) = \left[A + \frac{(a_1\Delta^*_1 + a_2\Delta^*_2)}{\Delta^*} \right]^{-1} \times 100\% \tag{6.1.2}$$

Let $d_1^{(2)}$ be an estimator making use of the population variances $S^2_{x_1}$ and $S^2_{x_2}$, then

$$\text{RE}(d_1^{(2)}, \hat{R}^*_{(\alpha)}) = \left[A + \frac{(\lambda_1\Delta^*_3 + \lambda_2\Delta^*_4)}{\Delta^*_0} \right]^{-1} \times 100\% \tag{6.1.3}$$

Let $d_1^{(3)}$ be an estimator making use of population mean $\bar{X}_1$ and population variance $S^2_{x_1}$ of only one auxiliary variable, then

$$\mathrm{RE}(d_1^{(3)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[A + \frac{(a_1 D_1 + a_2 D_2)}{D} \right]^{-1} \times 100\% \tag{6.1.4}$$

Let $d_1^{(4)}$ be an estimator making use of only the population mean $\bar{X}_1$ of one auxiliary variable, then

$$\mathrm{RE}(d_1^{(4)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[A - \frac{a_1^2}{C_{x_1}^2} \right]^{-1} \times 100\% \tag{6.1.5}$$

Let $d_1^{(5)}$ be an estimator making use of population variance $S_{x_1}^2$ of the first auxiliary variable, then

$$\mathrm{RE}(d_1^{(5)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[A - \frac{\lambda_1^2}{\beta_2^*(x_1)} \right]^{-1} \times 100\% \tag{6.1.6}$$

Let $d_1^{(6)}$ be an estimator making use of population mean $\bar{X}_2$ and population variance $S_{x_2}^2$ of only the second auxiliary variable, then

$$\mathrm{RE}(d_1^{(6)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[A + \frac{(a_2 D_3 + \lambda_2 D_4)}{D^*} \right]^{-1} \times 100\% \tag{6.1.7}$$

Let $d_1^{(7)}$ be an estimator making use of the population mean $\bar{X}_2$ of the second auxiliary variable, then

$$\mathrm{RE}(d_1^{(7)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[A - \frac{a_2^2}{C_{x_2}^2} \right]^{-1} \times 100\% \tag{6.1.7}$$

Let $d_1^{(8)}$ be an estimator making use of population variance $S_{x_2}^2$ of the second auxiliary variable, then

$$\mathrm{RE}(d_1^{(8)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[A - \frac{\lambda_2^2}{\beta_2^*(x_2)} \right]^{-1} \times 100\% \tag{6.1.8}$$

where

$$A = [C_{y_0}^2 + \alpha(\alpha C_{y_1}^2 - 2\rho_{y_0 y_1} C_{y_0} C_{y_1})],\ a_1 = (a_{y_0 x_1} - \alpha a_{y_1 x_1}),$$

$$a_{y_0 x_1} = \rho_{y_0 x_1} C_{y_0} C_{x_1},\ a_{y_1 x_1} = \rho_{y_1 x_1} C_{y_1} C_{x_1},\ a_2 = (a_{y_0 x_2} - \alpha a_{y_1 x_2}),$$

$$a_{y_0x_2}=\rho_{y_0x_2}C_{y_0}C_{x_2}\,,\ a_{y_1x_2}=\rho_{y_1x_2}C_{y_1}C_{x_2}\,,\ \lambda_1=(\lambda_{y_0x_1}-\alpha\lambda_{y_1x_1})\,,$$

$$\lambda_2=(\lambda_{y_0x_2}-\alpha\lambda_{y_1x_2})\,,\ \lambda_{y_0x_1}=\sum_{j=1}^{N}(Y_{0j}-\bar{Y}_0)(X_{1j}-\bar{X}_1)^2\Big/((N-1)\bar{Y}_0S_{x_1}^2)\,,$$

$$\lambda_{y_1x_1}=\sum_{j=1}^{N}(Y_{1j}-\bar{Y}_1)(X_{1j}-\bar{X}_1)^2\Big/((N-1)\bar{Y}_1S_{x_1}^2)\,,$$

$$\lambda_{y_0x_2}=\sum_{j=1}^{N}(Y_{0j}-\bar{Y}_0)(X_{2j}-\bar{X}_2)^2\Big/((N-1)\bar{Y}_0S_{x_2}^2)$$

$$\lambda_{y_1x_2}=\sum_{j=1}^{N}(Y_{1j}-\bar{Y}_1)(X_{2j}-\bar{X}_2)^2\Big/((N-1)\bar{Y}_1S_{x_2}^2)$$

$$\gamma_{x_1x_2}=\sum_{j=1}^{N}(X_{1j}-\bar{X}_1)^2(X_{2j}-\bar{X}_2)^2\Big/((N-1)S_{x_1}^2S_{x_2}^2)-1\,,$$

$$\Delta=C_{x_1}^2\Delta_{(1)}-a_{x_1x_2}\Delta_{(2)}+f_{x_1x_1}\Delta_{(3)}-f_{x_1x_2}\Delta_{(4)}$$

$$\Delta_{(1)}=C_{x_2}^2(\beta_2^*(x_1)\beta_2^*(x_2)-\gamma_{x_1x_2}^2)-f_{x_2x_1}(f_{x_2x_1}\beta_2^*(x_2)-f_{x_2x_2}\gamma_{x_1x_2})+$$
$$f_{x_2x_2}(f_{x_2x_1}\gamma_{x_1x_2}-f_{x_2x_2}\beta_2^*(x_1))$$

