ON THE WEIBULL RECORD STATISTICS AND ASSOCIATED INFERENCES

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1. INTRODUCTION

The record values were introduced by Chandler (1952). Let $X_1, X_2, ..., X_n$ be a sequence of i.i.d absolutely continuous random variables with cumulative distribution function (cdf), $F(.)$, and probability density function (pdf), $f(.)$. Let $Y_n = \max(\min)\{X_1, X_2, ..., X_n\}$ for $n \geq 1$. One defines Y_j as a upper (lower) record value of the sequence .. if $Y_j > (<) Y_{j-1}$ for $j > 1$. By definition the first observation X_1, is the first lower and also the upper record value.

Record statistics is widely used in many real life applications involving data relating to weather, sport, economics and life testing studies. We refer the readers, for example, to Chandler (1952), Ahsanullah (1995, 2004) and Arnold *et al.* (1992). Statistical inference about distributions through record values also has a long history. Some inferential methods based on record have been discussed by Ahsanullah (1988, 1990), Balakrishnan and Chan (1993), Balakrishnan *et al.* (1995), Sultan and Balakrishnan (1999), Sultan and Moshref (2000), Raqab (2002) and Soliman *et al.* (2006). The distributions considered by these authors include: exponential, Gumbel, Rayleigh, Weibull, logistic, generalized exponential and generalized Pareto.

In this paper we construct confidence interval and compute the coefficient of skewness of upper/lower record statistics. First in section 2 we give some preliminary results. In sections 3 and 4, we give the confidence interval and the quantile of records. In section 5 we propose a confidence interval for nth upper/lower records for Weibull model and a point estimation for the shape of this family. Also, in this section, some discussions are given on the behavior of skewness of nth upper/lower records for Weibull distribution which is useful for making inference about the nth upper/lower record. Finally, we conclude the paper in section 6.

2. PRELIMINARIES

In the following, some preliminaries will be given. The cdf of the two-parameter Weibull distribution is given by (Johnson *et al.*, 1994; Murthy *et al.*, 2004):

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\} \tag{1}$$

for $x > 0, \alpha > 0$ and $\beta > 0$. The parameters .. and β are known as the shape and scale parameters, respectively. Also, a continuous random variable G_n is said to have the gamma distribution with shape parameter $n > 0$ if its pdf is:

$$f_{G_n}(x) = \frac{x^{n-1}\exp(-x)}{\Gamma(n)} \tag{2}$$

where $x > 0$. Let $X_{U(n)}$ and $X_{L(n)}$ for $n \geq 1$ denote, respectively, the n th upper and lower record statistics of a given cdf $F(.)$ and pdf $f(.)$. Then the pdf of $X_{U(n)}$ and $X_{L(n)}$ are given in Ahsanullah (1995, 2004):

$$f_{X_{U(n)}}(x) = \frac{[-\log(1-F(x))]^{n-1}}{\Gamma(n)} f(x) \tag{3}$$

and

$$f_{X_{L(n)}}(x) = \frac{[-\log F(x)]^{n-1}}{\Gamma(n)} f(x) \tag{4}$$

where $n \geq 1$. Transforming $u = -\log(1-F(x))$ we derive

$$F_{X_{U(n)}}(y) = \int_0^{y'} \frac{u^{n-1}\exp(-u)}{\Gamma(n)} du = F_{G_n}(y') \tag{5}$$

where $y' = -\log(1-F(y))$ and $F_{G_n}(.)$ denotes the cdf of G_n. Thus,

$$\begin{aligned} F_{X_{U(n)}}(y) &= P(G_n < -\log(1-F(y)) \\ &= P(\exp(-G_n) > 1-F(y)) \\ &= P(F^{-1}(1-\exp(-G_n)) < y) \end{aligned} \tag{6}$$

Therefore $X_{U(n)}$ and $F^{-1}(1-\exp(-G_n))$ are distributed identically or $X_{U(n)} \overset{d}{=} F^{-1}(1-\exp(-G_n))$ for $n \geq 1$. Now assuming that the family of distributions $F(.)$ with quantile function $F^{-1}(.)$ are given, to simulate from n th upper record statistic from this family one can do as follows (Arnold *et al.*, 1992, pp. 243).

1. simulate a generation, such as g from gamma distribution with shape parameter n.

2. evaluate the quantile function at point $1-\exp(-g)$.

3. accept the $F^{-1}(1-\exp(-G_n))$ as a generation of n th upper record from the given family $F(.)$.

Refer to (4) and making a change of variable $u=-\log F(x)$, we see that:

$$F_{X_{L(n)}}(y)=\int_{y'}^{\infty}\frac{u^{n-1}\exp(-u)}{\Gamma(n)}du=1-G_{n-1}(y') \tag{7}$$

where $y'=-\log F(y)$. Thus,

$$\begin{aligned} F_{X_{L(n)}}(y) &= P(G_n>-\log F(y)) \\ &= P(\exp(-G_n)<F(y)) \\ &= P(F^{-1}(\exp(-G_n))<y) \end{aligned} \tag{8}$$

Now suppose that a family of distributions F with quantile function $F^{-1}(.)$ is available. To generate observation from n th lower record statistic of this family proceed as follows (Arnold *et al.*, 1992, pp. 243).

1. simulate a generation, such as g from gamma distribution with shape parameter n.

2. evaluate the quantile function at point $\exp(-g)$.

3. accept the $F^{-1}(\exp(-G_n))$ as a generation of n th upper record from the given family $F(.)$.

