STATISTICA, anno LXXII, n. 1, 2012

ON BOUNDS OF SOME DYNAMIC INFORMATION DIVERGENCE MEASURES

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1. INTRODUCTION

Discrimination and inaccuracy measures play a key role in information theory, reliability and other related fields. There are many discrimination measures (relative entropy) available in literature and are used as a measure of the distance between two distributions or functions. Renyi's information divergence of order α is one such popular discrimination measures used by many researchers (see Asadi *et al.*, 2005a, 2005b and references therein). Let X and Y be two absolutely continuous non negative random variables (rv's) that describe the lifetimes of two items. Denote by f, F and $\bar{F}$, the probability density function (pdf), cumulative distribution function (cdf) and survival function (sf) of X respectively and g, G and $\bar{G}$, the corresponding functions of Y. Also, let $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$ be the hazard (failure) rates and $\lambda_X = f/F$ and $\lambda_Y = g/G$ be the reversed hazard rates of X and Y respectively. Then Renyi's information divergence of order α between two distributions f and g is defined by

$$I_{X,Y} = \frac{1}{\alpha-1}\ln\int_0^{\infty} f^{\alpha}(x)g^{(1-\alpha)}(x)dx = \frac{1}{\alpha-1}\ln E_f\left[\frac{f(X)}{g(X)}\right]^{\alpha-1}$$

for α such that $0<\alpha\neq 1$.

However, in many applied problems viz., reliability, survival analysis, economics, business, actuary etc. one has information only about the current age of the systems, and thus are dynamic. Then the discrimination information function between two residual lifetime distributions based on Renyi's information divergence of order α is given by

$$I_{X,Y}(t) = \frac{1}{\alpha-1}\ln\int_t^{\infty} \frac{f^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\frac{g^{(1-\alpha)}(x)}{\bar{G}^{(1-\alpha)}(t)}dx \tag{1}$$

for α such that $0<\alpha\neq 1$. Note that $I_{X,Y}(t)=I_{X_t,Y_t}$, where $X_t=(X-t\,|\,X>t)$ and $Y_t=(Y-t\,|\,Y>t)$ are residual lifetimes associated to X and Y. Another set of interest that leads to the dynamic information measures is the past lifetime of the individual. In the context of past lifetimes, (Asadi *et al.*, 2005b) defined Renyi's discrimination implied by F and G between the past lives $(t-X\,|\,X\leq t)$ and $(t-Y\,|\,Y\leq t)$ as

$$\bar{I}_{X,Y}(t)=\frac{1}{\alpha-1}\ln\int_0^t \frac{f^{\alpha}(x)}{F^{\alpha}(t)}\frac{g^{(1-\alpha)}(x)}{G^{(1-\alpha)}(t)}dx \tag{2}$$

for α such that $0<\alpha\neq 1$. Given that at time t, two items have been found to be failing, equation (2) measures the disparity between their past lives.

Recently, the inaccuracy measure due to (Kerridge, 1961) is also widely used as a useful tool to measure the inaccuracy between two distributions f and g. It is given by

$$K_{X,Y}=-\int_0^{\infty} f(x)\ln g(x)dx\,.$$

It can be expressed as

$$K_{X,Y}=D(X,Y)+H(X)$$

where $D(X,Y)=\int_0^{\infty} f(x)\ln(f(x)/g(x))dx$ is the Kullback-Leibler (KL) divergence between X and Y and $H(X)=-\int_0^{\infty} f(x)\ln f(x)dx$ is Shannon measure of information of X. (Taneja *et al.*, 2009) introduced a dynamic version of Kerridge measure, given by

$$K_{X,Y}(t)=-\int_t^{\infty}\frac{f(x)}{\bar{F}(t)}\ln\frac{g(x)}{\bar{G}(t)}dx \tag{3}$$

Note that $K_{X,Y}(t)=K_{X_t,Y_t}$. Clearly, when $X=Y$, equation (3) becomes the popular dynamic measure of uncertainty (residual entropy) due to (Ebrahimi, 1996). A similar expression for the inactivity times is available in (Vikas Kumar *et al.*, 2011) and given by

$$\bar{K}_{X,Y}(t)=-\int_0^{t}\frac{f(x)}{F(t)}\ln\frac{g(x)}{G(t)}dx \tag{4}$$

For a wide variety of research for the study of these dynamic information measures, we refer to (Ebrahimi and Kirmani, 1996a, 1996b; Di Crescenzo and Longobardi, 2002, 2004; Asadi *et al.*, 2005a, 2005b; Taneja *et al.*, 2009; Vikas Kumar *et al.*, 2011) and references therein.

