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REALISTIC UTILITY VERSUS GAME UTILITY: A PROPOSAL FOR DEALING WITH THE SPREAD OF UNCERTAIN PROSPECTS

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1. INTRODUCTION. CONCEPTS AND MEASURES OF UTILITY

This article is a contribution towards modelling the preference behaviour of individuals, along the lines that in recent decades have abandoned – or generalized – the classical Expected Utility (EU) approach of von Neumann-Morgenstern-Savage (N-M-S for short). The viewpoint accepted in this paper starts from the recognition – in agreement with many important scholars – that the utility of certain or sure prospects is relative to the prospects themselves, and cannot obey probabilistic constraints, and also that the attitude toward risk taking is separated from the strength of preference concerning certain outcomes (see e.g. Bernoulli, 1738/1954; Allais, 1953, 1988; Dyer and Sarin, 1982; Hansson, 1988; Wakker, 1994). As a consequence, in evaluating and predicting risk behaviours, one must consider a *realistic* utility function, and the evaluation of a risky prospect must be based – in principle – on the whole probability distribution of the prospect (Allais, 1953, pp. 504, 509).

Along these lines, a *utility function* u_g for the game (or wager, or random variable offered to the subject) can be assumed, which allows to assess the importance or eagerness of the game on the utility scale. In order to achieve a simple and tractable operational device, beside the mean of utilities, as in the N-M-S approach of Expected Utility, a measure of dispersion is also introduced, following the suggestion of Allais: "l'erreur fondamentale de toute l'école américaine, c'est de négliger indirectement et inconsciemment, la dispersion des valeurs psychologiques" (1953, p. 544). The operational approach consists, quite simply, that the evaluation function (of an uncertain prospect) raises with the mean of utilities, and – for a risk averse subject – diminishes with the *mean absolute deviation* (MA) of utilities (formula (5) of this paper); this measure of dispersion is conveniently multiplied by a parameter λ, whose sign and magnitude depend on the contingent attitude of the subject in evaluating a given prospect, starting from a given asset position; it will be seen that such a parameter must be constrained between limits 0 and 0.5 for a risk averse subject, in order to comply with the dominance requisite. This is admittedly a simplistic and first approximation approach – however much less simplistic and much more realistic than the Expected Utility approach. A more refined, but also more demanding technique, aiming all the same at disentangling the two main contributions to the explanation of risk behaviour, has been studied by Wakker (1994), in the contest of rank-dependent utility models.

This article follows a previous proposal by Frosini (1997), and is itself an abridged version of an extended paper (Frosini, 2010), available in the pre-print series of the Department of Statistics of the Catholic University of Milan. The availability of both papers makes it easy to concentrate the formal developments to new issues, while referring to one or other of these papers for specific topics here only mentioned.

Let $h_1, ..., h_n$ be certain (or sure) prospects, and $p = (p_1, ..., p_n)$ a probability distribution over them ($p_i \geq 0 \ \ \forall \ i; \sum p_i = 1$); a general finite prospect can be written $d = [h_1,p_1; ...; h_n,p_n]$. The above certain prospects can be related to final personal wealth, or else to changes of the present status quo; different treatments of the two cases will be resumed later on. Each sure prospect h_i is endowed with a *utility* $u_i = u(h_i)$, formally defined except for an affine transformation; let $U = [u(h_1), p_1; ...; u(h_n),p_n] = [u_1,p_1; ...; u_n,p_n]$ be the rv (= random variable) which matches the utilities with their respective probabilities. Continuous distributions could be employed as well, with no essential modifications in the sequel. Both classical approaches by Daniel Bernoulli and by von Neumann and Morgenstern define the utility of a risky prospect d as the arithmetic mean, or expectation,

$$u(d) = \mathrm{E}(U) = \bar{u} = \sum u(h_i)p_i; \tag{1}$$

among two or more prospects, one should choose the one which maximizes expected utility (Lindley, 1985, p. 59).

J. Quiggin (1982, 1985, 1993) suggested transformation functions (see Kahneman and Tversky, 1979) from single probabilities to cumulative probabilities, thus saving most desirable properties of EU (Expected Utility) theory; his approach is usually condensed in the acronym RDEU (= Rank-Dependent Expected Utility). The expected value of EU theory was replaced by an evaluation function (with like appearance)

$$\mathrm{V}(\boldsymbol{x},\boldsymbol{p}) = \sum u(x_i)k_i(\boldsymbol{p}) \tag{2}$$

where

$$k_i(\boldsymbol{p}) = q(\textstyle\sum_{r=1}^{i} p_r) - q(\sum_{r=1}^{i-1} p_r).$$

In other words, the original probabilities p_i are changed by means of a *probability-perception* function so as to reflect the personal appreciation and risk attitude of the decision maker through the function q applied to the cumulative probabilities.

A special and common case of prospect is concerned with sums of money; the generic prospect of this kind will be symbolized by $d = [x_1,p_1; ...; x_n,p_n]$ $(x_i \leq x_{i+1})$, being x_i a real number, expressing a sum of money (negative, null or positive) in a given scale or currency. As d is formally a discrete rv, it will usually be named X.