$$\Delta_{(2)}=a_{x_1x_2}(\beta_2^*(x_1)\beta_2^*(x_2)-\gamma_{x_1x_2}^2)-f_{x_2x_1}(f_{x_1x_1}\beta_2^*(x_2)-f_{x_1x_2}\gamma_{x_1x_2})+$$
$$f_{x_1x_2}(f_{x_1x_1}\gamma_{x_1x_2}-f_{x_1x_2}\beta_2^*(x_1))$$

$$\Delta_{(3)}=a_{x_1x_2}(f_{x_2x_1}\beta_2^*(x_2)-f_{x_1x_2}\gamma_{x_1x_2})-C_{x_2}^2(f_{x_1x_1}\beta_2^*(x_1)-f_{x_1x_2}\gamma_{x_1x_2})+$$
$$f_{x_2x_2}(f_{x_1x_1}f_{x_2x_2}-f_{x_1x_2}f_{x_2x_1})$$

$$\Delta_{(4)}=a_{x_1x_2}(f_{x_2x_1}\gamma_{x_1x_2}-f_{x_2x_2}\beta_2^*(x_1))-C_{x_2}^2(f_{x_1x_1}\gamma_{x_1x_2}-f_{x_1x_2}\beta_2^*(x_1))+$$
$$f_{x_2x_1}(f_{x_1x_1}f_{x_2x_2}-f_{x_1x_2}f_{x_2x_1})$$

$$\Delta_1=-a_1\Delta_{1(1)}-a_{x_1x_2}\Delta_{1(2)}+f_{x_1x_2}\Delta_{1(3)}-f_{x_1x_2}\Delta_{1(4)}$$

$$\Delta_{1(1)}=C_{x_2}^2(\beta_2^*(x_1)\beta_2^*(x_2)-\gamma_{x_1x_2}^2)-f_{x_2x_1}(f_{x_2x_1}\beta_2^*(x_2)-f_{x_2x_2}\gamma_{x_1x_2})+$$
$$f_{x_2x_2}(f_{x_2x_1}\gamma_{x_1x_2}-f_{x_2x_2}\beta_2^*(x_1))$$

$$\Delta_{1(2)} = -a_2(\beta_2^*(x_1)\beta_2^*(x_2) - \gamma_{x_1x_2}^2) - f_{x_2x_1}(\lambda_2\gamma_{x_1x_2} - \lambda_1\beta_2^*(x_2)) + f_{x_2x_2}(\lambda_2\beta_2^*(x_1) - \lambda_1\gamma_{x_1x_2})$$

$$\Delta_{1(3)} = -a_2(f_{x_2x_1}\beta_2^*(x_2) - f_{x_2x_2}\gamma_{x_1x_2}) - C_{x_2}^2(\lambda_2\gamma_{x_1x_2} - \lambda_1\beta_2^*(x_2)) + f_{x_2x_2}(\lambda_2 f_{x_2x_1} - \lambda_1 f_{x_2x_2})$$

$$\Delta_{1(4)} = -a_2(f_{x_2x_1}\gamma_{x_1x_2} - f_{x_2x_2}\beta_2^*(x_1)) - C_{x_2}^2(\lambda_2\beta_2^*(x_1) - \lambda_1\gamma_{x_1x_2}) + f_{x_2x_1}(\lambda_2 f_{x_2x_1} - \lambda_1 f_{x_2x_2})$$

$$\Delta_2 = C_{x_1}^2\Delta_{2(1)} + a_1\Delta_{2(2)} + f_{x_1x_1}\Delta_{2(3)} - f_{x_1x_2}\Delta_{2(4)}$$

$$\Delta_{2(1)} = -a_2(\beta_2^*(x_1)\beta_2^*(x_2) - \gamma_{x_1x_2}^2) - f_{x_2x_1}(\lambda_2\gamma_{x_1x_2} - \lambda_1\beta_2^*(x_2)) + f_{x_2x_2}(\lambda_2\beta_2^*(x_1) - \lambda_1\gamma_{x_1x_2})$$

$$\Delta_{2(2)} = a_{x_1x_2}(\beta_2^*(x_1)\beta_2^*(x_2) - \gamma_{x_1x_2}^2) - f_{x_2x_1}(f_{x_1x_1}\beta_2^*(x_2) - f_{x_1x_2}\gamma_{x_1x_2}) + f_{x_2x_2}(f_{x_1x_1}\gamma_{x_1x_2} - \beta_2^*(x_1) f_{x_1x_2})$$

$$\Delta_{2(3)} = a_{x_1x_2}(\lambda_2\gamma_{x_1x_2} - \lambda_1\beta_2^*(x_2)) + a_2(f_{x_1x_2}\beta_2^*(x_2) - f_{x_1x_2}\gamma_{x_1x_2}) + f_{x_2x_2}(\lambda_1 f_{x_1x_2} - \lambda_2 f_{x_1x_1})$$

$$\Delta_{2(4)} = a_{x_1x_2}(\lambda_2\beta^*{}_2(x_1) - \lambda_1\gamma_{x_1x_2}) + a_2(f_{x_1x_2}\gamma_{x_1x_2} - f_{x_1x_2}\beta_2^*(x_1)) + f_{x_2x_1}(\lambda_1 f_{x_1x_2} - \lambda_2 f_{x_1x_1})$$

$$\Delta_3 = C_{x_1}^2\Delta_{3(1)} - a_{x_1x_2}\Delta_{3(2)} - a_1\Delta_{3(3)} - f_{x_1x_2}\Delta_{3(4)}$$

$$\Delta_{3(1)} = C_{x_2}^2(\lambda_2\gamma_{x_1x_2} - \lambda_1\beta_2^*(x_2)) + a_2(f_{x_2x_1}\beta_2^*(x_2) - f_{x_2x_2}\gamma_{x_1x_2}) + f_{x_2x_2}(\lambda_1 f_{x_2x_2} - \lambda_2 f_{x_2x_1})$$