The concept of weighted distributions was introduced by (Rao, 1965) in connection with modeling statistical data and in situations where the usual practice of employing standard distributions for the purpose was not found appropriate. If the pdf of X is f and $w(.)$ is a non-negative function satisfying $\mu_w = E(w(X)) < \infty$, then the pdf f_w, df F_w sf $\bar{F}_w$ of the corresponding weighted rv X^w are respectively $f_w(x) = \dfrac{w(x) f(x)}{\mu_w}$, $\bar{F}_w(x) = \dfrac{E(w(x) \mid X > t)}{\mu_w} \bar{F}(x)$ and $F_w(x) = \dfrac{E(w(x) \mid X \le t)}{\mu_w} F(x)$. An important distribution which arises as a special case of weighted distributions is the equilibrium models, obtained when $w(.) = \bar{F} / f$. It is also arises naturally in renewal theory as the distribution of the backward or forward recurrence time in the limiting case. Associated with X, a equilibrium rv X_E can be defined as with pdf $f_E = \bar{F} / \mu$, with sf and failure rate function are given respectively by $\bar{F}_E(x) = r(x)\bar{F}(x)/\mu$ and $h_E(x) = 1/r(x)$, where $\mu = E(X)$ and $r(x) = E(X - x \mid X > x)$. For various applications and recent works on weighted and equilibrium distributions, we refer to (Gupta and Kirmani, 1990; Navarro *et al.*, 2001; Di Crescenzo and Longobardi, 2006; Gupta, 2007; Maya and Sunoj, 2008; Navarro *et al.*, 2011; Sunoj and Sreejith, press).

Although a wide variety of research has been carried out for studying these dynamic information measures (1) to (4) in the context of modeling and analysis, however, very little has been studied to obtain its bounds with regard to some stochastic ordering. Accordingly in the present paper, we obtain certain bounds/inequalities on these dynamic discrimination measures (1) to (4) for rv's X and Y and subsequently between X and X_w, using likelihood ordering. More importantly, some close relationships between these dynamic discrimination measures, reliability measures and residual information measures are obtained in terms of bounds.

2. RENYI'S DISCRIMINATION MEASURE OF ORDER α

Renyi's discrimination measure for the residual lives of the original and weighted rv's is given by

$$I_{X,X_w}(t) = \frac{1}{\alpha - 1} \ln \int_t^\infty \frac{f^\alpha(x)}{\bar{F}^\alpha(t)} \frac{f_w^{(1-\alpha)}(x)}{\bar{F}_w^{(1-\alpha)}(t)} dx \,, \tag{5}$$

for α such that $0 < \alpha \ne 1$, and that for past lives is given by

$$\bar{I}_{X,X_w}(t) = \frac{1}{\alpha - 1} \ln \int_0^t \frac{f^\alpha(x)}{F^\alpha(t)} \frac{f_w^{(1-\alpha)}(x)}{F_w^{(1-\alpha)}(t)} dx \,. \tag{6}$$

for α such that $0<\alpha\neq 1$. Equations (5) and (6) measures the discrepancy between the residual (past) lives of original rv X and weighted rv X_w. More importantly, $I_{X,X_w}(t)$ may be a useful tool for measuring how far the true density is distant from a weighted density. On the other hand, when the original and weighted density functions are equal then, $I_{X,X_w}(t)=0$ *a.e.*

Remark 1. Equations (5) and (6) may be useful in the determination of a weight function and therefore for the selections of a suitable weight function in an observed mechanism, we can choose a weight function for which (5) or (6) are small. Moreover, (5) and (6) are asymmetric in f and f_w, therefore, for reversing the roles of f and f_w in (5), say $I_{X_w,X}(t)$ and equate with (5) for a symmetric measure implies the weight function is unity, *i.e.*, when $f_w = f$ (see Maya and Sunoj, 2008).

In many instances in applications, stochastic orders and inequalities are very useful for the comparison of two distributions. In the univariate case, several notions of stochastic orders are popular in literature. It is well known that likelihood ratio order is more important than the other orders such as usual stochastic order or the hazard rate order (see Shaked and Shanthikumar, 2007), as it implies the other two. Accordingly, in the following theorems, we use the likelihood ratio ordering to obtain some bounds and inequalities on Renyi's discrimination measure of order α between X and Y and subsequently between X and X_w. We say X is said to be smaller than Y in likelihood ratio ($X\leq_{LR} Y$) if $f(x)/g(x)$ is decreasing in x over the union of the supports of X and Y. The results are quite similar to KL information divergence given in (Di Crescenzo and Longobardi, 2004). In a similar way for the Renyi's information divergence of order α, likelihood ratio ordering provides some simple upper or lower bounds which are functions of important reliability measures and/or Shannon information measure. The following theorem provides a simple upper bound for Renyi information of order α with bounds are functions of hazard rates of X and Y.