If $d' = [x_1', p_1'; ...; x_n', p_n']$ is another prospect, such that $x_i' \geq x_i$, and/or there is a transfer in probabilities towards larger x_i values, then $d \lesssim d'$. This property, objectively based, is usually named *monotonicity* property, or (first) *dominance* property. All cases, discrete and continuous, can be included in a definition which refers to the cdf's (cdf = cumulative distribution function) of the rv's being compared: if X_1 and X_2 are two monetary prospects, with respective cdf's F_1 and F_2, X_1 dominates X_2 ($X_2 \lesssim X_1$) (or, equivalently, F_1 dominates F_2) if $F_1(x) \leq F_2(x)$ for all x (Quiggin, 1993, pp. 16, 21-25). The cited "dominance property" is often referred to in the literature as *first stochastic dominance* (FSD); the fact that X_1 dominates X_2 is usually symbolized by X_1 FSD X_2.

Although maintaining the (first) dominance principle as absolutely requisite, other two connected criteria appear also adequate in order to establish a reasonable and interesting partial ordering between probability prospects: the mean-preserving spread (Rothschild and Stiglitz, 1970, pp. 226-230; 1971; Diamond and Stiglitz, 1974), and the mean-preserving monotone spread (Quiggin, 1991; 1993 p. 85). A much wider comment about these and other related criteria is referred to Frosini (2010, pp. 3-12).

2. AN OPERATIVE SPECIFICATION OF THE UTILITY OF THE GAME u_g

2.1 *The influence of risk attitudes on the utility of uncertain prospects*

The definition of a risk neutral behaviour demands a strong, although reasonable, *convention*, which goes back to Daniel Bernoulli (1738/1954), meaning that one is indifferent between maintaining a game (a random variable) or accepting the certain value given by its expectation. Such neutral behaviour acts as a watershed between the common risk averse behaviour (the preference goes to the expectation) and the infrequent risk prone behaviour (maintenance of the game). As one is usually interested in the characterization of risk averse behaviour, this kind of behaviour could be called *weak risk aversion* (in agreement with Chateauneuf *et al.*, 2005, p. 650; a partially different definition is the one given by Quiggin and Chambers, 1998, p. 124).

Such final, or external, or revealed behaviour appears as the conjoint result of (at least) two independent factors, the curvature of the (realistic) utility function and the attitude toward risk. The former can be assumed concave for practically all individuals, while it can be assumed reasonably linear for many firms, companies and institutions. The final behaviour obviously depends on the interaction between these two factors: "a DM [decision maker] with diminishing marginal utility on certain wealth may be risk-seeking" (Chateauneuf *et al.*, 2005, p. 650).

As the real attitude toward risk cannot be derived from the utilities of certain prospects, it is expedient to introduce the concept of *g-risk attitude* (*g* for game), which depends on the uncertainty of the game for a subject; concerning risk aversion, this concept is practically overlapping with "aversion to mean-preserving increase in risk (MPIR) in the sense of Rothschild and Stiglitz" (Chateauneuf *et al.* 2005, p. 650, where this kind of attitude is called *strong risk aversion*. Actually, the words "weak" and "strong" are not most suitable for the above characterization: they can be taken as conventional symbols, devoid of proper meaning).

Following Frosini (1997, p. 442), the reference to the expected value of utilities can be advantageously maintained in the model for the case of *g*-risk neutral behaviour, while *g*-risk averse and *g*-risk prone behaviours are related to the divergence of the utility of the gamble $d = [h_1,p_1; ...; h_n,p_n]$ (uncertain prospect) from the expectation. The essential feature of this approach consists in assuming a utility $u_g(d)$ which mirrors the individual attitude towards risky prospects, and does not match – except in special cases – the expectation of utilities. Thus we have the following three specifications:

(a) an individual is *g-risk neutral* for the prospect d if the utility of the prospect equals the expected value of utilities, i.e. if $u_g(d) = \overline{u}$;
(b) an individual is *g-risk averse* for the prospect d if its utility is less than the average utility, i.e., for $\delta(d) > 0$, $u_g(d) = \overline{u} - \delta(d)$;
(c) an individual is *g-risk prone* for the prospect d if its utility is greater than the average utility, i.e. if, for $\delta(d) > 0$, $u_g(d) = \overline{u} + \delta(d)$.

The simplest structure – and corresponding graphical display, see Figure 1 – of this approach, is related to the evaluation of a binary prospect $d = [0,1 - p; 1,p]$; in this case the realistic utility $u(x)$ (x = money, according to an appropriate scale), and the utility for risky prospects $u_g(p)$, can both be plotted in a square of side one (with a different meaning for the abscissas of the two functions). For practically all individuals the realistic utility is a concave function (with decreasing marginal utility), while the utility $u_g(p)$ is a convex function for risk averse individuals. The *certainty equivalent* $x = \mathrm{CE}(d)$ of the binary prospect d results from the equation

$$u(x) = u_g(p) \qquad \Rightarrow \qquad x = u^{-1} \circ u_g(p)$$

and for a general prospect d:

$$u(x) = u_g(d) \qquad \Rightarrow \qquad x = u^{-1} \circ u_g(d). \tag{3}$$

Note that the certainty equivalent x uttered by an individual is of course independent of any theory contrived to explain the relation between x and the risky prospect d; anyway, the *apparent* relation displayed by equalizing the expectation p of $d = d(p) = [0,1 - p;1,p]$ to the N-M-S utility $v(x)$ becomes

$$\overline{u} = p = u_g^{-1} \circ u(x) = v(x) \tag{4}$$

thus showing that the N-M-S utility equals – under the above theory – the composition $u_g^{-1} \circ u$. For most people u_g is convex, which implies that application of u_g^{-1} to u boosts the concavity of the utility function for sure outcomes.

