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SPECTRAL DENSITY ESTIMATION FOR SYMMETRIC STABLE P-ADIC PROCESSES

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1. INTRODUCTION

In recent years, there has been a growing interest on p-adic numbers. The latter may answer some questions in Physics. Besides the number of papers in this area shows the interest of p-adic numbers to answer some questions in physics, as in string theory (connected with p-adic quantum field) and in the other natural sciences in which there are complicated fractal behaviors and hierarchical structures (turbulence theory, dynamical systems, statistical physics, biology, see (Kozyrev (2008), Hua-chieh (2001), Dragovich (2009)). Particularly 2-adic numbers will be useful for computer construction see (Klapper (1994)). Cianciri (1994) presented the main ideas to interpret a quantum mechanical state by means of p-adic statistics. He was interested in limits of probabilities when the number of trials approaches infinity. However, these limits are considered with respect to the p-adic metric. Khrennikov (1998) found a new asymptotic of the classical Bernoulli probabilities. In Khrennikov (1993), he developed the theory of p-adic probability to describe the statistical information processes. Kamizono (2007) defined the symmetric stochastic integrals with respect to p-adic Brownian motion and provided a sufficient condition for its existence. The properties of the trajectories of a p-adic Wiener process were studied using Vladimirov's p-adic differentiation operator, see (Bikulov and Volovichb (1997)).

Brillinger (1991) studied central limits theorems for finite Fourier transforms and for a family of quadratic statistics based on stationary processes $\mathrm{X}(t) \quad t \in \mathrm{Q_p}$ where $\mathrm{Q_p}$ is the field of p-adic numbers. He also studied the spectral representation of theses processes and gave spectral density estimation by constructing the periodogram similarly to the real case. Rachdi and Monsan (1999) give another estimator built from discrete-time observations $\mathrm{X}(\tau_\mathrm{k})_{\mathrm{k}\in Z}$ where $(\tau_\mathrm{k})_{\mathrm{k}\in Z}$ is sequence of random variables taking their values in $\mathrm{Q_p}$, associated to a Poisson process.

In this paper, we consider a class of stationary symmetric α-stable processes. The estimate of the spectral density of theses processes is given by Masry and

Cambanis (1984) when the time of processes is continuous real, and by Sabre (1994, 1995) when the time is discrete. Our goal is to extend these works if the process is p-adic time. Precisely,we consider a harmonizable p-adic process: $X(t)=\int_{Q_p} e^{it\lambda} dM(\lambda), \quad t \in Q_p$ where Q_p is the field of the p-adic numbers and M is a symmetric α stable random measure with control measure m. Assume that the measure m is absolutely continuous with respect to Haar measure: $dm(x)=f(x)dH(x), \quad x \in Q_p$. The density function f is called spectral density of the process X. The aim of this paper is to give an estimator of the spectral density f by observing the process X on the p-adic ball U_n. We start by constructing a periodogram that we smooth for obtaining nonparametric estimates of the spectral density which is asymptotically unbiased and consistent.

The paper is organized as follows: Section 2, introduces p-adic processes. The periopdogram based on the observations of the process is defined in section 3. Section 4 gives an asymptotically unbiased and consistent estimate by smoothing the periodogram. Section 5 contains the concluding remarks, the potential applications and the open research problems.

2. PRELIMINARIES

In this section, we define the field Q_p of p-adic numbers and give some properties. Let p be a prime number. Define the following norm: for $a, b \neq 0 \in Z$, $|a/b|_p = p^{n-m}$ where m is the highest power of p dividing a and n is the highest power of p dividing b. The norm of zero is vanishing. Define Q_p as the completion of Q in the metric defined by the norm $|.|_p$. The addition, product, quotient operations are carried over from Q. It follows that the p-adic norm, $|.|_p$, has the following characteristic properties: $|x|_p=0$ is equivalent to $x=0$; $|xy|_p=|x|_p|y|_p$ and $|x+y|_p \leq max(|x|_p,|y|_p)$. Note that $|x|_p$ can take only the countably many values p^m, $m \in Z$. An important result given in Ostrowski's theorem namely the Euclidean and the p-adic norms are the only possible non-trivial norms on the field of rational numbers Q Koblitz (1980) and Valdimirov (1988). All $x \neq 0 \in Q_p$ can be represented in a unique form (Hansel representation) $x=\sum_{i \geq m} x_i p^i$, with $x_i \in \{0,1,...,p-1\}$ and $m \in Z$. If $x_m \neq 0$ then the norm of this p-adic number x is defined to be $|x|_p=p^{-m}$ and the fractional part of a p-adic number x, denoted $<x>$, is defined by: $<x>=\sum_{i<0} x_i p^i$. Note that $<x>\in(0,1)$ and $<x>\leq p|x|_p$. The ring of p-adic integers, Z_p is given by the elements of Q_p satisfying $|x|_p \leq 1$. The ball with center x_0 and radius p^n is defined by: $U_n(x_0)=\{x\in Q_p / |x-x_0|_p \leq p^n\}$. In particular when $x_0=0$ we denote $U_n(0)=U_n$ and $U_0=\{x\in Q_p / |x|_p \leq 1\}$.