Theorem 1. If $X\leq_{LR} Y$ *, then* $I_{X,Y}(t)\leq(\geq)\dfrac{\alpha}{\alpha-1}\ln\left[\dfrac{h_X(t)}{h_Y(t)}\right]$ *if* $\alpha>1$ $(0<\alpha<1)$.

Proof. Since $X\leq_{LR} Y$, $\dfrac{f(x)}{g(x)}$ is decreasing in x, *i.e.*, $\dfrac{f(x)}{g(x)}\leq\dfrac{f(t)}{g(t)}$ for all $x > t$.

$$I_{X,Y}(t)=\frac{1}{\alpha-1}\ln\int_t^{\infty}\frac{f^{\alpha}(x)}{\overline{F}^{\alpha}(t)}\frac{g^{(1-\alpha)}(x)}{\overline{G}^{(1-\alpha)}(t)}dx=\frac{1}{\alpha-1}\ln\int_t^{\infty}\frac{f^{\alpha}(x)}{g^{\alpha}(x)}\frac{g(x)}{\overline{F}^{\alpha}(t)\overline{G}^{(1-\alpha)}(t)}dx\,,$$

then

$$I_{X,Y}(t) \le (\ge) \frac{1}{\alpha-1} \ln \int_t^{\infty} \frac{f^{\alpha}(t)}{g^{\alpha}(t)} \frac{g(x)}{\bar{F}^{\alpha}(t)\bar{G}^{(1-\alpha)}(t)} dx \text{, for } \alpha > 1 \ (0<\alpha<1).$$

$$= \frac{1}{\alpha-1} \ln \left[\frac{h_X^{\alpha}(t)}{h_Y^{\alpha}(t)} \right] = \frac{\alpha}{\alpha-1} \ln \left[\frac{h_X(t)}{h_Y(t)} \right].$$

Corollary 1. If $X \le_{LR} X_w$*, hen* $I_{X,X_w}(t) \le (\ge) \frac{\alpha}{\alpha-1} \ln \left[\frac{E(w(X) \mid X > t)}{w(t)} \right]$ *if* $\alpha > 1$ $(0<\alpha<1)$.

Proof: Using the relationship for hazard rate between X and X_w, we have $\frac{h_X(t)}{h_{X_w}(t)} = \frac{E(w(X) \mid X > t)}{w(t)}$, from which the corollary follows.

Example 1. The Pareto distribution has played a very important role in the investigation of city population, occurrence of natural resources, insurance risk and business failures and has been a useful model in many socio economic studies (see Abdul Sathar *et al.*, 2005). Accordingly, we consider Pareto I distribution with pdf $f(x) = ck^c x^{-c-1}$, $x > k, k > 0, c > 1$ to illustrate the above theorem. Using the weight function $w(x) = x$, we have $X \le_{LR} X_w$ and X_w also follows a Pareto distribution.

$$I_{X,X_w}(t) = \frac{1}{\alpha-1} \ln \left[\frac{c^{\alpha}(c-1)}{(c-1)^{\alpha}(c+\alpha-1)} \right] = \frac{\alpha}{\alpha-1} \ln \left(\frac{c}{c-1} \right) + \frac{1}{\alpha-1} \ln \left(\frac{c-1}{c+\alpha-1} \right).$$

So $I_{X,X_w}(t) \le (\ge) \ \frac{\alpha}{\alpha-1} \ln \left(\frac{c}{c-1} \right) = \frac{\alpha}{\alpha-1} \ln \left[\frac{E(w(X) \mid X > t)}{w(t)} \right]$ according as $\alpha > 1$ $(0<\alpha<1)$.

Corollary 2. If $X \le_{LR} X_E$, then $I_{X,X_E}(t) \le (\ge) \frac{\alpha}{\alpha-1} \ln[1 + r'(t)]$ if $\alpha > 1$ $(0<\alpha<1)$.

Proof: Since $\frac{h_X(t)}{h_{X_E}(t)} = 1 + r'(t)$ for the equilibrium rv, the corollary follows.

Even if the past lifetime information measures appears to be a dual of its residual version, however, (Di Crescenzo and Longobardi, 2004) has shown the importance of past lifetime discrimination measures in comparison with residual lifetime and thus a separate study of these discrimination measures for past lifetime is quite worthwhile. Accordingly, in the rest of paper, we include the bounds for these discrimination measures for the past lifetimes as well. The following theorem provides a lower (upper) bound for $\bar{I}_{X,Y}(t)$ using likelihood ordering.