The above equalities and related comments are exemplified in Figure 1, which is referred to the functions

$$u_g(p) = p(1 + p)/2 \qquad \text{(with } \lambda = 0.25 \text{ in formula (6));} \ u(x) = x^{0.7}$$
$$x(p) = u^{-1}(p) = [p(1 + p)/2]^{1/0.7}$$
$$v(x) = u_g^{-1} \circ u(x) = -0.5 + (0.25 + 2x^{0.7})^{0.5} \text{ for } 0 \le x,p \le 1.$$

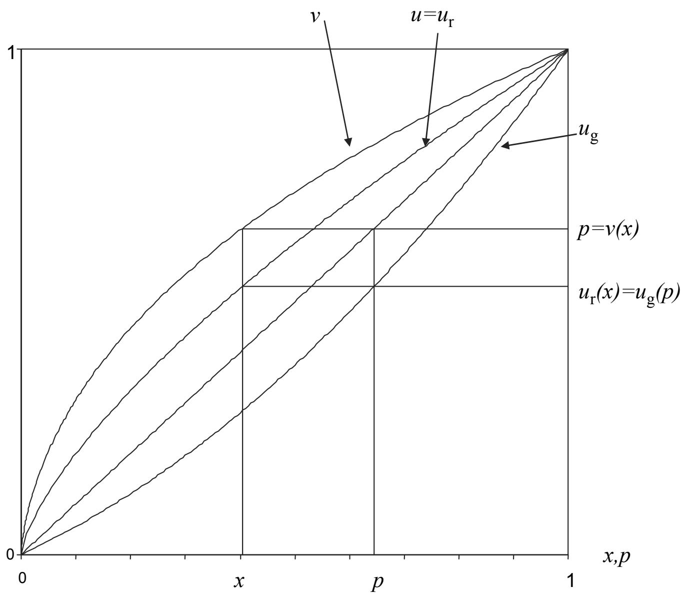

Figure 1 – Example of relevant functions for binary games in the case of risk averse behaviour. x = monetary outcome, p = probability of success for the game $d(p) = [0,1 - p; 1,p]$, $u_g(p)$ = utility of the game, $u(x) = u_r(x)$ = realistic utility of x, $v(x)$ = von Neumann-Morgenstern utility of x (not a realistic utility).

The parametric specification of u_g proposed by Frosini (1997) for the case of monetary outcomes $x_1, ..., x_n$, mainly in view of its simplicity, is the following:

$$u_g(d) = \sum u(x_i)p_i - \lambda\sum \mid u(x_i) - \overline{u} \mid p_i = \overline{u} - \lambda \text{MA}(U) \quad (5)$$

i.e. by subtracting from the expectation $\overline{u}$ a term proportional the the mean absolute deviation MA of utilities. This criterion is invariant with respect to linear transformations of utilities, meaning that, if $U = [u_1,p_1; ...; u_n,p_n]$ and $V = a + bU$, the following relation holds:

$$u_g(V) = u_g(a + bU) = a + bu_g(U).$$

2.2 *Satisfying the first stochastic dominance*

We assume throughout - as a quite natural requirement - the validity of (first) stochastic dominance, namely that the risky prospect d_2 is preferred to d_1 when the following inequality takes place between the distribution function F_1 of the random variable $d_1 = [u_1,p_1; ...; u_n,p_n]$ and the distribution function F_2 of the random variable $d_2 = [u_1,p_1'; ...; u_n,p_n']$ $(u_i \leq u_{i+1})$:

$$F_2(x) \leq F_1(x) \text{ for } -\infty < x < \infty$$

and strict inequality for some x.

In the case of a binary prospect $d(p) = [0,1-p;1,p]$, stochastic dominance is displayed by increasing p values. On account of this fact, the range of λ admissible values is easily established for the case of binary prospects; actually, in this case

$$u_g(d) = f(p) = p - 2\lambda p(1-p) \qquad 0 \le p \le 1 \tag{6}$$

being λ positive for risk averse individuals, and negative for risk prone individuals. If we want the criterion u_g to be consistent with stochastic dominance, this convex function must be increasing from $p = 0$ to $p = 1$, which occurs when $0 < |\lambda| < 0.5$; for example, when $\lambda = 0.75$, $f(p) = p(1.5p - 0.5)$ has roots 0 and 1/3, and is decreasing for $0 < p < 1/6$. A conjectured "mean risk aversion" is related to $\lambda = 1/4$, which implies $u_g(d) = p(1+p)/2$ $(0 \le p \le 1)$.

This same result for the range of λ values ($|\lambda| < 1/2$) can be achieved by looking for the admissible values in the general formula (5). Let us start by rewriting the mean absolute deviation MA as follows:

$$\text{MA} = \sum |u_i - \overline{u}|\, p_i = 2\sum_{u_i \le \overline{u}} (\overline{u} - u_i) p_i = 2\sum_{u_i > \overline{u}} (u_i - \overline{u}) p_i = 2\left(\sum_{u_i > \overline{u}} u_i p_i - \overline{P}\overline{u}\right)$$

being $\overline{P} = \sum_{u_i > \overline{u}} p_i$. As the values u_i are in ascending order, and we may conventionally assume that $0 \le u_1 \le \ldots \le u_n = 1$, the difference in (5) takes its minimum, given $\overline{u}$ and λ, when MA is a maximum, which happens when

$$\frac{1}{\overline{P}} \sum_{u_i > \overline{u}} u_i p_i - \overline{u} = \text{E}(u_i \mid u_i > \overline{u}) - \overline{u}$$

is a maximum. Given $0 \le \overline{u} \le 1$, max $[\text{E}(u_i \mid u_i > \overline{u}) - \overline{u}]$ is clearly attainable when the only probabilities different from zero are $p_1 = 1 - \overline{u}$ and $p_n = \overline{u}$; therefore, the maximizing value of MA is $2\overline{u}(1 - \overline{u})$, and the minimum value of $u_g(d)$ is

$$u_g(d) = \overline{u} - 2\lambda\, \overline{u}\, (1 - \overline{u}) \tag{7}$$

which looks like formula (6) – and actually, in this case, $\overline{u} = p$; the maximum value allowable for λ is therefore 0.5.

