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THE MULTIVARIATE ASYMMETRIC SLASH LAPLACE DISTRIBUTION AND ITS APPLICATIONS

Bindu Punathumparambath

1. INTRODUCTION

Symmetric distributions generalizing normality have got lot of attention in the statistical literature. Regression models applied in the field of biology, economics, psychology, sociology, need not obey Gaussian law in all situations. It is suggested that error structures in these cases should be handled beyond the normality frame work. Hence there is a special interest in constructing distributions that could describe symmetry, skewness and heavy tails observed in the data. One such family of distributions is the slash distribution proposed by Kafadar (1982 and 1988). The slash distribution is the ratio of a standard normal random variable to an independent uniform random variable U on the interval $(0,1)$ raised to the power $1/q$, $q>0$ and has heavier tails than normal distribution.

Genton and Wang (2006) generalized the univariate slash normal of Kafadar to multivariate skew-slash normal and investigated its properties. An alternative to multivariate skew-slash distribution is introduced by Arslan (2008 & 2009). Tan and Peng (2005) introduced multivariate slash Student's t and skew slash Student's t distributions and studied their properties. A generalization of the slash distribution using the scale mixture of the exponential power distribution was introduced by Ali Gen (2007). Jose and Lishamol (2007) introduced the slash Laplace distribution. A new family of slash distributions with eliptical contours was proposed by Go'mez *et al.* (2007). Also Arslan and Genc (2009) introduced a generalization of the multivariate slash distribution. Recently Bindu (2011, 2012(a)) introduced a family of skew-slash distributions generated by normal and Cauchy kernels. Also Bindu (2012(b)) introduced a new family of multivariate skew-slash t and skew-slash Cauchy distributions.

In this paper we study the multivariate asymmetric slash distribution and derive its various properties. The standard classical slash Laplace distribution is obtained as the distribution of the ratio $X=(Y/U^{1/q})$, where Y is a standard classical Laplace random variable, U is an independent uniform random variable over the interval $(0,1)$ and $q>0$. The probability density function (pdf) is given below,

$$h(x;q) = \frac{q}{2}\int_0^1 v^q \ e^{-|xv|} \ \mathrm{d}v, -\infty < x < \infty .$$

For $q = 1$ we obtain the canonical slash Laplace density as,

$$h(x,1) = \begin{cases} \dfrac{f(0) - f(x) - |x| f(x)}{x^2}, & x \neq 0, \\ \dfrac{f(0)}{2}, & x = 0, \end{cases}$$

where $f(\cdot)$ *is the standard classical Laplace pdf given by*, $f(x) = (1/2)e^{-|x|}, -\infty < x < \infty.$

The pdf of the univariate canonical slash Laplace distribution is symmetric about the origin and has the same tail heaviness as the Cauchy distribution. In the present work we study the multivariate symmetric and asymmetric slash Laplace distribution and several of its properties were explored. This article is organized as follows. Section 2, introduces multivariate slash Laplace distribution and describes various properties. In Section 3, multivariate asymmetric slash Laplace distributions are developed and properties are studied. In section 4, we illustrate the application of the asymmetric slash Laplace distribution to the microarray gene expression data.

2. MULTIVARIATE SLASH LAPLACE DISTRIBUTION

In this section, we define a multivariate slash Laplace distribution and derive its pdf. Also an alternative definition of the multivariate slash Laplace distribution based on elliptically contoured distribution is also given.

Definition 1 *A random vector* $X \in \mathrm{R}^d$ *has a d-dimensional slash Laplace (*SLL_d*) distribution with location parameter* $\mu = 0$*, positive definite scale matrix parameter* Σ *and tail parameter* $q > 0$*, denoted by* $X \sim SLL_d(0, \Sigma, q)$*, if* $X = \frac{Y}{U^{1/q}} + \mu$*, where* Y *is a Laplace random vector with characteristic function (cf) given by* $\phi_Y(t) = \frac{1}{1 + \frac{1}{2}t'\Sigma t}, t \in \mathrm{R}^d$ *and* $U \sim U(0,1)$ *independent of* Y.