Theorem 2. If $X \le_{LR} Y$, then $\bar{I}_{X,Y}(t) \ge (\le) \frac{\alpha}{\alpha-1} \ln\left[\frac{\lambda_X(t)}{\lambda_Y(t)}\right]$ if $\alpha > 1$ $(0 < \alpha < 1)$.

Now we extend the above theorem to weighted models.

Corollary 3. If $X \le_{LR} X_w$, then $\bar{I}_{X,X_w}(t) \ge (\le) \frac{\alpha}{\alpha-1} \ln\left[\frac{E(w(X) \mid X \le t)}{w(t)}\right]$ if $\alpha > 1$ $(0 < \alpha < 1)$.

Example 2. Suppose X is a finite range rv with pdf $f(x) = cx^{c-1}, 0 < x < 1, c > 0$, and taking $w(x) = x^{\beta} (\beta > 0)$ we have $\frac{f(x)}{f_w(x)} = \frac{c}{\beta + c} x^{-\beta}$ is decreasing in x (*i.e.*, $X \le_{LR} X_w$). It is easy to show that

$$\bar{I}_{X,X_w}(t) = \frac{\alpha}{\alpha-1} \ln\left(\frac{c}{\beta+c}\right) + \frac{1}{\alpha-1} \ln\left(\frac{\beta+c}{\beta+c-\beta\alpha}\right) \ge (\le) \frac{\alpha}{\alpha-1} \ln\left(\frac{c}{\beta+c}\right)$$

$$= \frac{\alpha}{\alpha-1} \ln\left[\frac{E(w(X) \mid X \le t)}{w(t)}\right]$$

according as $\alpha > 1$ $(0 < \alpha < 1)$, provided $\beta + c > \beta\alpha$.

In the study of relative entropies, it is quite useful if we find some close relationships between its different measures and other important reliability/information measures. Therefore, in the following theorem we derive a lower bounds for $I_{X,Y}(t)$, which are functions of both hazard rate and Shannon information measure.

Theorem 3. If $g(x)$ is decreasing in x then $I_{X,Y}(t) \ge -\ln h_Y(t) - I_X(t)$, $\alpha \ne 1$ where $I_X(t) = \frac{1}{1-\alpha} \ln \int_t^{\infty} \frac{f^{\alpha}(x)}{\bar{F}^{\alpha}(t)} dx$, the residual Renyi's entropy function.

Corollary 4. If $f_w(x)$ is decreasing in x, then $I_{X,X_w}(t) \ge -\ln\left(\frac{w(t) h_X(t)}{E(w(X) \mid X > t)}\right) - I_X(t)$, $\alpha \ne 1$.

Example 3. Applying the same pdf and weight function used in example 3, we can easily illustrate corollary 4.

The analogous results are straightforward for the past life times, the statements are as follows:

Theorem 4. If $g(x)$ is increasing in x then $\overline{I}_{X,Y}(t) \geq -\ln \lambda_Y(t) - \overline{I}_X(t)$, $\alpha \neq 1, \alpha > 0$ where $\overline{I}_X(t) = \frac{1}{1-\alpha} \ln \int_0^t \frac{f^\alpha(x)}{F^\alpha(t)} dx$, the past Renyi's entropy function.

Corollary 5. If $f_w(x)$ is increasing in x, then $\overline{I}_{X,X_w}(t) \geq -\ln\left(\frac{w(t)\lambda_X(t)}{E(w(X) \mid X \leq t)}\right) - \overline{I}_X(t)$, $\alpha \neq 1, \alpha > 0$.

Example 4. It is easy to show that for the power function rv with pdf $f(x) = cx^{c-1}, 0 < x < 1, c > 1$ and taking $w(x) = x$, we have $f_w(x) = (c+1)x^c$ increasing in x and hence corollary 5 follows.

In the following theorems, we establish an upper (lower) bound for $I_{X,Y}(t)$ for more than two rv's.

Theorem 5. Let X_1, X_2 and Y be 3 non negative absolutely continuous rv's with densities f_1, f_2 and g, sf's $\overline{F}_1$, $\overline{F}_2$ and $\overline{G}$ and hazard rates h_{X_1}, h_{X_2} and h_Y respectively. If $X_1 \leq_{LR} X_2$, then $I_{X_1,Y}(t) \leq (\geq) \frac{\alpha}{\alpha-1} \ln\left[\frac{h_{X_1}(t)}{h_{X_2}(t)}\right] + I_{X_2,Y}(t)$ if $\alpha > 1$ $(0 < \alpha < 1)$.