$$\Delta_{3(2)} = a_{x_1x_2}(\lambda_2\gamma_{x_1x_2} - \lambda_1\beta_2^*(x_2)) + a_2(f_{x_1x_2}\beta_2^*(x_2) - f_{x_1x_2}\gamma_{x_1x_2}) + f_{x_2x_2}(\lambda_1 f_{x_1x_2} - \lambda_2 f_{x_1x_1})$$

$$\Delta_{3(3)} = a_{x_1x_2}(f_{x_2x_1}\beta_2^*(x_2) - \gamma_{x_1x_2} f_{x_2x_2}) - C_{x_2}^2(f_{x_1x_1}\beta_2^*(x_2) - \gamma_{x_1x_2} f_{x_1x_2}) + f_{x_2x_2}(f_{x_1x_1} f_{x_2x_2} - f_{x_1x_2} f_{x_2x_1})$$

$$\Delta_{3(4)} = a_{x_1x_2}(\lambda_1 f_{x_2x_2} - \lambda_2 f_{x_2x_1}) - C_{x_2}^2(\lambda_1 f_{x_1x_2} - \lambda_2 f_{x_1x_1}) - a_2(f_{x_1x_1}f_{x_2x_2} - f_{x_1x_2}f_{x_2x_1})$$

$$\Delta_4 = C_{x_1}^2\Delta_{4(1)} - a_{x_1x_2}\Delta_{4(2)} + f_{x_1x_1}\Delta_{4(3)} + a_1\Delta_{4(4)}$$

$$\Delta_{4(1)} = C_{x_2}^2(\lambda_1\gamma_{x_1x_2} - \lambda_3\beta_2^*(x_1)) - f_{x_2x_2}(\lambda_1 f_{x_2x_2} - \lambda_2 f_{x_2x_1}) - a_2(f_{x_2x_1}\gamma_{x_1x_2} - f_{x_2x_2}\beta_2^*(x_1))$$

$$\Delta_{4(2)} = a_{x_1x_2}(\lambda_1\gamma_{x_1x_2} - \lambda_2\beta_2^*(x_1)) - f_{x_2x_1}(\lambda_1 f_{x_1x_2} - \lambda_2 f_{x_2x_1}) - a_2(f_{x_1x_1}\gamma_{x_1x_2} - f_{x_1x_2}\beta_2^*(x_1))$$

$$\Delta_{4(3)} = a_{x_1x_2}(\lambda_1 f_{x_2x_2} - \lambda_2 f_{x_2x_1}) - C_{x_2}^2(\lambda_1 f_{x_1x_2} - \lambda_2 f_{x_1x_1}) - a_2(f_{x_1x_1}f_{x_2x_2} - f_{x_1x_2}f_{x_2x_1})$$

$$\Delta_{4(4)} = a_{x_1x_2}(f_{x_2x_1}\gamma_{x_1x_2} - f_{x_2x_2}\beta_2^*(x_1)) - C_{x_2}^2(f_{x_1x_1}\gamma_{x_1x_2} - f_{x_1x_2}\beta_2^*(x_1)) + f_{x_2x_1}(f_{x_1x_2}f_{x_2x_2} - f_{x_1x_2}f_{x_2x_1})$$

$\beta_2^*(x_1) = \beta_2^*(x_1) - 1$, $\beta_2^*(x_2) = \beta_2^*(x_2) - 1$, $C_{x_1}^2 = S_{x_1}^2 / \bar{X}_1^2$, $C_{x_2}^2 = S_{x_2}^2 / \bar{X}_2^2$, $a_{x_1x_2} = \rho_{x_1x_2}C_{x_1}C_{x_2}$,

$$f_{x_1x_1} = \sum_{j=1}^{N}(X_{1j} - \bar{X}_1)^3 \Big/ ((N-1)\bar{X}_1 S_{x_1}^2),$$

$$f_{x_1x_2} = \sum_{j=1}^{N}(X_{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2)^2 \Big/ ((N-1)\bar{X}_1 S_{x_2}^2)$$

$$f_{x_2x_1} = \sum_{j=1}^{N}(X_{1j} - \bar{X}_1)^2(X_{2j} - \bar{X}_2) \Big/ ((N-1)\bar{X}_2 S_{x_1}^2),$$

$$f_{x_2x_2} = \sum_{j=1}^{N}(X_{2j} - \bar{X}_2)^3 \Big/ ((N-1)\bar{X}_2 S_{x_2}^2), \ \Delta^* = C_{x_1}^2 C_{x_2}^2 - a_{x_1x_2}^2,$$

$\Delta_1^* = a_2 a_{x_1x_2} - a_1 C_{x_2}^2$, $\Delta_2^* = a_1 a_{x_1x_2} - a_2 C_{x_1}^2$, $\Delta_0^* = \beta_2^*(x_1)\beta_2^*(x_2) - \gamma_{x_1x_2}^2$,

$\Delta_3^* = \lambda_2\gamma_{x_1x_2} - \lambda_1\beta_2^*(x_2)$, $\Delta_4^* = \lambda_1\gamma_{x_1x_2} - \lambda_2\beta_2^*(x_1)$, $D_1 = \lambda_1 f_{x_1x_1} - a_1\beta_2(x_1)$,

$D_2 = a_1 f_{x_1x_1} - \lambda_1 C_{x_1}^2$, $D = C_{x_1}^2\beta_2^*(x_1) - f_{x_1x_1}^2$, $D_3 = \lambda_2 f_{x_2x_2} - a_2\beta_2^*(x_2)$,

$D_4 = a_2 f_{x_2x_2} - \lambda_2 C_{x_2}^2$, and $D^* = C_{x_2}^2\beta_2^*(x_2) - f_{x_2x_2}^2$.