Here Σ is a $d \times d$ positive definite matrix. The pdf of the SLL_d random vector can be given as

$$g_d(x; O, \Sigma, q) = q\int_0^1 v^{q+d-1} f_d(xv, O, \Sigma)\mathrm{d}v, \ x \in \mathrm{R}^d, \tag{1}$$

where $f_d(\cdot)$ is the density function of the *d*-dimensional Laplace random vector, which is given by

$$f_d(y;O,\Sigma)=\frac{2}{(2\pi)^{d/2}\,|\Sigma|^{1/2}}\left(\frac{y'\Sigma^{-1}y}{2}\right)^{\nu/2}K_\nu(\sqrt{2y'\Sigma^{-1}y}),$$

where $\nu=\frac{(2-d)}{2}$ and $K_\nu(u)$ is the modified Bessel function of the third kind (see Kotz et al., 2001) and is given below,

$$K_\nu(u)=\frac{1}{2}\left(\frac{u}{2}\right)^{\nu}\int_0^\infty t^{-\nu-1}e^{\left(-t-\frac{u^2}{4t}\right)}\mathrm{d}t, u>0. \tag{2}$$

For the simulation purposes we consider Y as

$$Y\stackrel{d}{=}\sqrt{W}N,$$

where W is the standard exponential variate and N follows $N_d(0,\Sigma)$ independent of W. Then $X=\frac{Y}{U^{1/q}}$ will follow $SLL_d(0,\Sigma;q)$. The cumulative density function (cdf) of the SLL_d random variable can be obtained as

$$G_d(x;O,\Sigma,q)=q\int_0^1 v^{q+d-2}F_d(xv;O,\Sigma)\mathrm{d}v,\ x\in\mathrm{R}^d,$$

where $F_d(\cdot)$ is the cdf of the *d*-dimensional Laplace random vector.

Remark 1 *Note that the slash Laplace random vector in (1) is a scale mixture of the Laplace random vector and so it can be represented as,*

$$X\,|\,(U=u)\sim L_d(0,u^{-1/q}\Sigma),$$

where L_d is the *d*-dimensional Laplace distribution.

Remark 2 *The limiting distribution of multivariate slash Laplace,* $SLL_d(O,\Sigma,q)$*, as* $q\to\infty$ *is the multivariate Laplace distribution* $(L_d(O,\Sigma))$.

In the univariate case, i.e. for $d=1$ in equation (1) we get the univariate slash Laplace distribution and its pdf is given by

$$h(x;\mu,\sigma,q)=\frac{q}{2\sigma}\int_0^1 v^q\, e^{-\frac{|(vx-\mu)|}{\sigma}}\,\mathrm{d}v, -\infty<x<\infty, -\infty<\mu<\infty, q,\sigma>0. \tag{3}$$

Figure (1) gives the probability density curves of the slash Laplace distribution for various values of parameters and figure (2) gives the probability density curves of slash Laplace (SLL), slash normal (SLN), slash t (SLT) and slash Cauchy (SLC).

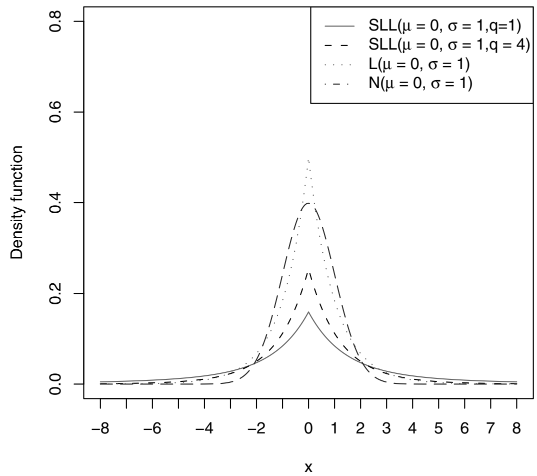

Figure 1 – Density functions for the univariate slash Laplace distribution for $\mu = 0$, $\sigma = 1$ and for two values of q, along with standard normal and Laplace densities.