Example 5. Let X_1 and X_2 be two independent exponential rv's with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively such that $\lambda_1 > \lambda_2$, then $\frac{f_1(x)}{f_2(x)} = \frac{\lambda_1}{\lambda_2} \exp[-(\lambda_1 - \lambda_2)x]$ is decreasing in x. Let $Y = \min(X_1, X_2)$, then

$$I_{X_1,Y}(t) = \frac{1}{\alpha-1} \ln\left[\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^\alpha \left(\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-\lambda_2\alpha}\right)\right]$$

$$= \frac{\alpha}{\alpha-1} \ln\left(\frac{\lambda_1}{\lambda_2}\right) + \frac{1}{\alpha-1} \ln\left[\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^\alpha \left(\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-\lambda_1\alpha}\right)\right] + \frac{1}{\alpha-1} \ln\left(\frac{\lambda_1+\lambda_2-\lambda_1\alpha}{\lambda_1+\lambda_2-\lambda_2\alpha}\right)$$

$$\leq (\geq) \frac{\alpha}{\alpha-1} \ln\left(\frac{\lambda_1}{\lambda_2}\right) + \frac{1}{\alpha-1} \ln\left[\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{\alpha}\left(\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-\lambda_1\alpha}\right)\right]$$

$$= \frac{\alpha}{\alpha-1} \ln\left[\frac{h_{X_1}(t)}{h_{X_2}(t)}\right] + I_{X_2,Y}(t)$$

according as $\alpha > 1$ $(0 < \alpha < 1)$, provided $\lambda_i + \lambda_j - \lambda_i \alpha > 0, i \neq j, i, j = 1, 2$.

Theorem 6. Let X_1, X_2 and Y be 3 non negative absolutely continuous rv's with densities f_1, f_2 and g, distribution functions F_1, F_2 and G and reversed hazard rates λ_{X_1}, λ_{X_2} and λ_Y respectively. If $X_1 \leq_{LR} X_2$, then $\bar{I}_{X_1,Y}(t) \geq (\leq) \frac{\alpha}{\alpha-1} \ln\left[\frac{\lambda_{X_1}(t)}{\lambda_{X_2}(t)}\right] + \bar{I}_{X_2,Y}(t)$ if $\alpha > 1$ $(0 < \alpha < 1)$.

Example 6. Let X_1 and X_2 be two independent Power function rv's with densities given by $f_1(x) = c_1 x^{c_1 - 1}; 0 < x < 1, c_1 > 0$ and $f_2(x) = c_2 x^{c_2 - 1}; 0 < x < 1, c_2 > 0$ respectively such that $c_1 < c_2$, so $\frac{f_1(x)}{f_2(x)} = \frac{c_1}{c_2} x^{c_1 - c_2}$ is decreasing in *x*. Letting $Y = \max(X_1, X_2)$, then it is easy to show that theorem 6 follows.

Theorem 7. Let X, Y_1 and Y_2 be 3 non negative absolutely continuous rv's with pdf's *f*, g_1 and g_2, sf's $\bar{F}$, $\bar{G}_1$ and $\bar{G}_2$ and hazard rates h_X, h_{Y_1} and h_{Y_2} respectively. If $Y_1 \leq_{LR} Y_2$, then $I_{X,Y_1}(t) \geq \ln\left[\frac{h_{Y_1}(t)}{h_{Y_2}(t)}\right] + I_{X,Y_2}(t)$ for $\alpha \neq 1, \alpha > 0$.

Example 7. Let Y_1 and Y_2 be two independent Pareto I rv's with densities given by $g_1(x) = c_1 k_1^{c_1} x^{-c_1 - 1}; x > k_1 > 0, c_1 > 0$ and $g_2(x) = c_2 k_2^{c_2} x^{-c_2 - 1}; x > k_2 > 0, c_2 > 0$ respectively such that $c_1 > c_2$, then $\frac{g_1(x)}{g_2(x)} = \frac{c_1 k_1^{c_1}}{c_2 k_2^{c_2}} x^{-(c_1 - c_2)}$ is decreasing in *x*. Consider $X = \min(Y_1, Y_2)$, then the theorem follows.