Q_p is a complete separable metric space, the stochastic process $X(t,w)$ for $t\in Q_p$ and $w\in\Omega$, $(\Omega,\mathcal{A},P)$ a probability space, is well defined as a map from $Q_p\times\Omega$ to R. $(Q_p,+)$ is an abelian locally compact group; from Haar's theorem there exists a positive measure $\mathcal{H}$ on Q_p, uniquely determined except for a constant. It has the properties:

$$d\mathcal{H}(t+a)=d\mathcal{H}(t) \quad \textit{and} \quad d\mathcal{H}(at)=|a|_p d\mathcal{H}(t) \tag{1}$$

The measure will be normalized by $\mathcal{H}(Z_p)=1$, see (Hewitt and al (1963)). Let t be in Z_p, then $t=t_0+t_1p+t_2p^2+\ldots$ writing $f(t)=g(t_1,t_2,\ldots)$ and taking $(T_0,T_1,\ldots)$ to be a sequence of i.i.d. random variables on the sample space $\{0,1,2,\ldots\}$ with equal probability. Then, it has $\int_{Z_p} f(t)d\mathcal{H}(t)=g(T_0,T_1,T_2,\ldots)$ and

$$\int_{Q_p} f(t)d\mathcal{H}(t)=\lim_{n\to\infty}\int_{|t|_p\leq p^n} f(t)d\mathcal{H}(t)=\lim_{n\to\infty} p^n\int_{Z_p} f(p^{-n}s)d\mathcal{H}(s).$$

3. PERIODOGRAM AND SPECTRAL DENSITY ESTIMATION

Consider a process $X=\{X_t / t\in Q_p\}$ where Q_p is the field of p-adic numbers having the following integral representation

$$X_t=\int_{Q_p} e^{i<t\lambda>}dM(\lambda) \qquad \forall t\in Q_p \tag{2}$$

where M is a symmetric α stable $S\alpha S$ random measure with independent and isotropic increments. There exists a control measure m that is defined by:

$$m(A)=[M(A),M(A)]_\alpha^{1/\alpha}.$$

Assume that the measure m is absolutely continuous with respect to Haar measure: $dm=\Phi(x)d\mathcal{H}(x)$ where $\mathcal{H}$ is Haar measure.

This density is estimated in Masry and Cambanis (1984) when the process has continuous real time, and in Sabre (1994, 1995) when the process and the random field have a discrete time.

The goal of this work is to give an asymptotically and consistent estimate of the density Φ called the spectral density of the process $X=\{X_t/t\in Q_p\}$. For that, we take the ball $U_n=\{x\in Q_{p;;}|x|_p\le p^n\}$ as the observation of the process. Consider the following periodogram:

$$d_n(\lambda)=A_n Re\int_{U_n} e^{-i<t\lambda>}p^{-n}h(tp^n)X(t)d\mathcal{H}(t) \qquad \lambda\in Q_p \tag{3}$$

$$H(\lambda)=\int_{Z_p} h(t)e^{-i<t\lambda>}d\mathcal{H}(t) \tag{4}$$

$$B_\alpha=\int_{Q_p}|H(\lambda)|^\alpha d\mathcal{H}(\lambda)<+\infty \tag{5}$$

$$H_n(\lambda)=\left(\frac{p^n}{B_\alpha}\right)^{\frac{1}{\alpha}}H(p^{-n}\lambda)=A_nH(p^{-n}\lambda) \tag{6}$$

Therefore, $A_n=\left(\frac{p^n}{B_\alpha}\right)^{1/\alpha}$ where

$$\int_{Q_p}|H_n(\lambda)|^\alpha d\mathcal{H}(\lambda)=\int_{Q_p}\frac{p^n}{B_\alpha}|H(p^{-n}\lambda)|^\alpha d\mathcal{H}(\lambda)$$
$$=\frac{p^n}{B_\alpha}p^{-n}\int_{Q_p}|H(v)|^\alpha d\mathcal{H}(v)$$

Since $|p^{-n}|_p=p^{+n}$, we obtain $\int_{Q_p}|H_n(\lambda)|^\alpha d\mathcal{H}(\lambda)=1$.

Proposition 3.1 *Let* $\Psi_n(\lambda)\triangleq\int_{Q_p}|H_n(\lambda-u)|^\alpha\Phi(u)d\mathcal{H}u$. *If* Φ *is continuous and bounded function, then* $B_\alpha(\Psi_n(\lambda)-\Phi(\lambda))$ *converges to zero as n tends to infinity.*