7. SUGGESTED STRATEGY-II

Here we are considering the situation when information on study variables y_0, y_1 could not be obtained for z units while information on 'm'(>1) auxiliary variables X_1, X_2, ..., X_m are available. The population means $\bar{X}_i$ and variances $S^2_{x_i}, (i=1,2,...,m)$ of auxiliary variables X_1, X_2, ..., X_m are known. Thus we define a family of estimators of $R_{(\alpha)}$ as

$$d_2 = S(\hat{R}^*_{(\alpha)}, u^T), \tag{7.1}$$

where $S(\hat{R}^*_{(\alpha)}, u^T)$ is a parametric function of $(\hat{R}^*_{(\alpha)}, u^T)$ such that

$$S(R_{(\alpha)}, e^T) = R_{(\alpha)} \text{ for all } \hat{R}^*_{(\alpha)}$$

which implies that

$$\frac{\partial S(\bullet)}{\partial \hat{R}^*_{(\alpha)}}\Big|_{(R_{(\alpha)}, e^T)} = 1 \tag{7.2}$$

and satisfies certain conditions as given for $\hat{R}^{(\alpha)}_{mg}$. To terms of order n^{-1}, it can be easily shown that:

$$E(d_2) = R_{(\alpha)} + O(n^{-1}) \tag{7.3}$$

and

$$\begin{aligned} MSE(d_2) = MSE(\hat{R}^*_{(\alpha)}) + \theta[2R_{(\alpha)} d^T S^{(1)}(R_{(\alpha)}, e^T) + \\ (S^{(1)}(R_{(\alpha)}, e^T))^T D(S^{(1)}(R_{(\alpha)}, e^T))] \end{aligned} \tag{7.4}$$

where

$$S^{(1)}{}_{\left(R_{(\alpha)}, e^T\right)} = \frac{\partial S(\bullet)}{\partial u}\Big|_{(R_{(\alpha)}, e^T)}$$

The MSE(d_2) at (7.4) is minimized for

$$S^{(1)}(R_{(\alpha)}, e^T) = -R_{(\alpha)} D^{-1} d \tag{7.5}$$

and hence the resultant (minimum) MSE of d_2 is given by

$$MSE(d_2) = MSE(\hat{R}^*_{(\alpha)}) - \theta R^2_{(\alpha)} d^T D^{-1} d$$
$$= \min.MSE(d_1) + (\theta^* - \theta) R^2_{(\alpha)} d^T D^{-1} d \tag{7.6}$$

where $\min.MSE(d_1)$ is given in (6.8). Now we state the following theorem:

Theorem 7.1. To the first degree of approximation

$$\min.MSE(d_2) \geq MSE(\hat{R}^*_{(\alpha)}) - \theta R^2_{(\alpha)} d^T D^{-1} d$$

with equality holding if

$$S^{(1)}(R_{(\alpha)}, e^T) = -R_{(\alpha)} D^{-1} d\,.$$

The following estimators: 1. $d_2^{(1)} = \hat{R}^*_{(\alpha)} \exp(\phi^T \log u)$, and 2. $d_2^{(2)} = \hat{R}^*_{(\alpha)}(1 + \phi^T(u - e))$ etc. are the members of the suggested class of estimators d_2. Proceeding in a similar manner as before, we state the following theorem.

Theorem 7.2. The family of estimators

$$d_2^* = S^*(\hat{R}^*_{(\alpha)}, u^T, \hat{\psi}^T_{(1)})$$

of $R_{(\alpha)}$ based on estimated optimum values of constants has MSE to the first degree of approximation equal to that of minimum MSE of d_2 that is

$$MSE(d_2^*) = \min.MSE(d_2) \tag{7.7}$$

where $\min.MSE(d_2)$ is cited in (7.6), $S^*(\hat{R}^*_{(\alpha)}, u^T, \hat{\psi}^T_{(1)})$ is a function of $(\hat{R}^*_{(\alpha)}, u^T, \hat{\psi}^T_{(1)})$ such that

$S^*(R_{(\alpha)}, e^T, \psi^T) = R_{(\alpha)}$ for all $\hat{R}^*_{(\alpha)}$ which implies that $\frac{\partial S^*(\bullet)}{\partial \hat{R}^*_{(\alpha)}}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = 1$,

$\frac{\partial S^*(\bullet)}{\partial u}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = -R_{(\alpha)}\psi$ and $\frac{\partial S^*(\bullet)}{\partial \hat{\psi}_{(1)}}\Big|_{(R_{(\alpha)}, e^T, \psi^T)} = 0$ and $\hat{\psi}_{(1)} = \hat{D}^{-1}\hat{d}^*$ is the estimator of $\psi = D^{-1}d$, where $\hat{D}$ and $\hat{d}^*$ are same as defined earlier.

7.1 *Special cases*

Let d_2 be an estimator making use of the population means $\bar{X}_1$ and $\bar{X}_2$ as well as known population variances $S_{x_1}^2$ and $S_{x_2}^2$. It can be easily seen that the percent relative efficiency of the estimator d_2 with respect to the control estimator $\hat{R}_{(\alpha)}^*$ is given by:

$$\mathrm{RE}(d_2, \hat{R}_{(\alpha)}^*) = \left[1 + \left(\frac{\theta}{\theta^*}\right)\frac{(a_1\Delta_1 + a_2\Delta_2 + \lambda_1\Delta_3 + \lambda_2\Delta_4)}{\Delta\, A}\right]^{-1} \times 100\% \qquad (7.1.1)$$

Let $d_2^{(1)}$ be an estimator making use of the population means $\bar{X}_1$ and $\bar{X}_2$, then the percent relative efficiency (PRE) of $d_2^{(1)}$ with respect to the control estimator $\hat{R}_{(\alpha)}^*$ is give by:

$$\mathrm{RE}(d_2^{(1)}, \hat{R}_{(\alpha)}^*) = \left[1 + \left(\frac{\theta}{\theta^*}\right)\frac{(a_1\Delta_1^* + a_2\Delta_2^*)}{A\,\Delta^*}\right]^{-1} \times 100\% \qquad (7.1.2)$$