This same range for λ values is required in order that the criterion (5) satisfies the First Stochastic Dominance (FSD) in the general case. Let us consider a displacement of probability from $d_1 = [u_1,p_1; \ldots; u_n,p_n]$ to $d_2 = [u_1,p_1'; \ldots; u_n,p_n']$ ($p_i, p_i' \ge 0$) such that $p_i' = p_i$ for $i \ne r,s$, $p_r' = p_r - \Delta$ (such that $p_r - \Delta \ge 0$), $p_s' = p_s + \Delta$, for $u_r < u_s$. The criterion FSD is satisfied if, in any case, $u_g(d_1) < u_g(d_2)$. In any case the expectation is increased from $\overline{u}$ to

$$\overline{u}' = \sum_{i \ne r,s} u_i p_i + u_r(p_r - \Delta) + u_s(p_s + \Delta) = \overline{u} + \Delta(u_s - u_r)$$

thus complying – as it is well known – with the criterion FSD. For the complete expression $u_g(d)$ let us consider three distinct cases, with subcases:

(a) $u_r < u_s \le \overline{u}$;

(a1) there are no u_i's such that $\overline{u} \le u_i \le \overline{u}$'; in this case the new MA' can be written

$$\text{MA'} = 2\sum_{u_i > \overline{u}} [u_i - \overline{u} - \Delta(u_s - u_r)]p_i \text{ , so that}$$

$$u_g(d_2) - u_g(d_1) = \overline{u} + \Delta(u_s - u_r) - 2\lambda\text{MA'} - \overline{u} + 2\lambda\text{MA}$$
$$= \Delta(u_s - u_r)(1 + 2\lambda\overline{P}) > 0, \text{ which holds in any case when } \lambda > -1/2.$$

(a2) There are values u_i's such that $\overline{u} \le u_i \le \overline{u}' = \overline{u} + \Delta(u_s - u_r)$; in this case MA' reduces to

$$\text{MA'} = 2\sum_{u_i > \overline{u}'} (u_i - \overline{u}')p_i$$

so that the previous difference $u_g(d_2) - u_g(d_1)$ is increased.

(b) $\overline{u} \le u_r < u_s$;

(b1) there are no u_i's such that $\overline{u} \le u_i \le \overline{u}$'; in this case the new MA' can be written

$$\text{MA'} = 2\sum_{u_i \le \overline{u}} [\overline{u} + \Delta(u_s - u_r) - u_i]p_i \text{ so that}$$

$$u_g(d_2) - u_g(d_1) = \overline{u} + \Delta(u_s - u_r) - 2\lambda\text{MA'} - \overline{u} + 2\lambda\text{MA}$$
$$= \Delta(u_s - u_r)[1 - 2\lambda(1 - \overline{P})]$$

which is greater than zero in any case when $\lambda < 0.5$.

(b2) There are values u_i's such that $\overline{u} \le u_i \le \overline{u}$'; in this case MA' is increased to

$$\text{MA'} = 2\sum_{u_i \le \overline{u}'} (\overline{u}' - u_i)p_i\text{'} \text{ so that}$$

$$u_g(d_2) - u_g(d_1) = \overline{u} + \Delta(u_s - u_r) - 2\lambda\text{MA'} - \overline{u} + 2\lambda\text{MA}$$
$$= \Delta(u_s - u_r)[1 - 2\lambda\sum_{u_i \le \overline{u}} p_i] - 2\lambda\sum_{\overline{u} < u_i \le \overline{u}'} (\overline{u}' - u_i)p_i\text{'} .$$

If, in the last summation, u_i is replaced by $\overline{u}$, the same summation is increased to

$$\Delta(u_s - u_r)\sum_{\overline{u} < u_i \le \overline{u}'} p_i\text{'} \text{ so that}$$

$$u_g(d_2) - u_g(d_1) = \Delta(u_s - u_r)[1 - 2\lambda \sum_{u_i \le \bar{u}'} p_i'] = \Delta(u_s - u_r)[1 - 2\lambda(1 - \bar{P}')]$$

thus we find anew that such a difference is greater than zero in any case when $\lambda < 0.5$.

(c) $u_r < \bar{u} < u_s$.

This case can be treated by means of two successive displacements of probability Δ, the first from p_r to the probability of $\bar{u}$ (possibly starting from a null value), and the second from the probability of $\bar{u}$ to p_s, thus by means of successive applications of the cases (a) and (b).