Proof: Using the definition of $\Psi_n(\lambda)$ and from (5) and (6), we have

$$B_\alpha[\Psi_n(\lambda)-\Phi(\lambda)]=$$

$$B_\alpha\left[\int_{Q_p}\left|H_n(\lambda-u)\right|^\alpha \Phi(u)\, d\mathcal{H}(u)-\int_{Q_p}\left|H_n(u)\right|^\alpha \Phi(\lambda)d\mathcal{H}(u)\right]=$$

$$B_\alpha\left[\int_{Q_p}\frac{p^n}{B_\alpha}\left|H(p^n(\lambda-u))\right|^\alpha \Phi(u)du-\int_{Q_p}\frac{p^n}{B_\alpha}\left|H(p^n u)\right|^\alpha \Phi(\lambda)d\mathcal{H}(u)\right]=$$

$$\left[p^n\int_{Q_p}p^{-n}\left|H(v)\right|^\alpha \Phi\left(\lambda-\frac{v}{p^n}\right)dv-p^n\int_{Q_p}p^{-n}\left|H(v)\right|^\alpha \Phi(\lambda)d\mathcal{H}(v)\right]=$$

$$\left[\int_{Q_p}\left|H(v)\right|^\alpha\left[\Phi\left(\lambda-\frac{v}{p^n}\right)-\Phi(\lambda)\right]d\mathcal{H}(v)\right].$$

Since Φ is continuous bounded function and from (5), we obtain

$$B_\alpha[\Psi_n(\lambda)-\Phi(\lambda)]\to 0.$$

Proposition 3.2 *Let $\lambda\in Q_p$ the characteristic function of $d_n(\lambda)$, $Eexp\{ird_n(\lambda)\}$ converges to $exp\{-C_\alpha\left|r\right|^\alpha \Phi(\lambda)\}$.*

Proof: From (2) and (3), the characteristic function of $d_n(\lambda)$ can be written as follows:

$$\begin{aligned}Eexp\{ird_n(\lambda)\}&=Eexp\{irA_n Re\int_{U_n}e^{-i<t\lambda>}h(tp^n)p^{-n}X(t)d\mathcal{H}(t)\}\\&=Eexp\{irA_n Re\int_{U_n}e^{-i<t\lambda>}p^{-n}h(tp^n)\int_{Q_p}e^{i<tu>}dM(u)\ \ d\mathcal{H}(t)\}\end{aligned}$$

Using the same argument used in Cambanis 1983, from the last equality we obtain:

$$Eexp\{ird_n(\lambda)\}=$$

$$exp\left\{-C_\alpha\left|r\right|^\alpha\left|A_n\right|^\alpha\int_{Q_p}\left|\int_{U_n}e^{-i<t(\lambda-u)>}p^{-n}h(tp^n)d\mathcal{H}(t)\right|^\alpha \Phi(u)\ \ d\mathcal{H}(u)\right\}$$

Let $x=tp^n$, we have

$$Eexp\{ird_n(\lambda)\}=$$

$$exp\left\{-C_\alpha\left|r\right|^\alpha\left|A_n\right|^\alpha\int_{Q_p}\left|\int_{Z_p}e^{-i<(\lambda-u)p^{-n}x>}h(x)d\mathcal{H}(x)\right|^\alpha \Phi(u)\ \ d\mathcal{H}(u)\right\}$$

$$E\,exp\{ir d_n(\lambda)\} = exp\left\{-C_\alpha |r|^\alpha \int_{Q_p} \left|A_n H(p^{-n}(\lambda - u))\right|^\alpha \Phi(u)\; d\mathcal{H}(u)\right\}$$

$$= exp\left\{-C_\alpha |r|^\alpha \int_{Q_p} \left|H_n(\lambda - u)\right|^\alpha \Phi(u)\; d\mathcal{H}(u)\right\}$$

$$= exp\{-C_\alpha |r|^\alpha \Psi_n(\lambda)\}.$$

From the proposition 3.1, we obtain the result.
We modify this periodogram as follows:

$$I_n(\lambda) = C_{q,\alpha} \left|d_n(\lambda)\right|^q, \tag{7}$$

where $0 < q < \dfrac{\alpha}{2}$ and the normalization constant is $C_{q,\alpha} = \dfrac{D_q}{F_{q,\alpha} C_\alpha^{\frac{q}{\alpha}}}$, where

$$D_q = \int \frac{1 - cos(u)}{|u|^{1+q}} du \;;\; F_{q,\alpha} = \int \frac{1 - e^{-|u|^\alpha}}{|u|^{1+q}} du \text{ and } C_\alpha = \frac{1}{2\pi} \int_0^\pi |cos(u)|^\alpha du\,.$$

Theorem 3.1 *Let* $\lambda \in Q_p$, *then* $E(I_n(\lambda)) = (\psi_n(\lambda))^{q/\alpha}$ *and* $I_n(\lambda)$ *is an asymptotically unbiased estimator of the spectral density but not consistent:* $E(I_n(\lambda)) - (\Phi(\lambda))^{q/\alpha} = o(1)$ *and* $Var(I_n(\lambda)) - V_{\alpha,q}\Phi^2(\lambda) = o(1)$, *with* $V_{\alpha,q} = \dfrac{C_{q,\alpha}^2}{C_{2q,\alpha}} - 1.$

The proof of this result is similar to that given in Masry and Cambanis (1984) and Sabre (1994, 1995). It is sufficient to use the equality:

$$|x|^q = D_q^{-1} \Re e \int \frac{1 - e^{-iux}}{|u|^{1+q}} du \quad \forall x \in R. \tag{8}$$