Let $d_2^{(2)}$ be an estimator making use of the population variances $S_{x_1}^2$ and $S_{x_2}^2$, then

$$\mathrm{RE}(d_2^{(2)}, \hat{R}_{(\alpha)}^*) = \left[1 + \left(\frac{\theta}{\theta^*}\right)\frac{(\lambda_1\Delta_3^* + \lambda_2\Delta_4^*)}{A\,\Delta_0^*}\right]^{-1} \times 100\% \qquad (7.1.3)$$

Let $d_2^{(3)}$ be an estimator making use of population mean $\bar{X}_1$ and population variance $S_{x_1}^2$ of only one auxiliary variable, then

$$\mathrm{RE}(d_2^{(3)}, \hat{R}_{(\alpha)}^*) = \left[1 + \left(\frac{\theta}{\theta^*}\right)\frac{(a_1 D_1 + a_2 D_2)}{DA}\right]^{-1} \times 100\% \qquad (7.1.4)$$

Let $d_2^{(4)}$ be an estimator making use of only the population mean $\bar{X}_1$ of one auxiliary variable, then

$$\mathrm{RE}(d_2^{(4)}, \hat{R}_{(\alpha)}^*) = \left[1 - \left(\frac{\theta}{\theta^*}\right)\frac{a_1^2}{AC_{x_1}^2}\right]^{-1} \times 100\% \qquad (7.1.5)$$

Let $d_2^{(5)}$ be an estimator making use of population variance $S_{x_1}^2$ of the first auxiliary variable, then

$$\mathrm{RE}(d_2^{(5)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[1 - \left(\frac{\theta}{\theta^*}\right)\frac{\lambda_1^2}{A\beta_2^*(x_1)}\right]^{-1} \times 100\% \tag{7.1.6}$$

Let $d_2^{(6)}$ be an estimator making use of population mean $\bar{X}_2$ and population variance $S_{x_2}^2$ of only the second auxiliary variable, then

$$\mathrm{RE}(d_2^{(6)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[1 + \left(\frac{\theta}{\theta^*}\right)\frac{(a_2 D_3 + \lambda_2 D_4)}{AD^*}\right]^{-1} \times 100\% \tag{7.1.7}$$

Let $d_2^{(7)}$ be an estimator making use of the population mean $\bar{X}_2$ of the second auxiliary variable, then

$$\mathrm{RE}(d_2^{(7)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[1 - \left(\frac{\theta}{\theta^*}\right)\frac{a_2^2}{AC_{x_2}^2}\right]^{-1} \times 100\% \tag{7.1.8}$$

Let $d_2^{(8)}$ be an estimator making use of population variance $S_{x_2}^2$ of the second auxiliary variable, then

$$\mathrm{RE}(d_2^{(8)}, \hat{\mathrm{R}}^*_{(\alpha)}) = \left[1 - \left(\frac{\theta}{\theta^*}\right)\frac{\lambda_2^2}{A\beta_2^*(x_2)}\right]^{-1} \times 100\% \tag{7.1.9}$$

8. SUGGESTED STRATEGY-III

We are again considering the situation when information on study variables y_0, y_1 could not be obtained for z units while information on auxiliary variables X_1, X_2,..., X_m are available for all the sample units. But the difference is that population means $\bar{X}_i$ and variances $S_{x_i}^2, (i = 1, 2, ..., m)$ of auxiliary variables X_1, X_2,..., X_m are not known. With this background, we define a family of estimators for $R_{(\alpha)}$ as:

$$d_3 = V(\hat{\mathrm{R}}^*_{(\alpha)}, w^T), \tag{8.1}$$

where

$w_i = \bar{x}_i^* / \bar{x}_i$, $\quad i = 1, 2, ..., m;$ and $\quad w_i = (s_{x_{i-m}}^{*2}) / (s_{x_{i-m}}^2)$, $\quad i = m+1, m+2, .., 2m$,

$w^T = (w_1, w_2, ..., w_{2m})$; $V(\hat{\mathrm{R}}^*_{(\alpha)}, w^T)$ is a function of $(\hat{\mathrm{R}}^*_{(\alpha)}, w^T)$ such that:

$$V(\mathrm{R}_{(\alpha)}, e^T) = \mathrm{R}_{(\alpha)} \text{ for all } \hat{\mathrm{R}}^*_{(\alpha)} \tag{8.2}$$

which implies that

$$\frac{\partial V(\bullet)}{\partial \hat{\mathrm{R}}^*_{(\alpha)}}\Big|_{(\mathrm{R}_{(\alpha)}, e^T)} = 1 \tag{8.3}$$

and satisfies certain conditions. To terms of order n^{-1}, it can be easily shown that

$$E(d_3) = \mathrm{R}_{(\alpha)} + O(n^{-1}) \tag{8.4}$$

and

$$\begin{aligned} MSE(d_3) = MSE(\hat{\mathrm{R}}^*_{(\alpha)}) + (\theta^* - \theta)[2\mathrm{R}_{(\alpha)} d^T V^{(1)}(\mathrm{R}_{(\alpha)}, e^T) + \\ (V^{(1)}(\mathrm{R}_{(\alpha)}, e^T))^T D(V^{(1)}(\mathrm{R}_{(\alpha)}, e^T))] \end{aligned} \tag{8.5}$$

where

$$V^{(1)}{}_{(\mathrm{R}_{(\alpha)}, e^T)} = \frac{\partial V(\bullet)}{\partial w}\Big|_{(\mathrm{R}_{(\alpha)} \cdot e^T)} \tag{8.6}$$