2.3 *Satisfying the independence condition*

A property maybe unexpected of criterion (5) is that it satisfies the so-called *independence condition*, which is introduced as an essential property or axiom in the orthodox Expected Utility theory (special case of (7) for $\lambda = 0$); this requisite is satisfied under the same constraint for λ ($|\lambda| < 1/2$), which ensures the compliance with the dominance property, just demonstrated. Such a condition says that, if prospect B is preferred to prospect A, a probability mixture of B with any other prospect C is always preferred to the same kind of mixture between A and C; in symbols:

$A \prec B \leftrightarrow d = [A,p;\ C,1-p] \prec d' = [B,p;\ C,1-p]$ for any prospect C and $0 < p \le 1$.

A similar condition is called *sure-thing principle* by Savage (1954) (cf Frosini and Giossi, 1994, par. 5).

Let a, b, c ($a < b$) be the utilities of the corresponding prospects A, B, C. With reference to the generic binary prospect $[A,p;\ B,1-p]$, the following expressions for the mean utility $\bar{u}$ and the mean absolute deviation MA hold:

$$\bar{u} = ap + b(1-p)$$

$$\mathrm{MA}(U) = (ap + b(1-p) - a)p + (b - ap - b(1-p))(1-p) = 2p(1-p)(b-a).$$

Now, let the following cases be examined: (I) $a < b \le c$; (II) $a \le c \le b$; (III) $c \le a < b$.

For case (I)

$$u_g(d) = ap + c(1-p) - 2\lambda p(1-p)(c-a)$$

$$u_g(d') = bp + c(1-p) - 2\lambda p(1-p)(c-b)$$

so that $u_g(d') > u_g(d)$ when

$$p(b-a)(1 + 2\lambda(1-p)) > 0. \quad (8)$$

When $\lambda > 0$ in (5), namely for a risk averse subject, this condition is naturally satisfied: the mixture of C with A ensures a lower expected utility and a larger dispersion than the mixture with B; both comparisons yield a larger utility $u_g(d^*)$. For a risk prone subject, i.e. when $\lambda < 0$, inequality (8) is satisfied when $|\lambda| < 1/(2(1-p))$, thus $|\lambda| < 1/2$ is the proper condition.

For case (II), $u_g(d^*) > u_g(d)$ when

$$p(b-a) - 2\lambda p(1-p)(a+b-2c) > 0. \tag{9}$$

If $(a+b)/2 \le c \le b$, it is immediately recognized that the second term of the algebraic sum (9) is positive or null for every $\lambda > 0$, so that the > sign is satisfied. When, to the contrary, $\lambda < 0$, the condition to be satisfied is $|\lambda| < 1/2$. If $a \le c \le (a+b)/2$, the most unfavourable situation for (9) to hold for $\lambda > 0$, is that $c = a$, in which case the sign > holds for $\lambda < 1/2$. Instead, when $\lambda < 0$ the most unfavourable situation is for $c = (a+b)/2$, hence the left hand side of (9) reduces to $p(b-a) > 0$ for every negative λ.

For case (III), $u_g(d^*) > u_g(d)$ when

$$p(b-a)(1 - 2\lambda(1-p)) > 0. \tag{10}$$

For $\lambda > 0$ we find anew the condition $\lambda < 1/2$. For $\lambda < 0$ no further constraint on λ arises.

2.4 *The shapes of indifference curves*

About the shapes of indifference curves inside the Marshak-Machina triangle, the interested reader is referred to Frosini (2010, par. 5.4).

2.5 *Why not to use the standard deviation of utilities*

Another natural application of the above criterion could be equating $\delta(u(x))$ to a value proportional to the standard deviation SD of the distribution U of utilities

$$u_g^*(d) = \sum u_i p_i - \lambda\,[\sum (u_i - \bar{u})^2 p_i]^{1/2} = \bar{u} - \lambda\,\mathrm{SD}(U). \tag{11}$$

Marshak (1950, p. 120, Note 10) reports that “As a measure of riskiness the standard deviation was suggested by I. Fisher as early as 1906”. However, if we want not to give up the stochastic dominance requirement, the criterion u_g^* presents serious shortcomings. In fact, if we choose for convenience – as usual – values $u(x_i) = u_i \ge 0$, the maximum reduction of $\bar{u}$ in (11) occurs in cases of maximum dispersion of the values u_i around a fixed mean $\bar{u}$; just as for the variability measure MA, if $0 \le u_i \le 1$ for $i = 1. ..., n$, maximum dispersion is achieved when the distribution is binary, i.e. when $U = [0,1-\bar{u}\,;\,1,\bar{u}]$, with mean $\bar{u}$ (cf. Frosini, 1984, p. 392); therefore the minimum $u_g^*(d)$ is

$$u_g^*(d) = \bar{u} - \lambda\,[\bar{u}\,(1-\bar{u})]^{1/2} \tag{12}$$

Problems arise for $\bar{u}$ in a neighbourhood of zero. In fact, as we approach zero from the right, the derivative of $\sigma(\bar{u}) = [\bar{u}\,(1-\bar{u})]^{1/2}$ tends to $+\infty$, whereas the derivative of $\bar{u}$ is 1; practically, for $0 < \bar{u} < \lambda^2/(1+\lambda^2)$ the negative term (for $\lambda > 0$) in (12) overcomes $\bar{u}$, thus there is an interval for $\bar{u}$ – for every $\lambda > 0$ – in which $u_g^*(\bar{u})$ decreases as $\bar{u}$ increases (contrary to the requirement of dominance).

2.6 *Other similar proposals*

We have seen above the validity of proposal (5) for $0 \leq \lambda \leq 0.5$ in case of risk averse subjects, and the rejection of (11) for any $\lambda \geq 0$, on the basis of the fundamental criterion of stochastic dominance. A class of similar criteria, for the rv $X = (x_1, p_1; ...; x_n, p_n)$, has been envisaged by Quiggin and Chambers (1998, p. 131; 2004, pp. 101-102); some comments about this class are made by Frosini (2010, par. 5.6).