Theorem 3.2 *Let* λ_1, λ_2 *be two different points in* Q_p *such that* $\Phi(\lambda_1) \neq 0$ *and* $\Phi(\lambda_2) \neq 0$*, then*

$$cov(I_n(\lambda_1), I_n(\lambda_2)) = o(1).$$

Proof: From (2), *we have*

$$EI_n(\lambda)-I_n(\lambda)=F_{q,\alpha}^{-1}C_\alpha^{-\frac{q}{\alpha}}\int\frac{\Re e\ e^{iud_n(\lambda)}-e^{-C_\alpha\Psi_n(\lambda)|u|^\alpha}}{|u|^{1+q}}du\ C(\lambda_1,\lambda_2)=$$

$$F_{q,\alpha}^{-2}(C_\alpha)^{-2q/\alpha}\iint E\left(\prod_{k=1}^{2}cos\left[u_k d_n(\lambda_k)\right]\right)\text{-}exp\left[-C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\,\Psi_n(\lambda_k)\right]\frac{du_1du_2}{|u_1u_2|^{1+q}}.$$

Where

$$\prod_{k=1}^{2}cos[u_k I_n(\lambda_k)]=\frac{1}{2}\Re e\left(exp\left\{i\sum_{k=1}^{2}u_k d_n(\lambda_k)\right\}\right)+$$

$$\frac{1}{2}\Re e\left(exp\left\{i\sum_{k=1}^{2}(-1)^{k-1}u_k d_n(\lambda_k)\right\}\right)$$

So that

$$E\prod_{k=1}^{2}cos[u_k d_n(\lambda_k)]=\frac{1}{2}exp\left\{-C_\alpha\int_{Q_p}\left|\sum_{k=1}^{2}u_k H_n(\lambda_k-u)\right|^\alpha\Phi(u)d\mathcal{H}(u)\right\}+$$

$$\frac{1}{2}exp\left\{-C_\alpha\int_{Q_p}\left|\sum_{k=1}^{2}(-1)^{k-1}u_k H_n(\lambda_k-w)\right|^\alpha\Phi(w)d\mathcal{H}(w)\right\}$$

Letting $u_2=-v_2$, we obtain

$$C(\lambda_1,\lambda_2)=F_{q,\alpha}^{-1}(C_\alpha)^{-2q/\alpha}\iint e^{-a}-e^{-b}\,\frac{du_1du_2}{|u_1u_2|}\tag{9}$$

where $a=C_\alpha\int_{Q_p}\left|\sum_{k=1}^{2}u_k H_n(\lambda_k-t)\right|^\alpha\Phi(t)d\mathcal{H}(t),$

$$b=C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\,\Psi_n(\lambda_k)$$

$$=C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\int_{Q_p}|H_n(\lambda_k-w)|^\alpha\,\Phi(w)d\mathcal{H}(w).$$

However,

$$\left|e^{-a}-e^{-b}\right| \le |a-b| e^{|a-b|-b} \tag{10}$$

$$|a-b| \le 2C_\alpha \left|u_1u_2\right|^{\alpha/2} \int_{Q_p} \left|H_n(\lambda_1-w)H_n(\lambda_2-w)\right|^{\alpha/2} \Phi(w)d\mathcal{H}(w)$$

$$|a-b|-b \le 2C_\alpha \left|u_1u_2\right|^{\alpha/2} sup(\Phi)L_n(\lambda_1,\lambda_2) - C_\alpha \sum_{k=1}^{2}\left|u_k\right|^{\alpha} \Psi_n(\lambda_k), \text{ where}$$

$$L_n(\lambda_1,\lambda_2)=\int_{Q_p} \left|H_n(\lambda_1-w)\right|^{\alpha/2} \left|H_n(\lambda_2-w)\right|^{\alpha/2} d\mathcal{H}(w). \tag{11}$$

Since $2\left|u_1u_2\right|^{\alpha/2} \le \left|u_1\right|^{\alpha} - \left|u_2\right|^{\alpha}$, thus we have:

$$|a-b|-b \le -C_\alpha \sum_{k=1}^{2}\left|u_k\right|^{\alpha} [\Psi_n(\lambda_k) - sup(\Phi)L_n(\lambda_1,\lambda_2)].$$

It is sufficient now to show that $L_n(\lambda_1,\lambda_2)$ converges to zero.

$$L_n(\lambda_1,\lambda_2)=\int_{Q_p} \left|H_n(\lambda_1-w)\right|^{\alpha/2} \left|H_n(\lambda_2-w)\right|^{\alpha/2} d\mathcal{H}(w)$$

$$=\int_{Q_p}\left(\frac{p^n}{B_\alpha}\right)\left|H(p^{-n}\lambda_1-p^{-n}w)\right|^{\alpha/2}\left|H(p^{-n}\lambda_2-p^{-n}w)\right|^{\alpha/2} d\mathcal{H}(w)$$

Setting $p^{-n}\lambda_1 - p^{-n}w = u'$, we get

$$L_n(\lambda_1,\lambda_2)=\frac{1}{B_\alpha}\int_{Q_p}\left|H(u')\right|^{\alpha/2}\left|H(p^{-n}\lambda_2-p^{-n}\lambda_1+u')\right|^{\alpha/2} d\mathcal{H}(u')$$

Assume that H verifying the following hypothesis.