The $MSE(d_3)$ at (8.5) is minimized for

$$V^{(1)}(R_{(\alpha)}, e^T) = -R_{(\alpha)} D^{-1} d \tag{8.7}$$

and hence the resultant (minimum) MSE of d_3 is given by

$$\min . MSE(d_3) = MSE(\hat{\mathrm{R}}^*_{(\alpha)}) - (\theta^* - \theta)\mathrm{R}^2_{(\alpha)} d^T D^{-1} d \tag{8.8}$$

which shows that the proposed class of estimators is more efficient than $\hat{R}^*_{(\alpha)}$. Now we state the following theorem:

Theorem 8.1. To the first degree of approximation

$$MSE(d_3) \geq MSE(\hat{\mathrm{R}}^*_{(\alpha)}) - (\theta^* - \theta)\mathrm{R}^2_{(\alpha)} d^T D^{-1} d$$

with equality holding if

$$V^{(1)}(R_{(\alpha)}, e^T) = -R_{(\alpha)} D^{-1} d \, .$$

The following estimators: 1. $d_3^{(1)} = \hat{\mathrm{R}}^*_{(\alpha)} \exp(\phi^T \log w)$, and 2. $d_3^{(2)} = \hat{\mathrm{R}}^*_{(\alpha)}(1 + \phi^T(w - e))$ etc. are the members of the suggested class of estimators d_3. Proceeding in a similar manner as section 4, we state the following theorem.

Theorem 8.2. The family of estimators

$$d_3^* = V^*(\hat{R}^*_{(\alpha)}, w^T, \hat{\psi}^T_{(2)})$$

of $R_{(\alpha)}$ based on estimated optimum values of constants has MSE to the first degree of approximation equal to that of minimum MSE of d_3 that is

$$MSE(d_3^*) = \min.MSE(d_3) \tag{8.9}$$

where $\min.MSE(d_3)$ is cited in (8.8), $V^*(\hat{R}^*_{(\alpha)}, w^T, \hat{\psi}^T_{(2)})$ is a function of $(\hat{R}^*_{(\alpha)}, w^T, \hat{\psi}^T_{(2)})$ such that $V^*(R_{(\alpha)}, e^T, \psi^T) = R_{(\alpha)}$ for all $\hat{R}^*_{(\alpha)}$ which implies that $\frac{\partial V^*(\bullet)}{\partial \hat{R}^*_{(\alpha)}}|_{(R_{(\alpha)}, e^T, \psi^T)} = 1$, $\frac{\partial V^*(\bullet)}{\partial w}|_{(R_{(\alpha)}, e^T, \psi^T)} = -R_{(\alpha)}\psi$ and $\frac{\partial V^*(\bullet)}{\partial \hat{\psi}_{(2)}}|_{(R_{(\alpha)}, e^T, \psi^T)} = 0$ and $\hat{\psi}_{(2)} = \hat{D}^{*-1}\hat{d}^*$ is the estimator of $\psi = D^{-1}d$, where $\hat{D}^*$ and $\hat{d}^*$ are same as defined earlier.

8.1 *Special cases*

Let d_3 be an estimator making use of the population means $\bar{X}_1$ and $\bar{X}_2$ as well as known population variances $S^2_{x_1}$ and $S^2_{x_2}$. It can be easily seen that the percent relative efficiency of the estimator d_3 with respect to the control estimator $\hat{R}^*_{(\alpha)}$ is give by:

$$\mathrm{RE}(d_3, \hat{R}^*_{(\alpha)}) = \left[1 + \left(1 - \frac{\theta}{\theta^*}\right)\frac{\left(a_1\Delta_1 + a_2\Delta_2 + \lambda_1\Delta_3 + \lambda_2\Delta_4\right)}{\Delta\, A}\right]^{-1} \times 100\% \tag{8.1.1}$$

Let $d_3^{(1)}$ be an estimator making use of the population means $\bar{X}_1$ and $\bar{X}_2$, then the percent relative efficiency (PRE) of $d_3^{(1)}$ with respect to the control estimator $\hat{R}^*_{(\alpha)}$ is given by:

$$\mathrm{RE}(d_3^{(1)}, \hat{R}^*_{(\alpha)}) = \left[1 + \left(1 - \frac{\theta}{\theta^*}\right)\frac{(a_1\Delta_1^* + a_2\Delta_2^*)}{A\,\Delta^*}\right]^{-1} \times 100\% \tag{8.1.2}$$

Let $d_3^{(2)}$ be an estimator making use of the population variances $S_{x_1}^2$ and $S_{x_2}^2$, then

$$\mathrm{RE}(d_3^{(2)}, \hat{R}_{(\alpha)}^*) = \left[1 + \left(1 - \frac{\theta}{\theta^*}\right)\frac{(\lambda_1\Delta_3^* + \lambda_2\Delta_4^*)}{A\Delta_0^*}\right]^{-1} \times 100\% \tag{8.1.3}$$

Let $d_3^{(3)}$ be an estimator making use of population mean $\bar{X}_1$ and population variance $S_{x_1}^2$ of only one auxiliary variable, then

$$\mathrm{RE}(d_3^{(3)}, \hat{R}_{(\alpha)}^*) = \left[1 + \left(1 - \frac{\theta}{\theta^*}\right)\frac{(a_1D_1 + a_2D_2)}{DA}\right]^{-1} \times 100\% \tag{8.1.4}$$

Let $d_3^{(4)}$ be an estimator making use of only the population mean $\bar{X}_1$ of one auxiliary variable, then

$$\mathrm{RE}(d_3^{(4)}, \hat{R}_{(\alpha)}^*) = \left[1 - \left(1 - \frac{\theta}{\theta^*}\right)\frac{a_1^2}{AC_{x_1}^2}\right]^{-1} \times 100\% \tag{8.1.5}$$