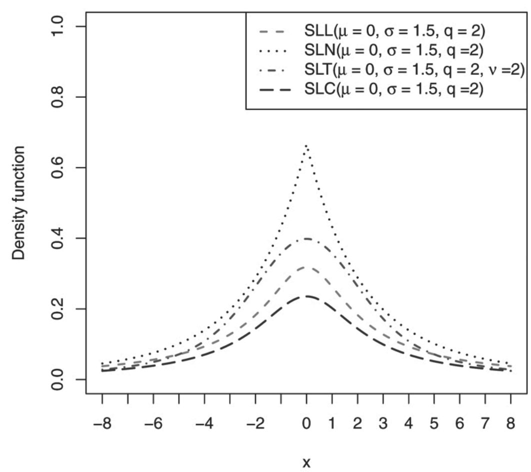

Figure 2 – Slash Laplace density function along with slash normal, slash t and slash Cauchy densities.

Figure (1) gives the density curves of the standard Laplace(the dotted line), normal (dashed line), slash Laplace with $q = 1$ (lowest one) and slash Laplace with $q = 4$ (second one). We can see that for $q = 1$ the slash Laplace density has heavy tails, hence the parameter q control the tail behaviour. A comparative study between the density curves of slash Laplace (highest one), slash normal, slash t and the slash Cauchy (lowest one) distributions, for $q = 2$ is given in Figure (2).

Now we define the multivariate slash Laplace distribution as a family of slash distribution generated from an elliptically contoured distribution. Ernst (1998) introduced a multivariate Laplace distribution via an elliptic contouring. Fang et al. (1990) defined an elliptically contoured random vector on R^d with characteristic function of the form

$$\phi(t) = e^{it'm}\, \psi(t'\Sigma t),$$

for some function ψ, m is a $d \times 1$ vector in R^d and Σ is a $d \times d$ positive-definite matrix.

Definition 2 *A random vector $X \in \mathsf{R}^d$ has a d-dimensional slash elliptical contoured Laplace ($SECL_d$) distribution with location parameter $\mu = 0$, positive definite scale matrix parameter Σ and tail parameter $q > 0$, denoted by $X \sim SECL_d(0,\Sigma,q)$, if $X = \dfrac{Y}{U^{1/q}} + \mu$, where $Y \sim ECL_d(O,\Sigma)$ is an elliptically contoured Laplace random vector with pdf given by $f_Y(y) = k_d\,|\Sigma|^{-1/2}\, e^{-[y'\Sigma^{-1}y]^{1/2}}$, k_d is a proportionality constant (see, Fang et al.(1990)) and $U \sim U(0,1)$ independent of Y.*

Remark 3 *The elliptically contoured Laplace random vector Y in definition 2 has the polar representation $Y \overset{d}{=} RHU^{(d)}$, where H is a $d \times d$ matrix such that $HH' = \Sigma$, R is a positive random vector independent of $U^{(d)}$ and $U^{(d)}$ is a random vector uniformly distributed on the surface of the hyper-ellipsoid $\{y \in \mathsf{R}^d : y'\Sigma^{-1} y = 1\}$.*

If $Y \sim L_d(O,\Sigma)$ then R has the density

$$f_R(x) = \frac{2x^{d/2} K_{d/2-1}(\sqrt{2}x)}{(\sqrt{2})^{d/2-1}\,\Gamma\left(\dfrac{d}{2}+1\right)}, x > 0,$$

where K_ν is the modified Bessel function of the third kind given in equation (2).

Proposition 1 *If $X \sim SLL_d(O,\Sigma,q))$, then the density of X can be expressed as*

$$f_X(x;\mu,\Sigma,q)) = \begin{cases} \dfrac{q}{(q+d)}\,|\Sigma|^{-1/2}\, h(0), & \textit{for } x = \mu, \\[2ex] \dfrac{q\,|\Sigma|^{-1/2}}{2\delta^{(q+d)/2}} \displaystyle\int_0^{\delta} w^{\frac{q+d-2}{2}}\, h(w)\,dw, & \textit{for } x \neq \mu, \end{cases}$$

where $\delta = (x-\mu)^T \, |\Sigma|^{-1} \, (x-\mu)$, $h(w)$ is the density generator function corresponding to the multivariate Laplace distribution with $w = \delta u^2$. If $h(w)$ is some non-negative function such that $\int_0^\infty u^{p-1} h(u^2) du < \infty$ (see, Fang et al. (1990)), then X has a multivariate slash-elliptical distribution proposed by $Go'mez$ et al. (2007). Hence multivariate slash Laplace distribution is a special case of the multivariate slash-elliptical distributions.