$$H(u)=0 \text{ si } |u|_p>1, \text{ et } |H(u)| \le |u|_p^{\frac{2(\alpha-1)}{\alpha}} \text{ si } |u|_p \le 1. \text{ Then}$$

$$L_n(\lambda_1,\lambda_2)=\frac{1}{B_\alpha}\int_{Q_p}\left|H(u')\right|^{\alpha/2}\left|H(p^{n}\lambda_2-p^{n}\lambda_1+u')\right|^{\alpha/2} d\mathcal{H}(u')$$

$$\le \frac{1}{B_\alpha}\int_{|u|\le 1}|u|_p^{\alpha-1}\left|u-p^n(\lambda_1-\lambda_2)\right|_p^{\alpha-1} d\mathcal{H}(u)$$

with $t=\dfrac{u}{p^n(\lambda_1-\lambda_2)}$, thus we obtain:

$$L_n(\lambda_1,\lambda_2) \le \frac{1}{B_\alpha}\left|p^n(\lambda_1-\lambda_2)\right|_p^{2\alpha-2}\int_D \left|p^n(\lambda_1-\lambda_2)\right|_p \left|t\right|_p^{\alpha-1}\left|t-1\right|_p^{\alpha-1} d\mathcal{H}(t) \text{ where}$$

$$D=\{\left|t\right|_p \left|p^n(\lambda_1-\lambda_2)\right|_p \le 1\},$$

$$L_n(\lambda_1,\lambda_2) \le \frac{\beta(\alpha,p)}{\beta_\alpha}\left|p^n(\lambda_1-\lambda_2)\right|_p^{2\alpha-1} \tag{12}$$

with $\beta(\alpha,p)=\int_{Q_p}\left|t\right|_p^{\alpha-1}\left|t-1\right|_p^{\alpha-1} d\mathcal{H}(t)$. From (9), (10) and the fact that $|p^n|_p=p^{-n}$, these complete the proof of the theorem.

4. THE SMOOTHING ESTIMATE

In order to have an asymptotically and consistent estimate, we smooth the periodogram that was modified using a spectral window.

$$f_n(\lambda)=\int_{Q_p} W_n(\lambda-u)I_n(u)d\mathcal{H}(u)$$

where $W_n(x)=\left|M_n\right|_p W(xM_n)$ such that

$$M_n \to \infty \;;\; \frac{M_n}{n}\to 0 \;;|M_n|_p \to 0 \text{ and } \frac{|M_n|_p}{|p^n|_p}\to\infty. \tag{13}$$

W is an even nonnegative function vanishing outside $[-1,1]$ and $\int_{Q_p} W(v)d\mathcal{H}(v)=1$

Proposition 4.1 *Let* $\lambda \in Q_p$ *and* $Bias(f_n(\lambda)) = E[f_n(\lambda)]-(\Phi(\lambda))^{p/\alpha}$ *then*

$$Bias(f_n(\lambda)) = o(1).$$

Moreover if Φ *verify* $\left|\Phi(x)-\Phi(y)\right| \le cste\left|x-y\right|_p^{-k}$, *then*

$$Bias(f_n(\lambda)) = O\left(\frac{1}{\left|M_n\right|_p^{-\frac{kp}{\alpha}}}\right).$$

Proof: We have

$$f_n(\lambda)=\int_{Q_p}\left|M_n\right|_p W(M_n(\lambda-u))I_n(u)d\mathcal{H}(u)$$

Letting $(\lambda - u)M_n = v$ and using (1), we obtain

$$f_n(\lambda) = \int_{Q_p} W(v) I_n\left(\lambda - \frac{v}{M_n}\right) d\mathcal{H}(u).$$

From the proposition 3.1, we have

$$E(f_n(\lambda)) = \int_{Q_p} W(v)\left\{\Psi_n\left(\lambda - \frac{v}{M_n}\right)\right\}^{p/\alpha} d\mathcal{H}(v).$$

As $p < \frac{\alpha}{2}$, then we obtain

$$\left|Bias(f_n(\lambda))\right| \le \int_{Q_p} W(v)\left|\left[\Psi_n\left(\lambda - \frac{v}{M_n}\right)^{p/\alpha} - (\Phi(\lambda))^{p/\alpha}\right]\right| d\mathcal{H}(v)$$

$$\le \int_{Q_p} W(v)\left|\Psi_n\left(\lambda - \frac{v}{M_n}\right) - (\Phi(\lambda))\right|^{p/\alpha} d\mathcal{H}(v)$$

On the other hand,

$$\Psi_n\left(\lambda - \frac{v}{M_n}\right) - (\Phi(\lambda)) =$$

$$\frac{1}{B_\alpha}\int_{Q_p} \left|H_n(u)\right|^\alpha \left[\Phi\left(\lambda - \frac{v}{M_n} - \frac{u}{p^n}\right) - \Phi(\lambda)\right] d\mathcal{H}(u)$$

Since Φ is uniformly continuous and from the proposition 3.1, we obtain $Bias(f_n(\lambda)) = o(1)$.