Let $d_3^{(5)}$ be an estimator making use of population variance $S_{x_1}^2$ of the first auxiliary variable, then

$$\mathrm{RE}(d_3^{(5)}, \hat{R}_{(\alpha)}^*) = \left[1 - \left(1 - \frac{\theta}{\theta^*}\right)\frac{\lambda_1^2}{A\beta_2^*(x_1)}\right]^{-1} \times 100\% \tag{8.1.6}$$

Let $d_3^{(6)}$ be an estimator making use of population mean $\bar{X}_2$ and population variance $S_{x_2}^2$ of only the second auxiliary variable, then

$$\mathrm{RE}(d_3^{(6)}, \hat{R}_{(\alpha)}^*) = \left[1 + \left(1 - \frac{\theta}{\theta^*}\right)\frac{(a_2D_3 + \lambda_2D_4)}{AD^*}\right]^{-1} \times 100\% \tag{8.1.7}$$

Let $d_3^{(7)}$ be an estimator making use of the population mean $\bar{X}_2$ of the second auxiliary variable, then

$$\mathrm{RE}(d_3^{(7)}, \hat{R}_{(\alpha)}^*) = \left[1 - \left(1 - \frac{\theta}{\theta^*}\right)\frac{a_2^2}{AC_{x_2}^2}\right]^{-1} \times 100\% \tag{8.1.8}$$

Let $d_3^{(8)}$ be an estimator making use of population variance $S_{x_2}^2$ of the second auxiliary variable, then

$$\mathrm{RE}(d_3^{(8)}, \hat{\mathrm{R}}_{(\alpha)}^*) = \left[1 - \left(1 - \frac{\theta}{\theta^*}\right)\frac{\lambda_2^2}{A\beta_2^*(x_2)}\right]^{-1} \times 100\% \tag{8.1.9}$$

In the next section, we show numerically computed values of the percent relative efficiency values based on a real dataset Daniel (1995).

9. NUMERICAL COMPARISONS

Note carefully that the percent relative efficiency values under Strategy-I, listed in Section 6.1, are free from the sample size n and non-response rate p, but depend only on the parameters of the population being studied. In contrast the values of the percent relative efficiencies under Strategies-II and III, listed in sections 7.1 and 8.1, depend upon the values of the sample size n and response rate p, and the population size N. From Daniel (1995), we consider the data set on page 424 in our empirical study where four variables: Allen Cognitive Level Test (ACL), scores on the Symbol-Digit Modalities Test (SDMT), scores on the vocabulary (V), and abstraction (A) are listed. As per notations of this paper, we consider the variable $Y_0 = ACL$, $Y_1 = SDMT$, $X_1 = V$ and $X_2 = A$. In this study we consider $\alpha = -1$ which means the problem of estimation of ratio of two types of scores. There are total $N = 69$ data values.

TABLE 9.1

Descriptive parameters of the population

Variable	Mean	StDev	Min	Median	Max	Skewness	Kurtosis
ACL	4.9435	0.7957	3.4	4.8	6.6	0.40	-0.71
SDMT	42.39	14.76	2.0	76	76	-0.24	-0.03
V	24.10	6.67	10	26	40	-0.23	-0.66
A	20.91	10.60	2.0	20	40	-0.01	-1.30

TABLE 9.2

Population Correlation Coefficients matrix

	ACL	SDMT	V	A
ACL	1	0.521	0.250	0.354
SDMT	-	1	0.556	0.577
V	-	-	1	0.698
A	-	-	-	1

We wrote FORTRAN code (See online Appendix) to compute the values of percent relative efficiencies. The percent relative efficiency values obtained from section 6.1 for Strategy I are given in Table 9.3.

TABLE 9.3

Percent Relative Efficiency values for Strategy-I

Percent Relative Efficiency $RE(*, \hat{R}_{(\alpha)})$ where * is the estimator given below:								
d_1	$d_1^{(1)}$	$d_1^{(2)}$	$d_1^{(3)}$	$d_1^{(4)}$	$d_1^{(5)}$	$d_1^{(6)}$	$d_1^{(7)}$	$d_1^{(8)}$
3254.79	2644.72	547.34	2904.06	2550.41	545.43	750.67	719.01	495.75

It is very interesting to note that although there is not very strong correlation between the variables, still the percent relative efficiency of the proposed estimators under strategy-I varies from 495.75% to 3254.79% depending upon the type of known auxiliary information as listed in section 6.1. Thus, it seems that the proposed estimators under Strategy-I are certainly efficient. For Strategy-II, the percent relative efficiency formulas listed in Section 7.1 depend upon the values of the sample size n and non-response rate p. Thus, we changed the sample size n between 10 to 25 with a step of 5, and the value of p is changed from 0.1 to 0.9 with a step of 0.2 to see the effect of both sample size and non-response rate. The results so obtained are listed in Table 9.4. From Table 9.4 two pictures are very clear that the estimator d_2 is always more efficient than other estimators, $d_2^{(j)}$, $j = 1, 2, .., 8$ which are in fact members of the generalized estimator d_2. Another point is clear that as the sample size n increases the relative gain decreases for given value of non-response rate. Also for a given sample size as the value of p increases from 0.1 to 0.9, the percent relative efficiency value also decreases. The least efficient strategy is $d_2^{(8)}$ which may be true because the known population variance of the auxiliary variable does not help in improving the estimator when the population is bivariate normal.