3. CONFIDENCE INTERVAL

To construct a confidence interval for the given upper record, first note that

$$P(\exp(-G_n^{-1}(1-\gamma/2))<\exp(-G_n)<\exp(-G_n^{-1}(\gamma/2))=1-\gamma \tag{9}$$

where the confidence level γ is partitioned in two equal parts (upper and lower) $\gamma/2$ and $\gamma/2$ and

$$G_n^{-1}(g)=\min\{x|G_n(x)>g\} \tag{10}$$

Thus,

$$\begin{aligned} &P(F^{-1}(1-\exp(-G_n^{-1}(\gamma/2))\ )<F^{-1}(1-\exp(-G_n)) \\ &\quad <F^{-1}(1-\exp(-G_n^{-1}(1-\gamma/2))))=1-\gamma \end{aligned} \tag{11}$$

and finally a confidence interval for the n th upper record statistic with probability $1-\gamma$ for $0<\gamma<1$ is

$$P(F^{-1}(1-\exp(-G_n^{-1}(\gamma/2))) < X_{U(n)} < F^{-1}(1-\exp(-G_n^{-1}(1-\gamma/2)))) \\ = 1-\gamma \tag{12}$$

Similarly, a confidence interval for the n th lower record statistic is given as follows.

$$P(F^{-1}(\exp(-G_n^{-1}(1-\gamma/2))) < X_{L(n)} < F^{-1}(\exp(-G_n^{-1}(\gamma/2)))) = 1-\gamma \tag{13}$$

Theorem 3.1 *Let of* $X_{U(n)}$ *and* $X_{L(n)}$ *be, respectively, the* n *th upper and lower record statistics from a family with cdf* $F(.)$. *Then, a* $100(1-\gamma)\%$ *confidence interval for upper and lower record statistics are given as follows.*

$$[F^{-1}(1-\exp(-G_n^{-1}(g/2))), F^{-1}(1-\exp(-G_n^{-1}(1-g/2)))] \tag{14}$$

$$[F^{-1}(\exp(-G_n^{-1}(1-\gamma/2))), F^{-1}(\exp(-G_n^{-1}(\gamma/2)))] \tag{15}$$

where $1-\gamma$.

4. RECORD QUANTILE

Quantiles play important role in characterizing a distribution. Some important statistical central, dispersion and combined summaries such as median, range and skewness coefficient are obtained readily through quantile function. For record statistic this function can be obtained easily. From (5) it can be seen that

$$\begin{aligned} p &= P(X_{U(n)} < q_p) \\ &= P(F^{-1}(1-\exp(-G_n)) < q_p) \\ &= P(G_n < -\log(1-F(q_p))) \end{aligned} \tag{16}$$

This means that

$$F^{-1}_{X_{U(n)}}(p) = F^{-1}(1-\exp(-G_n^{-1}(p))) \tag{17}$$

Then using (7), the quantile function of lower record statistic is derived as follows.

$$F^{-1}_{X_{L(n)}}(p) = F^{-1}(\exp(-G_n^{-1}(1-p))) \tag{18}$$

Theorem 4.1 *The* p *th quantile for* $0 < p < 1$ *of* n *th upper and lower record statistics, respectively, is given as follows.*

$$q_{U(n)}(p) = F^{-1}_{X_{U(n)}}(p) = F^{-1}(1-\exp(-G_n^{-1}(p))) \tag{19}$$

$$q_{L(n)}(p) = F^{-1}_{X_{L(n)}}(p) = F^{-1}(\exp(-G_n^{-1}(1-p))) \tag{20}$$

Thus, the skewness coefficient of n th upper and lower record statistics are, respectively,

$$Sk_{U(n)} = \frac{F^{-1}(1-e^{-G_n^{-1}(1-p)}) - 2F^{-1}(1-e^{-G_n^{-1}(0.5)}) + F^{-1}(1-e^{-G_n^{-1}(p)})}{F^{-1}(1-e^{-G_n^{-1}(1-p)}) - F^{-1}(1-e^{-G_n^{-1}(p)})} \tag{21}$$

$$Sk_{L(n)} = \frac{F^{-1}(e^{-G_n^{-1}(1-p)}) - 2F^{-1}(e^{-G_n^{-1}(0.5)}) + F^{-1}(e^{-G_n^{-1}(p)})}{F^{-1}(e^{-G_n^{-1}(1-p)}) - F^{-1}(e^{-G_n^{-1}(p)})} \tag{22}$$

where $p = 0.25$.

5. SOME INFERENCES ABOUT THE RECORDS OF WEIBULL MODEL

In the first subsection, a confidence interval is constructed for n th upper/ lower record statistic from Weibull distribution. In second subsection, on the basis of quantile function of n th upper/lower record of this family, a point estimator for shape parameter is given and some numerical results related to performance analysis is given. A simulation-based study also is provided in third subsection. Finally in the last subsection, behavior of skewness coefficient introduced in previous section is discussed for records of Weibull distribution.

5.1 *Confidence interval for the Weibull records*

Let $F(.)$ be the cdf of Weibull random variable given in (1). Therefore, using (14) and (15), a $100(1-\gamma)\%$ confidence of upper and lower record statistics, respectively, are constructed as follows.

$$\{\beta[G_n^{-1}(\gamma/2)]^{1/\alpha}, \beta[G_n^{-1}(1-\gamma/2)]^{1/\alpha}\} \tag{23}$$

$$\{\beta[-\log(1-e^{-G_n^{-1}(1-\gamma/2)})]^{1/\alpha}, \beta[-\log(1-e^{-G_n^{-1}(\gamma/2)}]^{1/\alpha}\} \tag{24}$$

where $0<\gamma<1$. The confidence interval for n th upper records of Weibull family for some levels of α when $\beta=1$ and confidence levels (CL): 0.90, 0.95 and 0.99 are given in Table 1.