Assume that Φ verify $\left|\Phi(x) - \Phi(y)\right| \le cste\left|x - y\right|_p^{-k} \quad \forall x, y \in Q_p$.

$$\left|\Psi_n\left(\lambda - \frac{v}{M_n}\right) - (\Phi(\lambda))\right| \le \frac{cste}{B_\alpha}\int_{|u| \le 1} \left|H(u)\right|^\alpha \left|\frac{v}{M_n} + \frac{u}{p^n}\right|_p^{-k} d\mathcal{H}(u)$$

$$\le \frac{cste}{B_\alpha}\int_{|u|_p \le 1} \left|u\right|_p^{2(\alpha-1)} \left|\frac{v}{M_n} + \frac{u}{p^n}\right|_p^{-k} d\mathcal{H}(u)$$

Since $|x+y|_p \leq Max(|x|_p, |y|_p)$, we have

$$\left|\Psi_n\left(\lambda - \frac{v}{M_n}\right) - (\Phi(\lambda))\right| =$$

$$Max\left(\frac{1}{B_\alpha}\int_{|u|_p\leq 1}|u|_p^{2(\alpha-1)}\left|\frac{v}{M_n}\right|_p^{-k} d\mathcal{H}(u), \frac{1}{B_\alpha}\int_{|u|_p\leq 1}|u|_p^{2(\alpha-1)}\left|\frac{u}{p^n}\right|_p^{-k} d\mathcal{H}(u)\right)$$

Using the following equality proved in Vladimirov (1988) page 25,

$$\int_{|x|_p<1}|1-x|_p^{r-1}dx = \frac{1-p^{-1}}{1-p^{-r}}$$, we obtain

$$\left|\Psi_n\left(\lambda - \frac{v}{M_n}\right) - (\Phi(\lambda))\right| \leq Max\left(\frac{1}{B_\alpha}\frac{|v|_p^{-k}}{|M_n|_p^{-k}}\frac{1-p^{-1}}{1-p^{-2\alpha+1}}, \frac{1}{B_\alpha |p^n|_p^{-k}}\frac{1-p^{-1}}{1-p^{-2\alpha+k+1}}\right).$$

Thus

$$|Bias(f_n(\lambda))| \leq O\left(max\left(\frac{1}{|p^n|_p^{-kp/\alpha}}, \frac{1}{|M_n|_p^{-kp/\alpha}}\right)\right).$$

From (13), we have $|M_n|_p > |p^n|_p$ for large n. Thus, we obtain

$$|Bias(f_n(\lambda))| = O(|M_n|_p^{kp/\alpha}).$$

Now we will show that f_n is an asymptotically consistent estimate.

Proposition 4.2 *Let* λ *be in* Q_p *and* $M_n = p^{\nu n}$ *where* $0<\nu<1$. *Assume that* $\Phi \in L^1_{Q_p}$. *Then* $Var(f_n(\lambda)) = O(p^{-n(3\nu+1)})$

Proof: From the definition of variance, we have

$$Var(f_n(\lambda)) = E(f_n(\lambda) - Ef_n(\lambda))^2 =$$

$$\int_{Q_p}\int_{Q_p} W_n(\lambda-u_1)W_n(\lambda-u_2)C(u_1,u_2)d\mathcal{H}(u_1)d\mathcal{H}(u_2)$$

where $C(u_1,u_2)=Cov(I_n(u_1),I_n(u_2))$

$$Var(f_n(\lambda))=\int_{Q_p}\int_{Q_p} W(x_1)W(x_2)C\left(\lambda-\frac{u_1}{M_n},\lambda-\frac{u_2}{M_n}\right)d\mathcal{H}(x_1)d\mathcal{H}(x_2)$$

$$=\iint_{\{|x_1-x_2|_p<\varepsilon_n\}}+\iint_{\{|x_1-x_2|_p>\varepsilon_n\}}\stackrel{\Delta}{=}J_1+J_2$$

where ε_n is a positive real converging to zero as n tends to infinity.

$$|J_2|\leq\iint_{\{|x_1-x_2|_p>\varepsilon_n\}}W(x_1)W(x_2)C\left(\lambda-\frac{x_1}{M_n},\lambda-\frac{x_2}{M_n}\right)d\mathcal{H}(x_1)d\mathcal{H}(x_2)$$

From (9) and (10), we get

$$C\left(\lambda-\frac{x_1}{M_n},\lambda-\frac{x_2}{M_n}\right)\leq cste\iint|a-b|e^{|a-b|-b}\frac{du_1du_2}{|u_1u_2|^{1+q}}$$

$$a=C_\alpha\int_{Q_p}\left|\sum_{k=1}^{2}u_kH_n\left(\lambda-\frac{x_k}{M_n}-v\right)\right|^\alpha\Phi(v)d\mathcal{H}(v)$$

$$b=C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\Psi_n\left(\lambda-\frac{x_k}{M_n}\right)=C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\int_{Q_p}\left|H_n\left(\lambda-\frac{x_k}{M_n}-v\right)\right|^\alpha\Phi(v)dv.$$

$$|a-b|\leq 2C_\alpha|u_1u_2|^{\alpha/2}\int_{Q_p}\left|H_n\left(\lambda-\frac{x_1}{M_n}-v\right)H_n\left(\lambda-\frac{x_2}{M_n}-v\right)\right|^{\alpha/2}\Phi(v)dv.$$