3. UTILITY FOR GAINS AND UTILITY FOR LOSSES. THE PROBLEM OF PROBABILISTIC INSURANCE

Unlike N-M-S utility theory, which attaches a utility function to the wealth – past, present or future, of any amount – of a subject, one of the main contributions of Kahneman and Tversky (1979) has been to reveal opposite features of the subject's behaviour in the presence of possible gains or else in the presence of possible losses (a similar viewpoint, although in a different context, had been suggested by Markowitz, 1952 and 1959). What has been said until now, as summarized in Figure 1, is essentially referred to games contemplating only gains, beyond the possibility of maintaining the status quo. Kahneman and Tversky (1979, p. 279) observe that "a salient characteristic of attitudes to changes in welfare is that losses loom larger than gains", and suggest that "the value function [u_r above] is (i) defined on deviations from the reference point; (ii) generally concave for gains and commonly convex for losses [*reflection effect*, or *mirror effect*]; (iii) steeper for losses than for gains".

A particular application of our model in the negative range (for losses) is successful for demonstrating the correctness of Kahneman and Tversky's result on the so called *probabilistic insurance* (1979, pp. 269, 270, 285). We quote the problem in its essence, referring to the original paper for a wider discussion. According to expected utility theory, with the assumption of a concave utility function, probabilistic insurance is superior to regular insurance. "If at asset position w one is just willing to pay a premium y to insure against a probability p of losing x, then one should definitely be willing to pay a smaller premium ry to reduce the probability of losing x from p to $(1-r)p$, $0 < r < 1$. Formally, if one is indifferent between $(w-x,p;\ w,1-p)$ and $(w-y)$, then one should prefer probabilistic insurance $(w-x,(1-r)p;\ w-y,rp;\ w-ry,1-p)$ over regular insurance $(w-y)$" (p. 270). Now if d_A is the prospect concerning regular insurance, $\bar{u}_A$ and MA(d_A) the location and dispersion parameters of model (5) for this prospect, with d_B, $\bar{u}_B$ and MA(d_B) referring analogously to the prospect of probabilistic insurance, the following equalities hold:

$$\overline{u}_A = u(w-x)p + u(w)(1-p) = u_1 p + u_2(1-p)$$

$$\overline{u}_B = u(w-x)(1-r)p + u(w-y)rp + u(w-ry)(1-p) = u_1(1-r)p + u_2'rp + u_3'(1-p)$$

$$\mathrm{MA}(d_A) = (\overline{u}_A - u_1)p + (u_2 - \overline{u}_A)(1-p)$$

$$\mathrm{MA}(d_B) = (\overline{u}_B - u_1)(1-r)p + (u_2' - \overline{u}_B)rp + (u_3' - \overline{u}_B)(1-p).$$

This last equality has assumed – as it is reasonable for the problem – that the values $u_2' = u(w-y)$ and $u_3' = u(w-ry)$ are not far from one another, and in any case that their distance from u_1 is much greater than the distance $(u_3' - u_2')$. As we move in the loss region with respect to the present position w, namely covering a *convex* function, the inequality $\overline{u}_B < \overline{u}_A$ holds (cf. Kahneman and Tversky 1979, p. 270). Besides, comparing the utility distributions, one observes that the range $(u_3' - u_1)$ for d_B is smaller than the range $(u_2 - u_1)$ for d_A, and the probability p relative to u_1 for d_A is distributed over this same value *and* an intermediate value for d_B, hence the spread is reduced: $\mathrm{MA}(d_A) > \mathrm{MA}(d_B)$. Finally, taking into account that in the negative domain λ is negative (at least for most people), both terms in formula (7) show changes in agreement with the "normal" appreciation of the two prospects, i.e. probabilistic insurance is estimated inferior to regular insurance.

It seems appropriate to call the attention to the correct "position" of the *reference point*, which serves as the zero point of the value scale, or discrimination point between gains and losses. This is important because the utility of a change (positive or negative) depends on the magnitude *and sign* of the change, hence on the asset position that serves as reference point (Kahneman and Tversky 1979, p. 277). Besides, the reference point can undoubtedly affect the same model for the utility of the game u_g; in particular, the parameter λ in model (5) is likely to depend, for a given individual, on the chosen reference point, that can be displaced with respect to the present asset position (Kahneman and Tversky, 1979, pp. 286-288).

4. NATURAL SOLUTIONS OF SOME PARADOXES FOR DECISIONS UNDER RISK

The above approach, condensed in formula (5),together with its generalization to the case of imprecise probabilities presented in Section 5, allows to avoid classical paradoxes for the N-M-S theory, such as those raised by Allais (1953) and Ellsberg (1961) (see also Savage 1954, pp. 101-102; Gärdenfors and Sahlin 1982, p. 12). Allais example considers two decision situations, each involving two gambles (the following figures refer to hundred thousand dollars).

Situation 1. Choose between
d_1 : 5 with certainty ; d_2 : [0,0.01 ; 5,0.89 ; 25,0.1]

Situation 2. Choose between
d_3 : [0,0.89 ; 5,0.11] ; d_4 : [0,0.9 ; 25,0.1].