Theorem 8. Let X, Y_1 and Y_2 be 3 non negative absolutely continuous rv's with densities *f*, g_1 and g_2, distribution functions F, G_1 and G_2 and reversed hazard rates λ_X, λ_{Y_1} and λ_{Y_2} respectively. If $Y_1 \leq_{LR} Y_2$, then $\bar{I}_{X,Y_1}(t) \leq \ln\left[\frac{\lambda_{Y_2}(t)}{\lambda_{Y_1}(t)}\right] + \bar{I}_{X,Y_2}(t)$ for $\alpha \neq 1, \alpha > 0$.

Example 8. Let Y_1 and Y_2 be two independent power function rv's with densities given by $g_1(x) = c_1 x^{c_1 - 1}; 0 < x < 1, c_1 > 0$ and $g_2(x) = c_2 x^{c_2 - 1}; 0 < x < 1, c_2 > 0$ respectively such that $c_1 < c_2$, so $\frac{g_1(x)}{g_2(x)} = \frac{c_1}{c_2} x^{c_1 - c_2}$ is decreasing in *x*. Using $X = \max(Y_1, Y_2)$, then we can illustrate the theorem.

3. DYNAMIC INACCURACY MEASURE

In this section we obtain bounds similar to that given in section 2 for the Kerridge inaccuracy measures. Let *X* and *Y* be the rv's defined in section 1. Then the dynamic inaccuracy measure for residual and past lives of the original and weighted distributions are given by

$$K_{X,X_w}(t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \ln\left[\frac{w(x) f(x)}{E(w(X) \mid X > t)\bar{F}(t)}\right] dx, \tag{7}$$

and

$$K_{X,X_w}(t) = -\int_0^{t} \frac{f(x)}{F(t)} \ln\left[\frac{w(x) f(x)}{E(w(X) \mid X \leq t) F(t)}\right] dx. \tag{8}$$

Remark 2. From the above definition, it is easy to obtain, $K_{X,X_E}(t) = 1 + \ln r(t)$.

The following theorem gives a simple lower bound for Kerridge inaccuracy measures using likelihood ordering.

Theorem 9. If *g*(*x*) is decreasing in *x*, then $K_{X,Y}(t) \geq -\ln h_Y(t)$.

Proof. Since *g*(*x*) is decreasing in *x*, we have $g(x) \leq g(t)$ for all *x* > *t*. Then,

$$K_{X,Y}(t) = -\frac{1}{\bar{F}(t)} \int_t^{\infty} f(x) \ln \frac{g(x)}{\bar{G}(t)} dx \geq -\frac{1}{\bar{F}(t)} \int_t^{\infty} f(x) \ln \frac{g(t)}{\bar{G}(t)} dx = -\ln h_Y(t).$$

Corollary 6. If $f_w(x)$ is decreasing in x, then $K_{X,X_w}(t) \geq \ln\left(\frac{E(w(X)\,|\,X>t)}{w(t)h_X(t)}\right)$.

Analogues results are obtained for past lifetimes in the following theorems.

Theorem 10. If $g(x)$ is increasing in x, then $\bar{K}_{X,Y}(t) \geq -\ln \lambda_Y(t)$.

Corollary 7. If $f_w(x)$ is increasing in x, then $K_{X,X_w}(t) \geq \ln\left(\frac{E(w(X)\,|\,X\leq t)}{w(t)h_X(t)}\right)$.

Example 10. Suppose X is a Uniform rv with pdf $f(x)=\frac{1}{a}; 0<x<a, a>0$. Taking the weight function $w(x)=x$, $f_w(x)=\frac{2x}{a^2}$ is increasing in x. Then, $\bar{K}_{X,Y}(t)=1+\ln(t/2) \geq \ln(t/2) \;=\ln\left(\frac{E(w(X)\,|\,X\leq t)}{w(t)h_X(t)}\right)$.

In the following theorem we have a simple bound for Kerridge inaccuracy measure between X and X_w which are functions of hazard rates of the same rv's and residual entropy of X.

Theorem 11. If the weight function $w(x)$ is increasing in x, then $K_{X,X_w}(t) \leq \ln\left(\frac{E(w(X)\,|\,X>t)}{w(t)}\right)+H_X(t)$, where $H_X(t)=-\frac{1}{\bar{F}(t)}\int_t^{\infty} f(x)\ln\frac{f(x)}{\bar{F}(t)}dx$ is the residual entropy function.