TABLE 1

Confidence interval for n th upper record of Weibull family

	CL=0.99		CL=0.95		CL=0.90	
$\alpha = 0.5$	L	U	L	U	L	U
Order, n 1	0.0000	28.0720	0.0006	13.6080	0.0026	8.9740
2	0.0107	55.2070	0.0587	31.0430	0.1263	22.5040
3	0.1142	86.0030	0.3828	52.1960	0.6686	39.6370
4	0.4519	120.5050	1.1878	76.8650	1.8668	60.1190
5	1.1619	158.6110	2.6357	104.8900	3.8815	83.7870
6	2.3621	200.2160	4.8483	136.1500	6.8278	110.5240
7	4.1507	245.2250	7.9206	170.5500	10.7933	140.2420
8	6.6106	293.5600	11.9290	208.0140	15.8470	172.8730
9	9.8119	345.1500	16.9363	248.4780	22.0452	208.3590
10	13.8155	399.9370	22.9958	291.8910	29.4350	246.6540
$\alpha = 1$	L	U	L	U	L	U
Order, n 1	0.0050	5.2983	0.0253	3.6889	0.0513	2.9957
2	0.1035	7.4301	0.2422	5.5716	0.3554	4.7439
3	0.3379	9.2738	0.6187	7.2247	0.8177	6.2958
4	0.6722	10.9775	1.0899	8.7673	1.3663	7.7537
5	1.0779	12.5941	1.6235	10.2416	1.9701	9.1535
6	1.5369	14.1498	2.2019	11.6683	2.6130	10.5130
7	2.0373	15.6597	2.8144	13.0595	3.2853	11.8424
8	2.5711	17.1336	3.4538	14.4227	3.9808	13.1481
9	3.1324	18.5782	4.1154	15.7632	4.6952	14.4346
10	3.7169	19.9984	4.7954	17.0848	5.4254	15.7052
$\alpha = 2$	L	U	L	U	L	U
Order, n 1	0.0708	2.3018	0.1591	1.9207	0.2265	1.7308
2	0.3217	2.7258	0.4921	2.3604	0.5961	2.1780
3	0.5813	3.0453	0.7866	2.6879	0.9043	2.5091
4	0.8199	3.3132	1.0440	2.9610	1.1689	2.7845
5	1.0382	3.5488	1.2742	3.2003	1.4036	3.0255
6	1.2397	3.7616	1.4839	3.4159	1.6165	3.2424
7	1.4274	3.9572	1.6776	3.6138	1.8125	3.4413
8	1.6035	4.1393	1.8584	3.7977	1.9952	3.6260
9	1.7699	4.3103	2.0286	3.9703	2.1668	3.7993
10	1.9279	4.4720	2.1898	4.1334	2.3293	3.9630
$\alpha = 3$	L	U	L	U	L	U
Order, n 1	0.1711	1.7433	0.2936	1.5451	0.3716	1.4416
2	0.4695	1.9513	0.6233	1.7728	0.7083	1.6803
3	0.6965	2.1010	0.8521	1.9332	0.9351	1.8465
4	0.8760	2.2225	1.0291	2.0620	1.1096	1.9793
5	1.0253	2.3266	1.1753	2.1717	1.2536	2.0918
6	1.1540	2.4187	1.3010	2.2681	1.3774	2.1907
7	1.2677	2.5019	1.4119	2.3549	1.4866	2.2794
8	1.3700	2.5780	1.5116	2.4342	1.5849	2.3602
9	1.4632	2.6485	1.6025	2.5074	1.6745	2.4348
10	1.5490	2.7144	1.6863	2.5756	1.7572	2.5043
$\alpha = 4$	L	U	L	U	L	U
Order, n 1	0.2661	1.5172	0.3989	1.3859	0.4759	1.3156
2	0.5672	1.6510	0.7015	1.5364	0.7721	1.4758
3	0.7624	1.7451	0.8869	1.6395	0.9509	1.5840
4	0.9055	1.8202	1.0217	1.7207	1.0812	1.6687
5	1.0189	1.8838	1.1288	1.7889	1.1847	1.7394
6	1.1134	1.9395	1.2181	1.8482	1.2714	1.8007
7	1.1947	1.9893	1.2952	1.9010	1.3463	1.8551
8	1.2663	2.0345	1.3632	1.9488	1.4125	1.9042
9	1.3304	2.0761	1.4243	1.9926	1.4720	1.9492
10	1.3885	2.1147	1.4798	2.0331	1.5262	1.9907
$\alpha = 5$	L	U	L	U	L	U
Order, n 1	0.3467	1.3958	0.4794	1.2983	0.5521	1.2454
2	0.6353	1.4935	0.7531	1.4099	0.8131	1.3653
3	0.8049	1.5612	0.9084	1.4851	0.9605	1.4448
4	0.9236	1.6147	1.0174	1.5437	1.0644	1.5063
5	1.0151	1.6597	1.1018	1.5925	1.1452	1.5571
6	1.0898	1.6988	1.1710	1.6346	1.2118	1.6008
7	1.1530	1.7336	1.2299	1.6718	1.2686	1.6394
8	1.2079	1.7651	1.2813	1.7053	1.3182	1.6741
9	1.2565	1.7939	1.3270	1.7359	1.3625	1.7056
10	1.3003	1.8205	1.3682	1.7641	1.4024	1.7346