Remark 4 *If the density generator function in the Proposition 1 is* $h(w) = \frac{1}{(2\pi)^{p/2}} e^{-w/2}$, *then the random vector* X *has the multivariate slash model introduced by Wang and Genton (2006).*

Theorem 1 *If* Y *be an elliptically symmetric distribution on* R^d *known as multivariate exponential power distribution (*$Ferna'ndez$ *et al. (1995)) with pdf given by,*

$$f_Y(y) = k_d \, |\Sigma|^{-1/2} \, e^{-[(y-m)'\Sigma^{-1}(y-m)]^{\lambda/2}},$$

where $m \in \mathsf{R}^d$, Σ is a $d \times d$ positive-definite matrix and k_d is the proportionality constant. Then $X = \frac{Y}{U^{1/q}}$, follow a multivariate slash exponential power distribution with pdf

$$g_d(x; 0, \Sigma, q) = q \, k_d \, |\Sigma|^{-1/2} \int_0^1 v^{q+d-1} e^{-[v'x'\Sigma^{-1}xv]^{\lambda/2}} \mathrm{d}v, \; x \in \mathsf{R}^d.$$

Remark 5 *If* $\lambda = 1$ *in theorem (1), then* Y *has the elliptical Laplace distribution (Haro-*$Lope'z$ *and Smith (1999)) and* X *has the multivariate slash elliptical Laplace distribution. For* $\lambda = 2$, X *has the multivariate elliptical slash distribution.*

Remark 6 *If* $d = 1$ *in theorem (1), then* Y *has the exponential power distribution and* X *has the univariate slash exponential power distribution. For* $d = 1$ *and* $\lambda = 1$, Y *has the univariate slash Laplace distribution. If* $d = 1$ *and* $\lambda = 2$, *then* Y *has the univariate slash distribution.*

STOCHASTIC REPRESENTATION

Here we give a stochastic representations for the multivariate slash Laplace distribution based on the stochastic representation of the multivariate Laplace distribution. Let Y is standardized Laplace random vector then it can be represented as

$$Y \stackrel{d}{=} \sqrt{W}.N,$$

where W is the standard exponential variate and N follows $N_d(0,\Sigma)$ independent of W. Then multivariate slash Laplace, $SLL_d(O,\Sigma,q)$ can be represented as,

$$X \stackrel{d}{=} \sqrt{W}.S,$$

where S is the multivariate slash normal distribution (Wang and Genton (2006)).

MOMENTS

If $X \sim SLL_d(O,\Sigma,q)$, then $X = \Sigma^{1/2}\dfrac{Y}{U^{1/q}}$, where Y is standardized Laplace random vector and $U \sim U(0,1)$.

Then the mean vector and dispersion matrix are

$$E(X) = O, q > 1 \textit{ and } D(X) = \frac{q}{(q-2)}\Sigma, q > 2.$$

CHARACTERISTIC FUNCTION

If $X \sim SLL_d(O,\Sigma,q)$, then the characteristic function is

$$\Phi_X(\tau) = E(e^{(i\tau)^T X}) = \int_0^1 \Phi_Y(\tau u^{-1/q})\mathrm{d}u,$$

where $\Phi_Y(.)$ is characteristic function of L_d given by

$$\Phi_Y(\tau) = \frac{1}{1+\frac{1}{2}\tau'\Sigma\tau}.$$

From the definition and the pdf Eq. (1) of $SLL_d(O,\Sigma,q)$, the following properties hold.

(i) If $X \sim SLL_d(O,\Sigma,q)$, then its linear transformation $V = b + AX \sim SLL_d(b, A\Sigma A^T, q)$, where b is a vector in R^d, A is a non-singular matrix. This property implies that the multivariate slash Laplace distribution is invariant under linear transformations.

(ii) The multivariate slash Laplace has heavier tails than the multivariate Laplace distribution.