Proof. Since $w(x)$ is increasing in x, we have $g(x) \geq g(t)$ for all $x > t$. Now using equation (7) we have

$$K_{X,X_w}(t) \leq -\frac{1}{\bar{F}(t)}\int_t^{\infty} f(x)\ln\left[\frac{w(t)f(x)}{E(w(X)\,|\,X>t)\bar{F}(t)}\right]dx = \ln\left(\frac{E(w(X)\,|\,X>t)}{w(t)}\right)+H_X(t).$$

Example 11. Let X be a Pareto I rv with pdf $f(x)=ck^c x^{-c-1}; c>1, x>k>0$. Take the weight function as $w(x)=x$, which is an increasing function in x. Then

$$K_{X,X_w}(t) = \ln\left(\frac{t}{c-1}\right)+1$$

$$= \ln\left(\frac{c}{c-1}\right)+\ln\left(\frac{t}{c}\right)+\left(\frac{c+1}{c}\right)-\frac{1}{c}\le \ln\left(\frac{c}{c-1}\right)+\ln\left(\frac{t}{c}\right)+\left(\frac{c+1}{c}\right)$$

$$= \ln\left(\frac{E(w(X)\,|\,X>t)}{w(t)}\right)+H_X(t).$$

The following theorem is an analogous result of theorem 11 for past lifetime.

Theorem 12. If the weight function $w(x)$ is decreasing in x, then $\bar{K}_{X,X_w}(t)\le \ln\left(\frac{E(w(X)\,|\,X\le t)}{w(t)}\right)+\bar{H}_X(t)$, where $\bar{H}_X(t) = -\frac{1}{F(t)}\int_0^t f(x)\ln\frac{f(x)}{F(t)}dx$ is the past entropy function.

Example 12. Consider a finite range rv X with density function given by $f(x)=cx^{c-1};0<x<1,c>1$. Let $w(x)=\frac{1}{x}$, a decreasing function in x, then using equation (8) we have

$$\bar{K}_{X,X_w}(t) = \ln\left(\frac{t}{c-1}\right)+\left(\frac{c-2}{c}\right) = \ln\left(\frac{c}{c-1}\right)+\ln\left(\frac{t}{c}\right)+\left(\frac{c-1}{c}\right)-\frac{1}{c}$$

$$\le \ln\left(\frac{c}{c-1}\right)+\ln\left(\frac{t}{c}\right)+\left(\frac{c-1}{c}\right) = \ln\left(\frac{E(w(X)\,|\,X\le t)}{w(t)}\right)+\bar{H}_X(t)$$

Theorem 13. If $X\le_{LR} Y$, then $K_{X,Y}(t)\le H_X(t)+\ln\left(\frac{h_X(t)}{h_Y(t)}\right)$.

Proof. From the definition (3), we have

$$K_{X,Y}(t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)}\ln\left(\frac{f(x)}{\bar{F}(t)}\frac{g(x)}{f(x)}\frac{\bar{F}(t)}{\bar{G}(t)}\right)dx = H_X(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)}\ln\left(\frac{g(x)}{f(x)}\frac{\bar{F}(t)}{\bar{G}(t)}\right)dx$$

$$\le H_X(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)}\ln\left(\frac{g(t)}{f(t)}\frac{\bar{F}(t)}{\bar{G}(t)}\right)dx = H_X(t) - \ln\left(\frac{g(t)}{f(t)}\frac{\bar{F}(t)}{\bar{G}(t)}\right)$$

where the inequality is obtained by using that $g(x)/f(x)$ is increasing.

A similar statement exists for the past lifetime.

Theorem 14. If $X \le_{LR} Y$, then $\bar{K}_{X,Y}(t) \ge \bar{H}_X(t) + \ln\left(\dfrac{\lambda_X(t)}{\lambda_Y(t)}\right)$.

Similar to theorems 5 to 8, in the following theorems we obtain some bounds for Kerridge's inaccuracy for more than two rv's.

Theorem 15. Let X, Y_1 and Y_2 be 3 non negative absolutely continuous rv's with pdf's f, g_1 and g_2, sf's $\bar{F}$, $\bar{G}_1$ and $\bar{G}_2$ and hazard rates h_X, h_{Y_1} and h_{Y_2} respectively. If $Y_1 \le_{LR} Y_2$, then $K_{X,Y_1}(t) \ge K_{X,Y_2}(t) + \ln\left[\dfrac{h_{Y_2}(t)}{h_{Y_1}(t)}\right]$.