TABLE 2

Confidence interval for n th lower record of Weibull family

	CL=0.99		CL=0.95		CL=0.90	
$\alpha = 0.5$	L	U	L	U	L	U
Order, n 1	0.000025	28.072160	0.000640	13.607830	0.002631	8.974411
2	0.000000	5.380253	0.000014	2.361184	0.000076	1.456955
3	0.000000	1.560734	0.000000	0.598485	0.000003	0.339201
4	0.000000	0.510561	0.000000	0.167991	0.000000	0.086689
5	0.000000	0.173030	0.000000	0.048251	0.000000	0.022550
6	0.000000	0.058625	0.000000	0.013736	0.000000	0.005797
7	0.000000	0.019514	0.000000	0.003821	0.000000	0.001455
8	0.000000	0.006325	0.000000	0.001033	0.000000	0.000355
9	0.000000	0.001988	0.000000	0.000271	0.000000	0.000084
10	0.000000	0.000605	0.000000	0.000069	0.000000	0.000019
$\alpha = 1$	L	U	L	U	L	U
Order, n 1	0.005013	5.298317	0.025318	3.688880	0.002630	8.974411
2	0.000593	2.319537	0.003811	1.536614	0.000070	1.456955
3	0.000094	1.249294	0.000729	0.773618	0.000000	0.339201
4	0.000017	0.714536	0.000156	0.409867	0.000000	0.086689
5	0.000003	0.415970	0.000036	0.219660	0.000000	0.022550
6	0.000001	0.242128	0.000009	0.117200	0.000000	0.005797
7	0.000000	0.139694	0.000002	0.061810	0.000000	0.001455
8	0.000000	0.079532	0.000001	0.016450	0.000000	0.000355
9	0.000000	0.044593	0.000000	0.008300	0.000000	0.000084
10	0.000000	0.024609	0.000000	0.000069	0.000000	0.000019
$\alpha = 2$	L	U	L	U	L	U
Order, n 1	0.070799	2.301810	0.159116	1.920650	0.226480	1.730820
2	0.024357	1.523000	0.061737	1.239600	0.093504	1.098660
3	0.009688	1.117720	0.026993	0.879560	0.042962	0.763160
4	0.004133	0.845300	0.012480	0.640210	0.020719	0.542610
5	0.001842	0.644960	0.005971	0.468680	0.010288	0.387510
6	0.000846	0.492060	0.002926	0.342350	0.005213	0.275930
7	0.000398	0.373760	0.001459	0.248630	0.002682	0.195310
8	0.000190	0.282010	0.000738	0.179260	0.001396	0.137280
9	0.000092	0.211170	0.000378	0.128280	0.000734	0.095820
10	0.000000	0.024609	0.000000	0.000069	0.000000	0.000019
$\alpha = 3$	L	U	L	U	L	U
Order, n 1	0.171140	1.743330	0.293636	1.545130	0.371553	1.441570
2	0.084028	1.323730	0.156206	1.153950	0.206010	1.064730
3	0.045445	1.077010	0.089985	0.918000	0.122666	0.835110
4	0.025754	0.894010	0.053804	0.742820	0.075435	0.665260
5	0.015025	0.746480	0.032914	0.603370	0.047304	0.531530
6	0.008946	0.623280	0.020457	0.489380	0.030067	0.423840
7	0.005408	0.518870	0.012866	0.395390	0.019304	0.336630
8	0.003308	0.430040	0.008168	0.317930	0.012492	0.266120
9	0.002041	0.354610	0.005224	0.254350	0.008135	0.209390
10	0.000000	0.024609	0.000000	0.000069	0.000000	0.000019
$\alpha = 4$	L	U	L	U	L	U
Order, n 1	0.266081	1.517170	0.398893	1.385870	0.475899	1.315610
2	0.156069	1.234100	0.248470	1.113370	0.305785	1.048170
3	0.098427	1.057220	0.164297	0.937850	0.207273	0.873590
4	0.064289	0.919400	0.111716	0.800130	0.143940	0.736620
5	0.042915	0.803090	0.077274	0.684600	0.101432	0.622510
6	0.029088	0.701470	0.054091	0.585100	0.072204	0.525290
7	0.019944	0.611360	0.038202	0.498620	0.051788	0.441940
8	0.013794	0.531050	0.027169	0.423390	0.037365	0.370520
9	0.009602	0.459530	0.019433	0.358160	0.027088	0.309540
10	0.000000	0.024609	0.000000	0.000069	0.000000	0.000019
$\alpha = 5$	L	U	L	U	L	U
Order, n 1	0.346746	1.395810	0.479386	1.298310	0.552093	1.245380
2	0.226284	1.183260	0.328261	1.089720	0.387554	1.038350
3	0.156492	1.045520	0.235778	0.949960	0.283945	0.897520
4	0.111304	0.934990	0.173177	0.836620	0.212102	0.783060
5	0.080555	0.839100	0.128952	0.738500	0.160302	0.684410
6	0.059016	0.753020	0.096940	0.651310	0.122138	0.597480
7	0.043636	0.674580	0.073395	0.573080	0.093623	0.520350
8	0.032489	0.602710	0.055881	0.502800	0.072106	0.451910
9	0.024316	0.536850	0.042739	0.439800	0.055747	0.391360
10	0.000000	0.024609	0.000000	0.000069	0.000000	0.000019

For n th lower record, these intervals are provided in Table 2. Note that each confidence interval is determined by its lower (L) and upper (U) bounds.

If α and β are unknown, the maximum likelihood estimation of confidence intervals given in (23) and (24) are, respectively as follows.

$$\{\hat{\beta}[G_n^{-1}(\gamma/2)]^{1/\hat{\alpha}}, \hat{\beta}[G_n^{-1}(1-\gamma/2)]^{1/\hat{\alpha}}\} \tag{25}$$

$$\{\hat{\beta}[-\log(1-e^{-G_n^{-1}(1-\gamma/2)})]^{1/\hat{\alpha}}, \hat{\beta}[-\log(1-e^{-G_n^{-1}(\gamma/2)})]^{1/\hat{\alpha}}\} \tag{26}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are maximum likelihood estimation of α and β.

5.2 *Point estimator for the shape parameter*

Refer to the result of Theorem 4.1 and use (1) instead of $F(.)$. Then we can write

$$q_{L(n)}(p) = \beta[-\log(1-\exp(-G_n^{-1}(1-p)))]^{1/\alpha} \tag{27}$$

and

$$q_{U(n)}(p) = \beta[G_n^{-1}(p)]^{1/\alpha} \tag{28}$$

As a result, use $p = 0.5$, then the $G_n^{-1}(0.5)$ or median of gamma distribution with shape parameter n can be explained fully by a polynomial of the form $-0.32710 + 0.9998n + .000002n^2$. Substituting this polynomial into (27) and (28) instead of $G_n^{-1}(0.5)$, a simple estimator of shape parameter of Weibull distribution on the basis of upper and lower record observation is introduced as follows.

Theorem 5.1 *Suppose that a sequence of n th upper record from Weibull family are observed. A simple estimator of shape parameter α is given by*

$$\hat{\alpha} = \frac{\log(0.000002n^2 + 0.9998n - 0.3271)}{\log(\hat{m}_{U(n)}) - \log(\beta)} \tag{29}$$

where $\hat{m}_{U(n)}$ is the sample median of n th upper record values.