Letting

$$\int_{Q_p}\left|H_n\left(\lambda-\frac{x_1}{M_n}-v\right)H_n\left(\lambda-\frac{x_2}{M_n}-v\right)\right|^{\alpha/2}\Phi(v)dv\stackrel{\Delta}{=}L_n\left(\lambda-\frac{x_1}{M_n},\lambda-\frac{x_2}{M_n}\right)$$

$$|a-b|\leq 2C_\alpha|u_1u_2|^{\alpha/2}L_n\left(\lambda-\frac{x_1}{M_n},\lambda-\frac{x_2}{M_n}\right)$$

$$|a-b|-b\leq -C_\alpha\sum_{k=1}^{2}|u_k|^\alpha\Delta_{k,n}\text{ , where}$$

$$\Delta_{k,n}=\left[\Psi_n\left(\lambda-\frac{x_k}{M_n}\right)-sup(\Phi)L_n\left(\lambda-\frac{x_1}{M_n},\lambda-\frac{x_2}{M_n}\right)\right].$$

We have $\iint e^{-(b-|a-b|)}\frac{du_1du_2}{|u_1u_2|^{1+q-\alpha/2}}\leq\prod_{k=1}^{2}\int e^{-C_\alpha|u_k|^\alpha\Delta_{k,n}}\frac{du_k}{|u_k|^{1+q-\alpha/2}}$.

By changing of the variable: $v=u_k(\Delta_{k,n})^{1/\alpha}$, we obtain

$$\iint e^{-(b-|a-b|)}\frac{du_1du_2}{|u_1u_2|^{1+q-\alpha/2}}\leq\frac{cste}{\prod_{k=1}^{2}(\Delta_{k,n})^{1/2-q/\alpha}}.$$

From (5), we have

$$L_n\left(\lambda-\frac{x_1}{M_n},\lambda-\frac{x_2}{M_n}\right)\leq cste\left|\frac{p^n}{M_n}\right|_p^{2\alpha-1}|x_1-x_2|_p^{2\alpha-1}\tag{14}$$

Since $\frac{|p^n|_p}{|M_n|_p}\to 0$, we obtain

$$L_n\left(\lambda-\frac{x_1}{M_n},\lambda-\frac{x_2}{M_n}\right)=0\left(\frac{|p^n|_p}{|M_n|_p}\right)^{2\alpha-1}.\tag{15}$$

Thus, from the proposition 3.1, we have $\Delta_{k,n}\to\Phi(\lambda)$. Therefore

$$J_2\leq\frac{cste}{(\Phi(\lambda))^{1-2q/\alpha}}\iint_{Q_p}w(x_1)w(x_2)L_n\left(\lambda-\frac{x_1}{M_n},\lambda-\frac{x_2}{M_n}\right)d\mathcal{H}(x_1)d\mathcal{H}(x_2)$$

$J_2\leq\frac{cste}{(\Phi(\lambda))^{1-\frac{2q}{\alpha}}}\Lambda$, where

$$\Lambda=\int_{Q_p}\Phi(v)\left\{\int_{Q_p}W(x)\left|H_n\left(\lambda-\frac{x}{M_n}-v\right)\right|^{\alpha/2}d\mathcal{H}(x)\right\}^2d\mathcal{H}(v)$$

Putting $u=\lambda-v$, we have

$$\Lambda=\int_{Q_p}\Phi(\lambda-u)\left\{\int_{Q_p}W(x)\left|H_n\left(u-\frac{x}{M_n}\right)\right|^{\alpha/2}d\mathcal{H}(x)\right\}^2 d\mathcal{H}(u)$$

Denote by:

$$G_n(u)=\int_{Q_p}W(x)\left|H_n\left(u-\frac{x}{M_n}\right)\right|^{\alpha/2}d\mathcal{H}(u)$$

and

$$S_n(\lambda)=\int_{Q_p}\Phi(\lambda-u)G_n^2(u)d\mathcal{H}(u)$$

Letting $u-\frac{x}{M_n}=v$ in the integral of $G_n(u)$, we get

$$G_n(u)=\int_{Q_p}|M_n|_p\, W[(u-v)M_n]|H_n(v)|^{\alpha/2}d\mathcal{H}(v)$$

Let $u=\frac{t}{M_n}$ in the integral of $S_n(\lambda)$,

$$S_n(\lambda)=\int_{Q_p}\Phi\left(\lambda-\frac{t}{M_n}\right)G_n^2\left(\frac{t}{M_n}\right)\frac{1}{|M_n|_p}d\mathcal{H}(t)$$

$$G_n\left(\frac{t}{M_n}\right)=\int_{Q_p}W(x)\left|H_n\left(\frac{t}{M_n}-\frac{x}{M_n}\right)\right|^{\alpha/2}d\mathcal{H}(x).$$

From (6), we obtain

$$G_n\left(\frac{t}{M_n}\right)=\int_{Q_p}\frac{W(x)}{B_\alpha^{1/2}}p^{n/2}\left|H\left(p^{-n}\left(\frac{t}{M_n}-\frac{x}{M_n}\right)\right)\right|^{\alpha/2}d\mathcal{H}(x)$$