Many people prefer d_1 to d_2, and d_4 to d_3; this implies the following inequalities for any hypothetical utility function u, by adopting the EU (= Expected Utility) criterion:

$$0.11\ u(5) > 0.1\ u(25) + 0.01\ u(0)$$

$$0.11\ u(5) < 0.1\ u(25) + 0.01\ u(0)$$

which are incompatible.

Now, let us assume that our subject K is risk averse, and adopts a criterion like formula (5); U_1, U_2, U_3, U_4 will be the random variables of utilities attached to the gambles d_1, d_2, d_3, d_4 respectively (the following results are resumed from Frosini, 1997, p. 448). Moreover, if we consider the traditional view of attaching utilities to final wealth (not really a discriminating hypothesis, as we are only envisaging possible gains, beside maintaining the status quo), and assume that the present property of K is \$ 200,000, we are faced to testing the compatibility of the following inequalities:

$$u(7) > 0.1\ u(27) + 0.89\ u(7) + 0.01\ u(2) - \lambda \mathrm{MA}(U_2) \tag{13}$$

$$0.1\ u(27) + 0.9\ u(2) - \lambda \mathrm{MA}(U_4) > 0.11\ u(7) + 0.89\ u(2) - \lambda \mathrm{MA}(U_3); \tag{14}$$

calling $\delta_i = \mathrm{MA}(U_i)$, after simplification the inequalities are compatible if $\delta_2 + \delta_3 - \delta_4 > 0$.

As a possible application, let us consider the following "constant risk averse" utility

$$u(x) = 1 - \exp(- cx) \qquad x > 0;\ c > 0$$

(cf Lindley, 1985, p. 86); if we adopt Lindley's suggestion of matching the utility ½ to the existing property (assumed to be \$ 200,000), so that $c = 0.34657$, we obtain the following values for the expectations and mean absolute deviations: $\overline{u}_1 = 0.91161$, $\delta_1 = 0$; $\overline{u}_2 = 0.91633$, $\delta_2 = 0.016717$; $\overline{u}_3 = 0.54528$, $\delta_3 = 0.080594$; $\overline{u}_4 = 0.54999$, $\delta_4 = 0.089985$; thus the inequalities (13)-(14) become

$$0.91161 > 0.91633 - \lambda \times 0.016717\ ;\ 0.54999 - \lambda \times 0.089985 > 0.54528 - \lambda \times 0.080594.$$

The necessary inequality $\delta_2 + \delta_3 - \delta_4 > 0$ is satisfied. Moreover, the range of λ values which allow to satisfy both inequalities appears comfortably large: $0.28199 < \lambda < 0.50201$, practically identical with the range of admissible values $0.28199 < \lambda < 0.5$.

Another, more general, paradox in N-M-S theory has been thoroughly investigated by Kahneman and Tversky (1979, pp. 266, 267, 282); the two authors call it the *paradox of the substitution axiom*. Actually, these authors call *substitution axiom* what is usually called *independence axiom* (or *condition*), as made above at Section 2.3, and *sure-thing principle* by Savage (1954): "The substitution axiom of utility theory asserts that if B is preferred to A, then any (probability) mixture (B,p) must be preferred to the mixture (A,p)" (p. 266). "The results suggest the following empirical generalization concerning the manner in which the substitution axiom is violated. If (y,pq) is equivalent to (x,p), then (y,pqr) is preferred to (x,pr), $0 < p,q,r < 1$" (p. 267). Note that in the quoted paper the probability prospect $[A,p; 0, 1-p]$ is simply written as (A,p), just neglecting the prospect *status quo*. With the aid of Figure 2 we shall show that our approach implies the "explanation" of this supposed paradox.

With our symbolism, let $d = [0,1 - pq; 1,pq]$, and $y' = \mathrm{CE}(d)$ (Certainty Equivalent of d). Moreover, let $d^* = [0,1 - p; x',p]$ such that $d^* \sim d$; obviously, x' must be < 1. Let u be the (realistic) utility framed in the square of side one for value $0 \leq x \leq 1$, u_g the utility of the game d, and u_g^* the utility of the game d^*. It seems reasonable – and this will be our assumption – that the curve describing u_g^* will equal the curve of u_g except for a scale change; thus

$$u_g^*(t) = x' u_g(t/x') \qquad 0 \leq t \leq x'$$

and for $t = px'$: $\quad u_g^*(t) = x' u_g(p)$.

Figure 2 shows the relevant curves, with the following choice of values and functions (with $\lambda = 0.4$ in formula (6)):

$$u_g(p) = 0.2p + 0.8p^2 \qquad 0 \leq p \leq 1$$

$$u_g^*(t) = 0.7[0.2\, t/0.7 + 0.8(t/0.7)^2] \qquad 0 \leq t \leq 0.7$$

$$u(x) = x^{0.7} \quad ; \quad p = 0.6 \quad ; \quad q = 0.9.$$

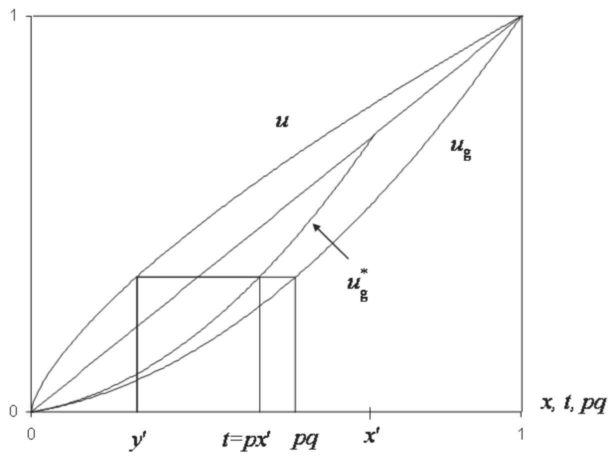

Figure 2 – Reference graph for the "paradox" of the substitution axiom (see text).