Corollary 5.1 Suppose $\{X_1, X_2, ..., X_n\}$ is a sequence m observations from the Weibull family with known scale parameter β. Then, for $n = 1$, the $\hat{\alpha}$ given in (29) is the percentile estimate of shape parameter. We have:

$$\hat{\alpha} = \frac{\log[0.6727]}{\log(\hat{m}) - \log(\beta)} \tag{30}$$

It should be noted that the value 0.6727 in numerator of (30) is obtained by substituting $n=1$ in $-0.32710+0.9998n+.000002n^2$ and an more accurate approximation is 0.6931. Therefore, it is suggested to use 0.6931 in numerator of (30). The percentile-based estimation of shape parameter of Weibull distribution has been discussed by Johnson et al. (1994) and Murthy et al. (2004).

5.3 *Performance analysis of the point estimator*

Here we study behavior of $\hat{\alpha}$ given in (29). Unfortunately checking the performance of $\hat{\alpha}$, theoretically is an exhaustive task. In contrast, it is easily done through simulations. We carried out a simulation study for some selected levels of parameters $\hat{\alpha}$ and $\hat{\beta}$ and record order for $n=2,3,...,10$. In this study for each order we compute the absolute relative error (ARE) of $\hat{\alpha}$ on the basis of a sample of size $N=50$ records. This process is repeated 50 times to obtain the boxplot of AREs. Figures 1-9 show the simulations results.

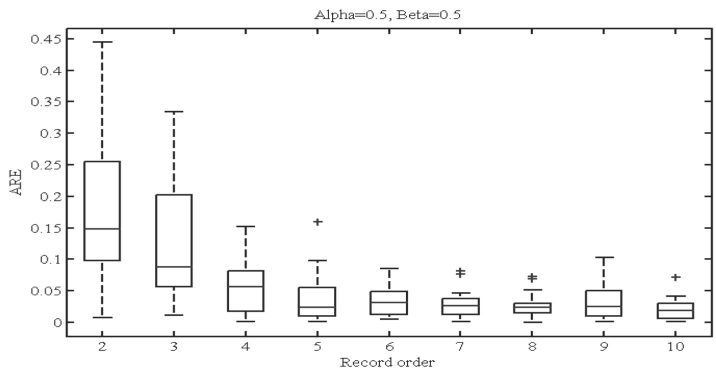

Figure 1 – Skewness coefficient of records.

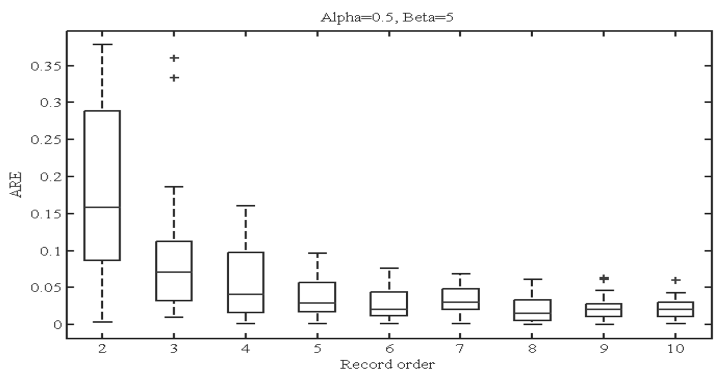

Figure 2 – Skewness coefficient of records.

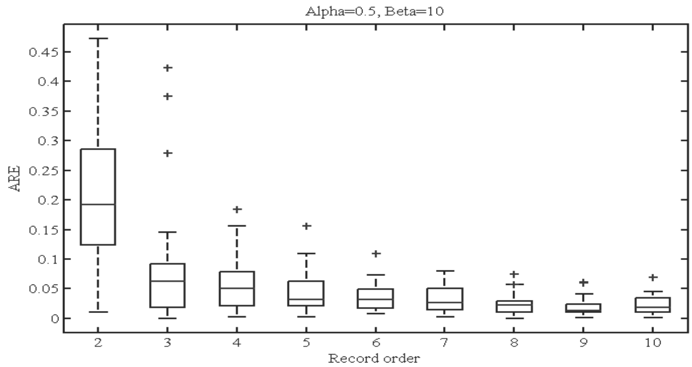

Figure 3 – Skewness coefficient of records.

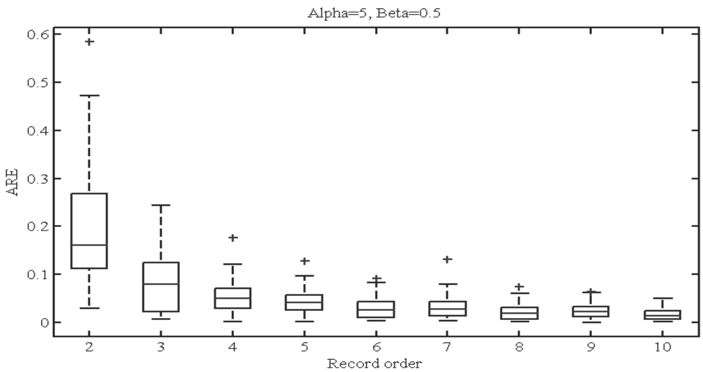

Figure 4 – Skewness coefficient of records.

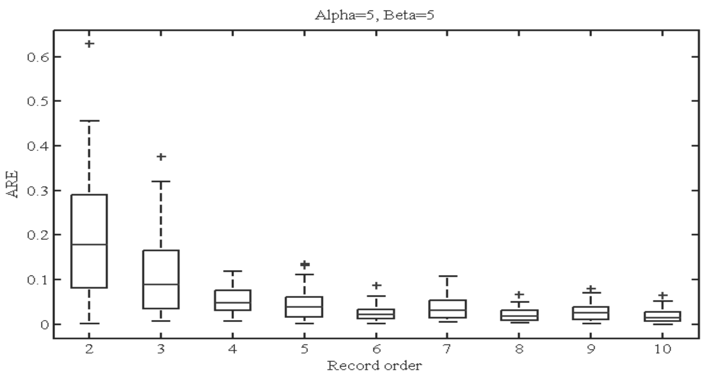

Figure 5 – Skewness coefficient of records.