(iii) The multivariate slash Laplace distribution tends to the multivariate Laplace distribution as $q \to \infty$. That is

$$\lim_{q\to\infty} g_d(x;O,\Sigma,q) = \frac{2}{(2\pi)^{d/2}\,|\Sigma|^{1/2}} \left(\frac{y'\Sigma^{-1}y}{2}\right)^{\nu/2} K_\nu(\sqrt{2y'\Sigma^{-1}y}),$$

where Σ is a $d \times d$ positive-definite matrix, $\nu = \frac{(2-d)}{2}$ and $K_\nu(u)$ is the modified Bessel function of the third kind given in equation (2).

(iv) The multivariate slash Laplace distribution is symmetric. Here Symmetric refers to the elliptically contoured or elliptically symmetric distribution.

(v) Star Unimodality property

We know that univariate standard Laplace and slash Laplace distributions are unimodal with the mode at zero. There are many nonequivalent notions of unimodality for probability distributions in $\mathbb{R}^d$ (see, Dharmadhikari and Joag-Dev (1988)). A natural extension of univariate unimodality is star unimodality in $\mathbb{R}^d$. This property requires that for a distribution with continuous density f the density be non-increasing along the rays emanating from zero.

Definition 3 *A distribution P with continuous density f on* $\mathbb{R}^d$ *is star unimodal about zero if and only if whenever* $0 < t < u < \infty$ *and* $x \neq 0$, then $f(ux) \leq f(tx)$.

The *d*-dimensional Laplace laws are star unimodal about zero (see, Kotz et al. 2001). The multivariate slash Laplace distribution is a scale mixture of the multivariate Laplace distribution, hence the multivariate slash Laplace laws are also star unimodal about zero.

3. MULTIVARIATE ASYMMETRIC SLASH LAPLACE DISTRIBUTION

In this section, we define a multivariate asymmetric slash Laplace distribution and derive its pdf. We discuss various properties and provide the stochastic representation of the multivariate asymmetric slash Laplace distribution, which is useful for simulation studies.

Definition 4 *A random vector* $X \in \mathbb{R}^d$ *has a d-dimensional asymmetric slash Laplace* (ASL_d) *distribution with location parameter* $\mu = O$, $d \times d$ *positive definite scale matrix* Σ, *skewness parameter* $m \in \mathbb{R}^d$, *and tail parameter* $q > 0$, *denoted by* $ASL_d(O,\Sigma,m,q)$, *if*

$$X = \frac{Y}{U^{1/q}}, \tag{4}$$

where $Y \sim AL_d(O, \Sigma, m)$ with characteristic function (cf) given by $\phi_Y(t) = \dfrac{1}{1 + \frac{1}{2}t'\Sigma t - im't}$, where $\mu = O$ is the location parameter, Σ is a $d \times d$ positive definite scale matrix, $m \in \mathrm{R}^d$ is the skewness parameter and $U \sim U(0,1)$ independent of Y.

The pdf of the random vector X in (4) is given by

$$f_d(x; O, \Sigma, m, q) = q\int_0^1 v^{q+d-1}\, g_d(xv; O, \Sigma, m)\, dv, \tag{5}$$

where $g_d(x; O, \Sigma, m)$ is the density of *d*-variate asymmetric Laplace (AL_d) random vector, which is given by

$$f_d(y) = \frac{2e^{(y'\Sigma^{-1}m)}}{(2\pi)^{d/2}\,|\Sigma|^{1/2}}\left(\frac{y'\Sigma^{-1}y}{2 + m'\Sigma^{-1}m}\right)^{\nu/2} K_\nu\left(\sqrt{(2 + m'\Sigma^{-1}m)(y'\Sigma^{-1}y)}\right),$$

where Σ is a $d \times d$ positive definite scale matrix, $m \in \mathrm{R}^d$ is the skewness parameter, $\nu = \dfrac{(2-d)}{2}$ and $K_\nu(.)$ is the modified Bessel function of the third kind given in equation (2).

For the simulation purposes, we consider Y as,

$$Y \overset{d}{=} mW + \sqrt{W}N,$$

where W is the standard exponential variate and N follows $N_d(O, \Sigma)$ independent of W. Then $X = \dfrac{Y}{U^{1/q}}$ will follow $ASL_d(O, \Sigma, m, q)$. Here we considered the more general non-central *d*-dimensional AL random vector with location centered at m.