Proof. From the definition (3), we have

$$K_{X,Y_1}(t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \ln\left(\frac{g_2(x)}{\bar{G}_2(t)} \frac{g_1(x)}{g_2(x)} \frac{\bar{G}_2(t)}{\bar{G}_1(t)}\right) dx = K_{X,Y_2}(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \ln\left(\frac{g_1(x)}{g_2(x)} \frac{\bar{G}_2(t)}{\bar{G}_1(t)}\right) dx$$

$$\ge K_{X,Y_2}(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \ln\left(\frac{g_1(t)}{g_2(t)} \frac{\bar{G}_2(t)}{\bar{G}_1(t)}\right) dx = K_{X,Y_2}(t) + \ln\left[\frac{h_{Y_2}(t)}{h_{Y_1}(t)}\right].$$

Example 13. Let Y_1 and Y_2 be two independent Pareto II rv's with pdf's $g_1(x) = ac_1(1+ax)^{-c_1-1}; x > 0, a, c_1 > 0$ and $g_2(x) = ac_2(1+ax)^{-c_2-1}; x > 0, a, c_2 > 0$ such that $c_1 > c_2$, then $\dfrac{g_1(x)}{g_2(x)} = \dfrac{c_1}{c_2}(1+ax)^{-(c_1-c_2)}$ is decreasing in x. Let $X = \min(Y_1, Y_2)$, then

$$K_{X,Y_1}(t) = \ln\left(\frac{1+at}{ac_1}\right) + \left(\frac{c_1+1}{c_1+c_2}\right) = \ln\left(\frac{c_2}{c_1}\right) + \ln\left(\frac{1+at}{ac_2}\right) + \left(\frac{c_2+1}{c_1+c_2}\right) - \left(\frac{c_1-c_2}{c_1+c_2}\right)$$

$$\ge \ln\left(\frac{c_2}{c_1}\right) + \ln\left(\frac{1+at}{ac_2}\right) + \left(\frac{c_2+1}{c_1+c_2}\right) = K_{X,Y_2}(t) + \ln\left[\frac{h_{Y_2}(t)}{h_{Y_1}(t)}\right].$$

Next we obtain an analogous result for the past lifetime.

Theorem 16. Let X, Y_1 and Y_2 be 3 non negative rv's with pdf's f, g_1 and g_2, df's F, G_1 and G_2 and reversed hazard rates λ_X, λ_{Y_1} and λ_{Y_2} respectively. If $Y_1 \le_{LR} Y_2$ then $\bar{K}_{X,Y_1}(t) \le \bar{K}_{X,Y_2}(t) + \ln\left[\dfrac{\lambda_{Y_2}(t)}{\lambda_{Y_1}(t)}\right]$.

Example 14. Let Y_1 and Y_2 be 2 independent finite range rv's with pdf's given by $g_1(x) = c_1 x^{c_1 - 1}; 0 < x < 1, c_1 > 0$ and $g_2(x) = c_2 x^{c_2 - 1}; 0 < x < 1, c_2 > 0$ such that $c_1 < c_2$, then $\frac{g_1(x)}{g_2(x)} = \frac{c_1}{c_2} x^{c_1 - c_2}$ is decreasing in x. Let $X = \max(Y_1, Y_2)$, we get

$$\overline{K}_{X,Y_1}(t) = \ln\left(\frac{t}{c_1}\right) + \left(\frac{c_1 - 1}{c_1 + c_2}\right) = \ln\left(\frac{c_2}{c_1}\right) + \ln\left(\frac{t}{c_2}\right) + \left(\frac{c_2 - 1}{c_1 + c_2}\right) + \left(\frac{c_1 - c_2}{c_1 + c_2}\right)$$

$$\leq \ln\left(\frac{c_2}{c_1}\right) + \ln\left(\frac{t}{c_2}\right) + \left(\frac{c_2 - 1}{c_1 + c_2}\right) = \overline{K}_{X,Y_2}(t) + \ln\left[\frac{\lambda_{Y_2}(t)}{\lambda_{Y_1}(t)}\right].$$

CONCLUSIONS

Bounds are derived for some important discrimination measures viz. Renyi information divergence of order α and Kerridge's inaccuracy for measuring distance between a true distribution and an arbitrary distribution, using likelihood ratio ordering. Further, likelihood ordering provides some simple upper or lower bounds to these discrimination measures, where the bounds are functions of certain important reliability measures such as hazard (reversed hazard) rates and residual (past) Shannon information measure(s). These bounds are also extended to the weighted case by assuming arbitrary distribution as a weighted one, useful for comparison of observed and original distributions. Examples are also given for these bounds for theoretical validation.

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ACKNOWLEDGEMENTS

Authors are thankful to the referees for their comments and suggestions that helped a substantial improvement of the earlier version of the paper.

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SUMMARY

On bounds of some dynamic information divergence measures

In this paper, we obtain certain bounds for some dynamic information divergences measures viz. Renyi's divergence of order α and Kerridge's inaccuracy, using likelihood ratio ordering. The results are also extended to weighted models and theoretical examples are given to supplement the results.