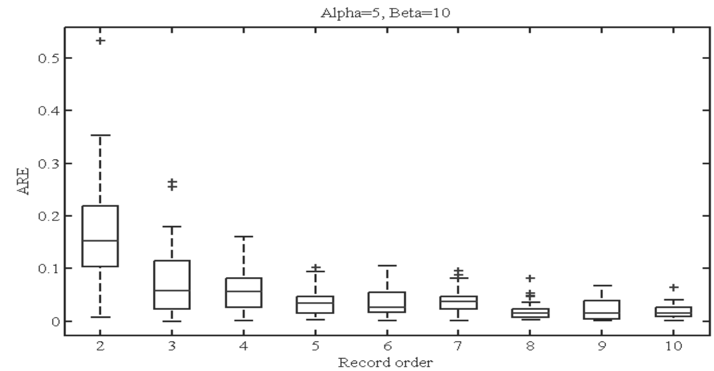

Figure 6 – Boxplot for ARE of $\hat{\alpha}$.

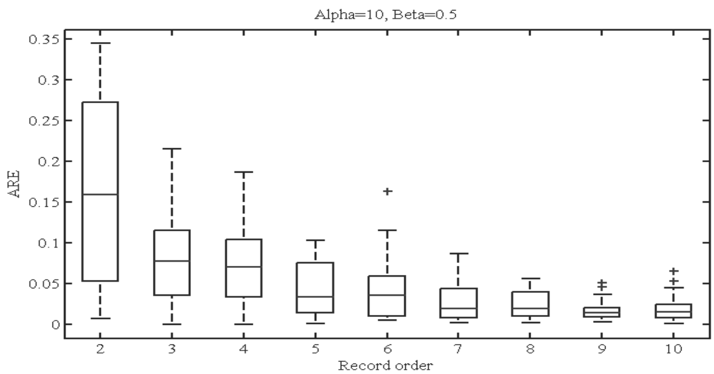

Figure 7 – Boxplot for ARE of $\hat{\alpha}$.

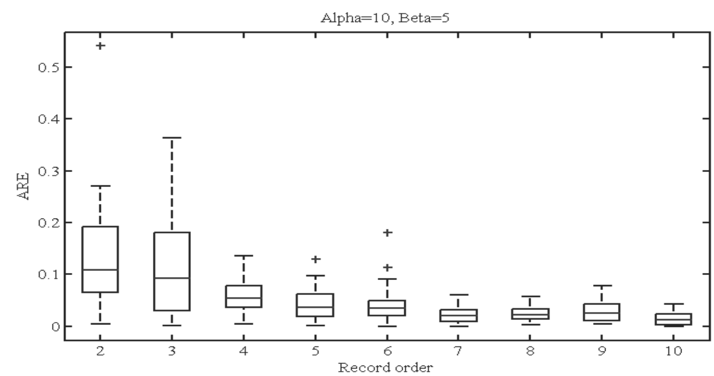

Figure 8 – Skewness coefficient of records.

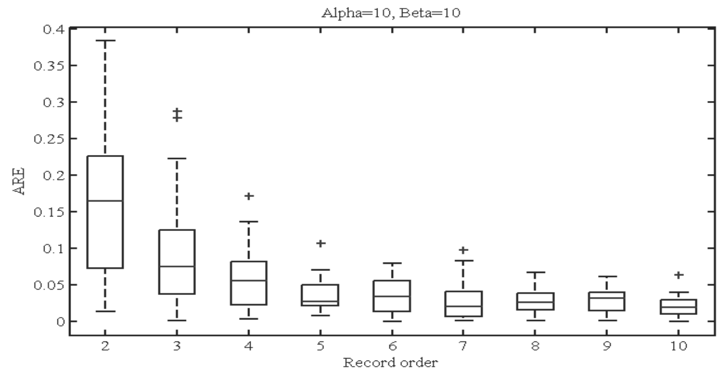

Figure 9 – Skewness coefficient of records.

The following observations can be made from these Figures:

1. generally the length of boxplot decreases as record order increases.

2. generally $\hat{\alpha}$ has longer right tail.

3. the levels of parameters α and β has no effect on the performance of $\hat{\alpha}$ for when record order increases.

Theorem 5.2 *Suppose that a sequence of n th lower record from Weibull family are observed. A simple and consistent estimator of shape parameter* $\hat{\alpha}$ *is given by*

$$\hat{\alpha} = \frac{\log[-\log(1-\exp(0.3271-0.9998n-0.000002n^2))]}{\log(\hat{m}_{L(n)})-\log(\beta)} \quad (31)$$

where $\hat{m}_{L(n)}$ *is the sample median of n th lower record values. The performance analysis of* $\hat{\alpha}$ *given in Theorm 5.2 involves almost the same results as obtained for* $\hat{\alpha}$ *given in Theorem 5.1.*

5.4 *Simulation study*

First, suppose that $X_{U(n)}$ and $X_{L(n)}$, respectively, are the upper and lower record statistics of Weibull $W_{\alpha,\beta}$, family with shape parameter α and scale parameter β. Using the algorithms given in section 2, we can simulate the n th upper record fromWeibull distribution as follows.

1. generate an observation from gamma distribution with shape parameter n.

2. accept the $\beta g^{1/\alpha}$ as an observation from n th upper record of Weibull distribution.

Also for simulating n th lower record from Weibull distribution we have:

1. generate an observation from gamma distribution with shape parameter n.

2. accept the $\beta(-\log(1-\exp(-g)))^{1/\alpha}$ as an observation from n th upper record of Weibull distribution.

In the following we shall describe how to extract a sequence of N second records, namely, $\{X_{U(2)}^{(1)}, X_{U(2)}^{(2)}, ..., X_{U(2)}^{(N)}\}$ from a sample of size m observations from Weibull distribution. Note that first observation is the first upper record. The next larger observation is $X_{U(2)}^{(1)}$ and the first observation after $X_{U(2)}^{(1)}$ can be regarded as the first upper record because observations are statistically independent. The next larger observation after this observation $X_{U(2)}^{(2)}$ and the next first observation is the first upper record and so on. Proceeding in this manner for a sample of size $m = 50$ from Weibull distribution with parameters $\alpha = 3$ and $\beta = 1$ given in Table 3, a sequence of $N = 17$ second upper records are obtained (to obtain the second upper records, read the observations of Table 3 row by row).