Remark 7 *Note that for* $m = O$ *, the distribution* $ASL_d(O, \Sigma, m)$ *reduces to multivariate symmetric Laplace Law and hence* X *reduces to symmetric multivariate slash Laplace denoted by* $SLL_d(O, \Sigma, q)$.

Remark 8 *The asymmetric slash Laplace random vector given in (4) is a scale mixture of the asymmetric Laplace random vector.*

Remark 9 *The limiting distribution of the multivariate asymmetric slash Laplace distribution as* $q \to \infty$*, is the asymmetric Laplace density. Also for* $m = 0$ *and* $q \to \infty$ *the multivariate asymmetric slash Laplace distribution tends to multivariate Laplace distribution.*

In the univariate case, i.e. for $d = 1$ in equation (5), we get the univariate asymmetric slash Laplace distribution and figure (3) gives probability density curves of the asymmetric slash Laplace distribution for various values of parameters.

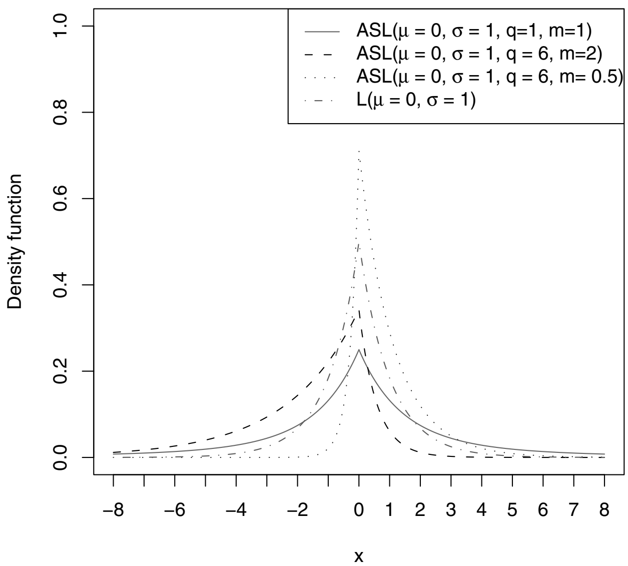

Figure 3 – Asymmetric slash Laplace densities along with Laplace density.

The density plot of asymmetric slash Laplace distribution for various values of the parameters are compared with the standard Laplace distribution, is given below. From the Figure (3), we can see that asymmetric slash Laplace distribution has heavier tails, asymmetry of varying degrees and peakedness than the Laplace distribution. We can see that the asymmetric slash Laplace density is symmetric, negatively skewed and positively skewed for $m = 0$, $m > 1$ and $m < 1$ respectively. Also for $q = 1$ asymmetric Laplace density has heavier tails like Cauchy distribution. The main feature of the asymmetric slash Laplace distribution in (4) is that the parameters m and q control skewness and tail behaviour.

STOCHASTIC REPRESENTATION

Here we give a stochastic representations for the multivariate asymmetric slash Laplace distribution based on the stochastic representation of the multivariate asymmetric Laplace distribution.

Let Y is multivariate asymmetric Laplace random vector then it can be represented as

$$Y \stackrel{d}{=} mW + \sqrt{W}N,$$

where W is the standard exponential variate and N follows $N_d(O,\Sigma)$ independent of W. Then $ASL_d(m,\Sigma,q)$ can be represented as,

$$X \stackrel{d}{=} mW\,U^{-1/q} + \sqrt{W}.S,$$

where S is the multivariate slash normal distribution (Wang and Genton (2006)).

Remark 10 *In the above representation of Y if W has a generalized inverse Gaussian distribution (Barndorff-Nielsen (1997)) with parameters (λ,χ,ψ) denoted by $GIG(\lambda,\chi,\psi)$ with pdf*

$$f(x) = \frac{(\chi/\psi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})} x^{\lambda-1}\, e^{-1/2(\chi x^{-1}+\psi x)},\ x > 0, \tag{6}$$

where K_λ is the modified Bessel function of the third kind given in equation (2), $\chi \geq 0$, $\psi > 0$ and $\lambda \in \mathrm{R}$. Then the random vector Y has the multivariate generalized hyperbolic distribution and hence the random vector X has the multivariate generalized slash hyperbolic distribution, with $m = \Sigma\beta$ and β is some *d*-dimensional vector.