Let $\frac{t-x}{M}p^{-n}=r$ we get

$$\frac{p^{n/2}}{|M_n|_p}G_n\left(\frac{t}{M_n}\right)=\int_{Q_p}W\left(t-\frac{M_n r}{p^n}\right)\frac{1}{B_\alpha^{1/2}}|H(r)|^{\alpha/2}d\mathcal{H}(r)$$

Since $\left|H(v)\right|^{\alpha} \in L^1_{Q_p}$ and W is bounded, we obtain

$$\frac{p^{n/2}}{\left|M_n\right|_p} G_n\left(\frac{t}{M_n}\right) \le B_{\alpha}^{-1/2} sup(W) \int_{Q_p} \left|H(r)\right|^{\alpha/2} d\mathcal{H}(r).$$

Thus, $\frac{p^n}{\left|M_n\right|_p^2} S_n(\lambda) = \int_{Q_p} \Phi\left(\lambda - \frac{t}{M_n}\right)\left(\frac{\left|p^n\right|^{1/2}}{\left|M_n\right|_p} G_n\left(\frac{t}{M}\right)\right)^2 d\mathcal{H}(t).$

$$\frac{p^n}{\left|M_n\right|_p^2} S_n(\lambda) \le \int_{Q_p} \Phi\left(-\frac{t}{M_n}\right) B_{\alpha}^{-1} (\sup(W))^2 \left(\int_{Q_p} \left|H(r)\right|^{\alpha/2} d\mathcal{H}(r)\right)^2 d\mathcal{H}(t)$$

$$\le c\mathit{ste} \int_{Q_p} \Phi\left(\lambda - \frac{t}{M_n}\right) d\mathcal{H}(t).$$

Let $\lambda - \frac{t}{M_n} = r$, we get $\frac{p^n}{\left|M_n\right|_p^2} S_n(\lambda) \le c\mathit{ste} \left| M_n\right|_p \int_{Q_p} \Phi(r) d\mathcal{H}(r)$. Since $\Phi \in L^1_{Q_p}$, we obtain

$$S_n(\lambda) = O\left(\frac{\left| M_n\right|_p^3}{p^n}\right). \tag{16}$$

$$J_1 = \iint_{\left|x_1 - x_2\right| \le \varepsilon_n} W(x_1) W(x_2) (V_1)^{1/2} (V_2)^{1/2} d\mathcal{H}(x_1) d\mathcal{H}(x_2).$$

where $V_1 = Var\left(I_n\left(\lambda - \frac{x_1}{M_n}\right)\right)$ and $V_2 = Var\left(I_n\left(\lambda - \frac{x_2}{M_n}\right)\right)$.

As $Var\left(I_n\left(\lambda - \frac{x}{M_n}\right)\right) = \left(V_{\alpha,p}\left(\Psi_n\left(\lambda - \frac{x}{M_n}\right)\right)\right)^{p/\alpha}$, for a large n we get

$$\left|J_1\right| \le c\mathit{onst} (\Phi(\lambda))^{2p/\alpha} \iint_{\left|x_1 - x_2\right| \le \varepsilon_n} W(x_1) W(x_2) d\mathcal{H}(x_1) d\mathcal{H}(x_2).$$

Since $\int_{Q_p} \int_{|t|_p < \varepsilon_n} W(x_1) W(t - x_1) d\mathcal{H}(t) d\mathcal{H}(x_1) \le \int_{|t|_p < \varepsilon_n} d\mathcal{H}(x_1)$, we have,

$\left|J_1\right| \leq const(\Phi(\lambda))^{2p/\alpha}\,\varepsilon_n$. Then $Var(f_n(\lambda))=O\left(\dfrac{|M_n|_p^3}{p^n}+\varepsilon_n\right)$.

Using the fact that $M_n=p^{\nu n}$ and choose $\varepsilon_n=p^{-(3\nu+1)n}$ with ν is satisfying $\nu<1$, thus we obtain

$$Var(f_n(\lambda))=O(p^{-n(3\nu+1)}).$$

5. CONCLUSION

In this paper, we have proposed in this paper some results about the estimation of the spectral density for symmetric stable p-adic processes. The approach was based on the technique used by Masry and Cambanis (1984) for stable processes combining estimates of p-adic spectrum introduced by Brillinger (1991). This work could be applied to several cases when processes have an infinite variance and have a discrete time, as for example:

- The segmentation of a sequence of images of a dynamic scene, detecting weeds in a farm field.
- The detection of possible structural changes in the dynamics of an economic structural phenomenon.
- The study of the rate of occurrence of notes in melodic music to simulate the sensation of hearing from afar.

This work could be supplemented by the study of optimal smoothing parameters using cross validation methods that have been proven in the field.

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SUMMARY

Spectral density estimation for symmetric stable p-adic processes

Applications of p-adic numbers ar beming increasingly important espcially in the field of applied physics. The objective of this work is to study the estimation of the spectral of p-adic stable processes. An estimator formed by a smoothing periodogram is constructed. It is shwon that this estimator is asymptotically unbiased and consistent. Rates of convergences are also examined.