TABLE 3

Simulated data from Weibull model with $\alpha = 3$, $\beta = 1$

0.64168	1.17691	0.75749	1.56636	0.44612	1.20635
0.59067	1.18106	1.14828	0.88969	0.74817	1.23842
0.88686	0.39042	1.25807	1.12559	0.51026	1.22784
0.78072	0.56409	1.4152	0.56353	0.82424	0.42325
0.99923	1.23484	1.11253	0.40959	1.07591	0.73145
1.15481	1.27428	0.55142	0.6519	0.38757	0.83189
0.86342	0.77948	1.01793	1.1998	0.21963	1.42924
0.98801	1.2252	1.04823	0.73359	0.95576	1.01923
0.79005	1.38088				

The extracted second upper records are given in Table 4.

TABLE 4

Simulated data from Weibull model with $\alpha = 3$, $\beta = 1$

1.17691	1.56636	1.20635	1.18106	1.23842	1.25807
1.22784	1.4152	0.82424	0.99923	1.27428	0.6519
0.83189	1.01793	1.42924	1.2252	1.38088	

The estimated α via maximum likelihood method based on complete data and based on the second upper records via (29) for $n = 2$, respectively, are 2.792 and 2.532. Figure 10 shows the complete data histogram and fitted Weibull models with estimated shape parameters.

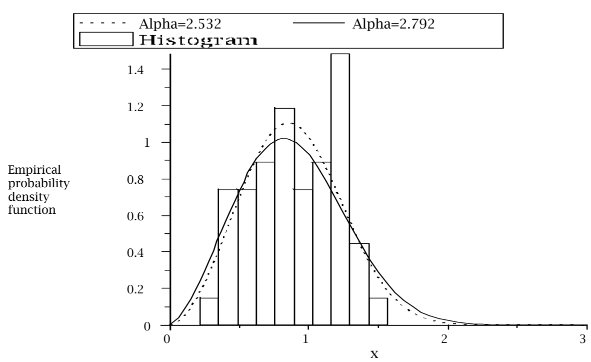

Figure 10 – Histogtram of simulated data for $\alpha = 3$, $\beta = 1$.

5.5 *Estimation of upper/lower records for the Weibull distribution*

The tabulated confidence intervals provided in subsection 5.1 can be used to introduce a point estimation for n th upper/lower record on the basis of a sample of size m observed from n th upper/lower record. Now, the degree of skewness of n th record plays a key role for this mean. For Weibull family the moments of n th upper/lower record are finite, see Arnold et al. (1992) and Sultan and Balakrishnan (1999). This means that if quantile-based skewness coefficient is negligible in magnitude, then midpoint of confidence interval can be regarded as a point estimation for n th upper/lower record. Our simulation-based studies show that sample mean of simulated n th suitably close to the midpoint of confidence interval when record skewness coefficient is negligible.

Now suppose that a sample of size m of n th upper/lower record of Weibull family is available. Refer to (21) and (22), when $p = 0.25$, the skewness coefficient of Weibull upper/lower record is obtained. This measure is displayed for $n = 1, 2, ..., 30$ in Figures 11-17.

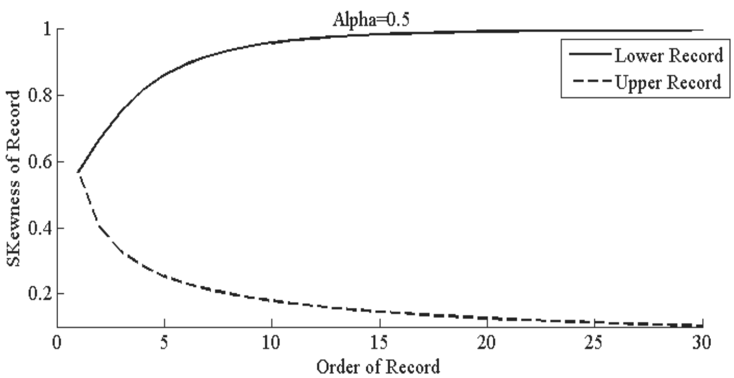

Figure 11 – Skewness coefficient of records.

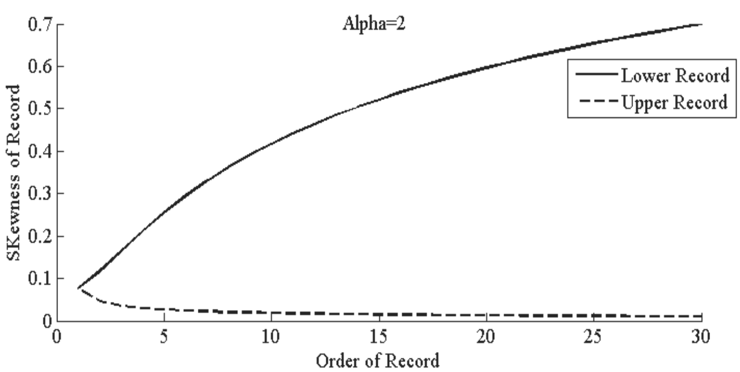

Figure 12 – Skewness coefficient of records.

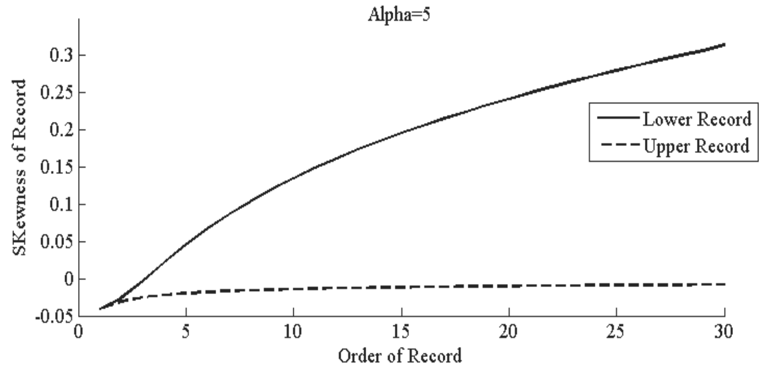

Figure 13 – Skewness coefficient of records.