Proposition 2 *If $X \sim ASL_d(O,\Sigma,m,q)$ and let $V = b + AX$, where b is a vector in R^d, A is a non-singular matrix. Then the random vector $V \sim ASL_d(b, A\Sigma A^T, A^{-T}m, q)$. This property implies that the multivariate asymmetric slash Laplace distribution is invariant under linear transformations.*

Remark 12 *Let $Y = (Y_1, Y_2, \ldots, Y_d)' \sim AL_d(O,\Sigma,m)$ for any $b = (b_1, b_2, \ldots, b_d)' \in \mathrm{R}_d$, the random variable $Y_b = \sum_{i=1}^{d} b_i Y_i$ is univariate $AL(\sigma,\mu)$ with $\sigma = \sqrt{b'\Sigma b}$ and $\mu = m'b$. Further, if Y is symmetric, then so is Y_b. Hence, if $X = (X_1, X_2, \ldots, X_d)' \sim ASL_d(m,\Sigma;q)$ for any $b = (b_1, b_2, \ldots, b_d)' \in \mathrm{R}$, the random variable $X_b = \sum_{i=1}^{d} b_i X_i$ is univariate $ASL(\sigma,\mu,q)$ with $\sigma = \sqrt{b'\Sigma b}$ and $\mu = m'b$.*

Remark 13 *The above remark implies that the sum $\sum_{i=1}^{d} X_i$ has an ASL distribution if all $X_{i'}s$ are components of a multivariate ASL random vector. Thus all $X_{i'}s$ are univariate ASL random variables.*

Remark 14 *If* X *has a univariate* $ASL(1,1,q)$ *law and* $m \in \mathrm{R}^d$ *, then the random vector* $X = mX$ *has the* $ASL_d(m,\Sigma,q)$ *,* $\Sigma = mm'$ *with characteristic function*

$$\Phi_X(\tau) = E(e^{(i\tau)^T X}) = \int_0^1 \phi(\tau u^{-1/q})\,\mathrm{d}u, \text{ where } \Phi(.) \text{ is given by}$$

$$\phi(\tau) = \frac{1}{1 + \frac{1}{2}\tau' mm'\tau - im'\tau}.$$

Remark 15 *Let* $Y \sim AL_d(m,O)$ *with characteristic function* $\psi_Y(t) = \dfrac{1}{1 - im't}$. *Let* $X \overset{d}{=} m\dfrac{Y}{U^{1/q}}$, then X has the multivariate slash Laplace distribution denoted by $X \sim ASL_d(m,O;q)$.

4. APPLICATIONS

In this section we will present an application of the asymmetric-slash Laplace distribution in univariate setting. We downloaded the cDNA dual dye microarray data sets (Experiment id-51401) from the Stanford Microarray Database. Each array chip contains approximately 42000 human cDNA elements, representing over 30000 unique genes. The data set was normalized using locally weighted linear regression (LOWESS) (Cleveland and Delvin, 1988). This method is capable of removing intensity dependence in $log_2(R_i/G_i)$ values and it has been successfully applied to microarray data (Yang et al., 2002). Where R_i is the red dye (for treatment) intensity and G_i is the green dye (control) intensity for the i^{th} gene. After normalization, each distribution of the gene expression has a similar shape and exhibits heavier tails compared to a Gaussian distribution and a certain degree of asymmetry. We use the maximum likelihood estimation method to estimate the parameters. The maximization of the likelihood is implemented using the optim function of the R statistical software, applying the BFGS algorithm (See R Development Core Team, 2006). Also the function nlminb in the S-Plus package can be used to locate the maximum point of the likelihood function assuming that all the parameters are unknown.