TABLE 9.4

Percent relative efficiency values for Strategy-II

		Percent Relative Efficiency $RE(*,\hat{R}_{(\alpha)})$ where * is the estimator given below:								
n	p	d_2	$d_2^{(1)}$	$d_2^{(2)}$	$d_2^{(3)}$	$d_2^{(4)}$	$d_2^{(5)}$	$d_2^{(6)}$	$d_2^{(7)}$	$d_2^{(8)}$
10	0.1	438.65	385.70	111.23	409.23	376.73	110.88	147.96	141.56	101.73
	0.3	263.95	247.50	108.84	255.08	244.50	108.57	135.28	130.92	101.39
	0.5	191.54	184.69	106.67	187.89	183.39	106.47	125.10	122.21	101.06
	0.7	151.93	148.80	104.68	150.27	148.19	104.54	116.75	114.94	100.76
	0.9	126.94	125.57	102.85	126.22	125.31	102.77	109.78	108.78	100.47
15	0.1	414.18	367.39	111.01	388.29	359.38	110.67	146.73	140.54	101.70
	0.3	242.13	228.94	108.31	235.04	226.51	108.06	132.70	128.73	101.31
	0.5	175.50	170.29	105.96	172.73	169.29	105.78	122.04	119.56	100.96
	0.7	140.11	137.87	103.89	138.93	137.44	103.78	113.66	112.22	100.63
	0.9	118.18	117.31	102.05	117.72	117.14	101.99	106.90	106.21	100.34
20	0.1	394.58	352.49	110.82	371.36	345.22	110.48	145.65	139.64	101.67
	0.3	226.89	215.79	107.89	220.94	213.73	107.65	130.68	127.01	101.25
	0.5	165.35	161.08	105.45	163.08	160.26	105.29	119.89	117.69	100.88
	0.7	133.39	131.61	103.38	132.45	131.26	103.29	111.74	110.52	100.55
	0.9	113.82	113.19	101.61	113.49	113.07	101.57	105.37	104.84	100.27
25	0.1	375.83	338.04	110.61	355.05	331.46	110.29	144.54	138.71	101.64
	0.3	213.82	204.39	107.48	208.78	202.63	107.26	128.78	125.38	101.19
	0.5	157.26	153.70	105.00	155.37	153.01	104.85	118.05	116.07	100.81
	0.7	128.48	127.02	102.99	127.71	126.73	102.90	110.26	109.21	100.49
	0.9	111.05	110.55	101.32	110.79	110.46	101.28	104.36	103.93	100.22

In the same way, Table 9.5 shows the percent relative efficiency of the proposed class of estimators under Strategy-III.

One thing is very interesting here that as the non-response increases, for a given sample size, the relative efficiency increases. Also as the sample size increases for a given value of non-response, the relative efficiency increases. These findings are just opposite of the case in Strategy-II. Thus we conclude that in case of higher non-response, preferable to use Strategy-III and in case of low response rate prefer to adapt Strategy-II.

TABLE 9.5

Percent relative efficiency values for Strategy-III

		Percent Relative Efficiency $RE(*,\hat{R}_{(\alpha)})$ where * is the estimator given below:								
n	p	d_3	$d_3^{(1)}$	$d_3^{(2)}$	$d_3^{(3)}$	$d_3^{(4)}$	$d_3^{(5)}$	$d_3^{(6)}$	$d_3^{(7)}$	$d_3^{(8)}$
10	0.1	108.52	108.15	101.04	108.32	108.07	101.01	103.41	103.08	100.17
	0.3	129.77	128.22	103.09	128.95	127.92	103.00	110.66	109.56	100.51
	0.5	159.39	155.65	105.12	157.41	154.93	104.97	118.55	116.51	100.83
	0.7	203.56	195.36	107.13	199.18	193.82	106.91	127.16	123.98	101.13
	0.9	276.46	258.01	109.11	266.48	254.65	108.83	136.61	132.05	101.43
15	0.1	110.13	109.68	101.22	109.89	109.59	101.18	104.02	103.62	100.20
	0.3	135.78	133.84	103.57	134.76	133.47	103.47	112.44	111.14	100.58
	0.5	172.52	167.59	105.82	169.90	166.65	105.64	121.43	119.02	100.93
	0.7	229.48	218.04	107.97	223.35	215.92	107.72	131.04	127.32	101.26
	0.9	329.75	301.66	110.02	314.42	296.67	109.71	141.35	136.04	101.56
20	0.1	111.60	111.08	101.38	111.33	110.98	101.34	104.56	104.12	100.23
	0.3	141.10	138.79	103.96	139.88	138.34	103.84	113.93	112.46	100.65
	0.5	183.60	177.58	106.33	180.40	176.44	106.14	123.63	120.94	101.01
	0.7	250.13	235.78	108.52	242.41	233.14	108.26	133.69	129.57	101.34
	0.9	369.12	332.82	110.54	349.18	326.48	110.21	144.11	138.36	101.63
25	0.1	113.20	112.60	101.55	112.88	112.48	101.50	105.15	104.64	100.26
	0.3	146.68	143.95	104.34	145.24	143.43	104.21	115.42	113.77	100.71
	0.5	194.71	187.51	106.79	190.87	186.15	106.59	125.66	122.69	101.08
	0.7	269.44	252.12	108.96	260.09	248.97	108.69	135.87	131.43	101.40
	0.9	401.69	357.92	110.89	377.51	350.38	110.55	146.05	139.97	101.68

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SUMMARY

Estimation of population ratio, product, and mean using multiauxiliary information with random non-response

In this paper, a family of estimators of population ratio R, product P and mean $\overline{Y}_0$ has been suggested using multi-auxiliary information under simple random sampling

without replacement (SRSWOR) and its properties have been discussed. We have further suggested three families of estimators in the presence of random non-response in different situations under an assumption that the number of sampling units on which information cannot be obtained due to random non-response follows some distribution. The estimators of the family involve unknown constants whose optimum values depend on unknown population parameters. When these population parameters are replaced by their consistent estimates, the resulting estimators are shown to have the same asymptotic mean squared error (MSE). The work of Singh et al. (2007) is shown as a special case. At the end, numerical comparisons are also made.