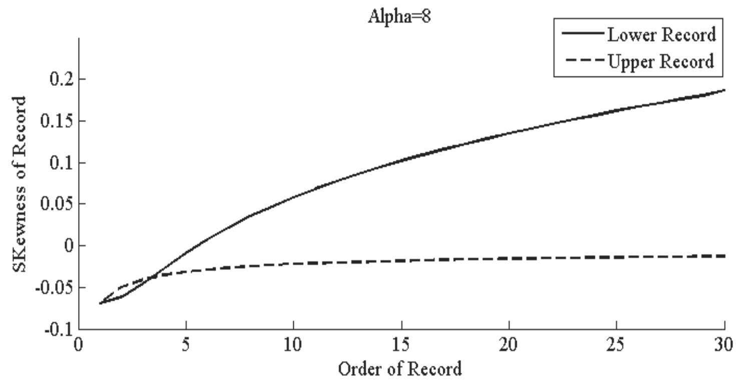

Figure 14 – Skewness coefficient of records.

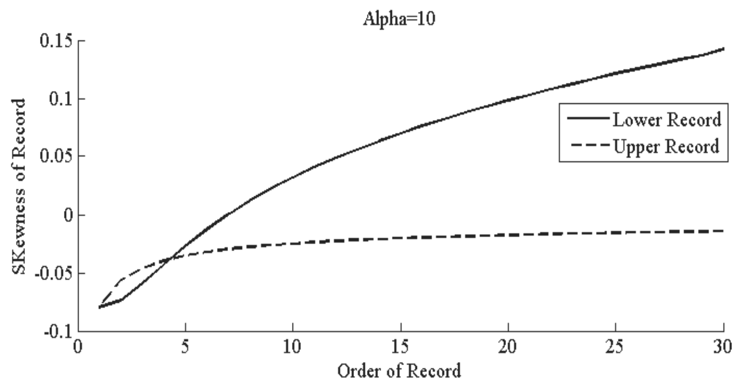

Figure 15 – Skewness coefficient of records.

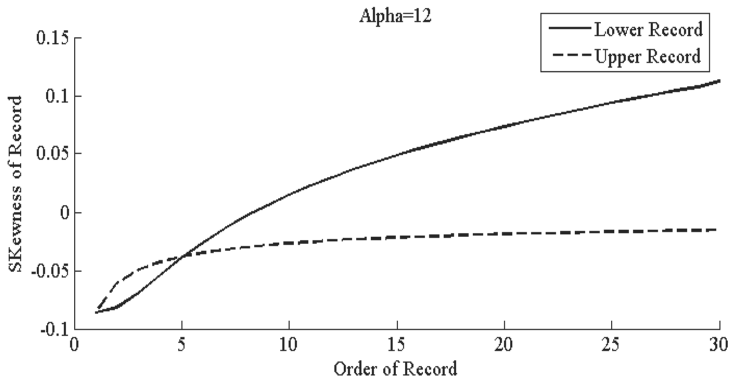

Figure 16 – Skewness coefficient of records.

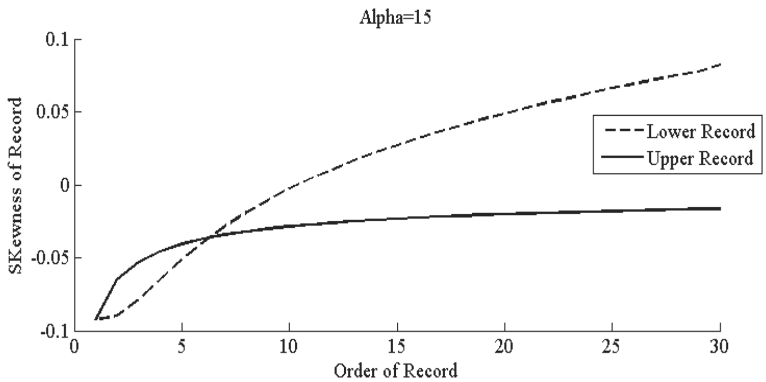

Figure 17 – Skewness coefficient of records.

The following results are concluded from these Figures.

1. lower record is symmetric for small orders and approaches to totally right skewed when record order increases. This means that sample mean of n th lower record is a good approximation for n th lower record.

2. upper record is skewed to the right when $\alpha \downarrow 0$. Otherwise, it behaves symmetrically. This means that sample mean of n th upper record is a good estimation for n th lower record when α is not small.

3. the skewness value is independent of scale parameter. This is means that β has no effect on the length of confidence interval (this is evident from (23) and (24)).

6. CONCLUSION

We propose a confidence interval for n th upper/lower record of two-parameter Weibull model. These intervals are tabulated in terms of levels of shape parameter and record order. A point estimator of shape parameter of this family is also given using n th upper/lower record statistic. Comprehensive simulation-based analysis as well as an illustrative example reveal that it is a good competitor for maximum likelihood estimator with respect to relative error criterion. If we accept that small values of quantile-based coefficient of skewness in magnitude is a sign of symmetry, then a point approximation of n th upper/ lower record statistic of Weibull family is obtained.

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SUMMARY

On the Weibull record statistics and associated inferences

On the basis of some characteristics such as quantile function and skewness coefficient of n th upper/lower record of a given absolutely continuous distribution, as well as a confidence interval for n th upper/lower record statistic of a two-parameter Weibull model, a point estimator for shape parameter of this family also is given. Furthermore, behavior of the skewness coefficient of n th upper/lower record statistic of this family is discussed. Tabulation of the confidence interval of the n th upper/lower record statistics of this family for some levels of confidence and parameters is useful for predicting the future nth upper/lower record. The performance of the given shape estimator is measured through a simulated data set comparison with the maximum likelihood estimation. Also, skewness analysis of the n th upper/lower record statistic is useful for estimating the n th upper/lower record statistic of this family as it will be discussed.