Figure (4) given below, depicts the histogram of the gene expression data and the fitted probability density function evaluated at the MLEs. We compared the empirical distribution function of the microarray gene expression data with the asymmetric slash Laplace (ASL), skew-slash, skew-slash t and skew-normal distributions evaluated at the MLEs. It can be clearly seen that the estimated density of the ASL fits the data quite well compare to skew-slash, slew-slash t and skew-normal densities. ASL captures skewness, peakedness and heavy tails. Hence, ASL provides the possibility of modelling impulsiveness and skewness required for gene expression data. From figure (4) it is observed that the gene expression data are asymmetric and the asymmetric slash Laplace distribution describes the data well. The parameter estimates together with the standard errors (given in brackets) are $\hat{\mu} = -0.154(0.003)$, $\hat{\sigma} = 0.501(0.016)$, $\hat{m} = 1.455(0.023)$ and $\hat{q} = 16.85(0.010)$. It can be clearly seen that the estimated density of the asymmetric-slash Laplace distribution fits the data quite well compare to skew slash and normal densities. It captures skewness, peakedness and heavy tails.

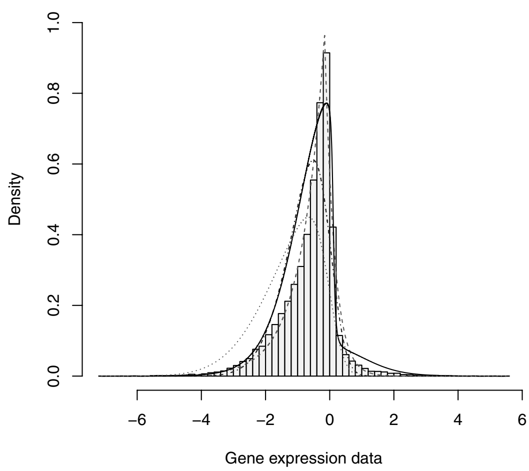

Figure 4 – Fitted asymmetric slash Laplace (ASL) probability density function (red dashed line (the peaked one)) to the microarray gene expression data along with skew slash t (black line), skew slash (blue dash and dot line) and skew normal (dotted line).

We have used Akaike's Information Criterion (AIC) (Akaike (1973), Burnham and Anderson (1998)) to evaluate the comparative appropriateness of ASL. Let $f(\theta)$ is our model, then AIC is given by

$$AIC = -2log(L_f(\hat{\theta} \mid x_1,...,x_n)) + 2K,$$

where K is the number of parameters being estimated, L is the likelihood function of the model f, and $\hat{\theta}$ is the maximum likelihood estimate of the parameters of f. A smaller value of AIC indicates a better fit. We found that $AIC_{ASL} - AIC_{skew-slasht} < 0$, $AIC_{ASL} - AIC_{skew-slash} < 0$ and $AIC_{ASL} - AIC_{skew-normal} < 0$, which implies a better fit for the ASL.

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Department of Statistics BINDU PUNATHUMPARAMBATH
St. Thomas College, Pala, Kerela, India

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SUMMARY

The multivariate asymmetric slash Laplace distribution and its applications

We have introduced a multivariate asymmetric-slash Laplace distribution, a flexible distribution that can take skewness and heavy tails into account. This distribution is useful in simulation studies where it can introduce distributional challenges in order to evaluate a statistical procedure. It is also useful in analyzing data sets that do not follow the normal law. We have used the microarray data set for illustration. The asymmetric slash Laplace distribution provides the possibility of modelling impulsiveness and skewness required for gene expression data. Hence, the probability distribution presented in this paper will be very useful in estimation and detection problems involving gene expression data. The multivariate asymmetric-slash Laplace distribution introduced in this article is clearly an alternative to multivariate skew-slash distributions because it can model skewness, peakedness and heavy tails. One interesting advantage of the multivariate asymmetric slash Laplace distribution is that its moments can be computed analytically by taking advantage of the moments of the multivariate asymmetric Laplace distribution, see the discussion in Section 3. Another attractive feature is that simulations from the multivariate asymmetric-slash Laplace distribution are straightforward from softwares that permit simulations from the multivariate asymmetric Laplace or Laplace distribution. We believe that the new class will be useful for analyzing data sets having skewness and heavy tails. Heavy-tailed distributions are commonly found in complex multi-component systems like ecological systems, biometry, economics, sociology, internet traffic, finance, business etc.