The above assumption $d^* \sim d$ is equivalent to saying, in our model (cf Figure 2) that

$$u_g(pq) = u_g^*(px') = x' u_g(p), \quad \text{hence} \quad x' = u_g(pq)/u_g(p).$$

Starting from the equivalence between d and

$$d^* = [0,1-p;\ x' = u_g(pq)/u_g(p), p]$$

it remains to check the preference relation between the prospects

$$b = [0,1-pqr;\ 1, pqr] \quad \text{and} \quad b^* = [0,1\text{-}pr;\ x' = u_g(pq)/u_g(p), pr].$$

The comparison between the ordinates of u_g and u_g^* gives rise to possible inequalities

$$u_g(pqr) > = < u_g^*(prx') = x'u_g(pr), \quad \text{hence} \quad u_g(p)u_g(pqr) > = < u_g(pq)u_g(pr). \qquad (15)$$

In passing we observe that, if u_g would be linear, as in N-M-S theory, left and right hands of (19) would be equal to p^2qr. Now, the evaluation of the above inequality depends on what happens when passing from $u_g(p)u_g(pr)$ to $u_g(p)u_g(prq)$ and to $u_g(pq)u_g(pr)$, respectively. As u_g is convex and increasing, its derivative is increasing with the argument, thus the reduction is larger when we start from a larger abscissa; hence the reduction is larger when we pass from $u_g(p)$ to $u_g(pq)$ than when passing from $u_g(pr)$ to $u_g(prq)$. Hence

$$u_g(p)u_g(pqr) > u_g(pq)u_g(pr), \text{ and, in conclusion,} \quad u_g(pqr) > u_g^*(x'pr)$$

in agreement with Kahneman and Tversky (1979, p. 282).

5. DECISIONS UNDER UNCERTAINTY AND ELLSBERG'S PARADOX

A simple and direct extension of the above approach to the case of decisions under uncertainty (practically: with no knowledge or only a limited knowledge of the probabilities relevant for a given decision problem) will now be derived and justified; as an immediate application, the Ellsberg's paradox will be simply explained. The following introductory remarks are resumed from Frosini (1997, pp. 453-454).

In order to approach an operational criterion, it will be admitted that our subject K is able to assess, at least as an acceptable workable approximation, a probability distribution over the vector $(p_1, ..., p_n)$, which is $(n-1)$-dimensional as $\sum p_i = 1$. Then, consistently with the definitions established in Section 2, we define the risky behaviour in the presence of an imprecise assessment of the probabilities $p_1, ..., p_n$ of the states of nature $s_1, ..., s_n$ as follows; it will be assumed that the prospects envisage both gains and losses, or only gains together with *status quo* (or null prospect):

– An individual is *g-risk neutral* if he or she is satisfied with replacing each p_i with its expected value $\bar{p}_i = \mathrm{E}(p_i)$ (note that $\sum \bar{p}_i = 1$).

– An individual is *g-risk averse* if he or she is satisfied with replacing for p_i a value p_i^* lower than $\mathrm{E}(p_i)$ when the prospect assured by the state s_i is a gain, and a value p_i^* greater than $\mathrm{E}(p_i)$ when the prospect is a loss or no gain.

– An individual is *g-risk prone* if he or she is satisfied with replacing for p_i a value p_i^* greater than $E(p_i)$ when the prospect assured by s_i is a gain, and a value p_i^* lower than $E(p_i)$ when the prospect is a loss or no gain.

For reasons of brevity, we shall make further observations only for the case of *g*-risk averse behaviour, both in the sense of uncertain prospects and in the sense of imprecise probabilities. If the prospects are sums of money x_i, and if we use the correction term $\lambda MA(U)$ applied to the expected probabilities, the expression

$$u_g(d) = \sum u(x_i)\,\overline{p}_i - \lambda \sum \mid u(x_i) - \overline{u} \mid \overline{p}_i \qquad (16)$$

is used when our subject is averse with respect to risky prospects (if $\lambda > 0$), and is neutral with respect to imprecise probabilities. If the subject is risk averse also with respect to the imprecise assessment of probabilities, the expression (16) can be used all the same, but by replacing $\overline{p}_i$ with a p_i^* smaller than $\overline{p}_i$ for positive prospects, and with a p_i^* greater than $\overline{p}_i$ for negative or null prospects. Also in this case, for the sake of simplicity, we could use a correction term proportional to the mean absolute deviation of the random variable p_i, so that

$$p_i^* = \overline{p}_i - \gamma_1 MA(p_i), \qquad \gamma_1 > 0, \text{ for a positive prospect,}$$

$$p_i^* = \overline{p}_i + \gamma_2 MA(p_i), \qquad \gamma_2 > 0, \text{ for a negative or null prospect.}$$

Ellsberg's paradox: An urn is known to contain 30 red balls and 60 black or yellow balls, the latter colours in unknown proportions p_2 and p_3, with sum $p_2 + p_3 = 2/3$. One ball is to be drawn at random from the urn. Most people prefer

Gamble 1: $ 100 if a red ball is drawn, with respect to
Gamble 2: $ 100 if a black ball is drawn;

and most people prefer

Gamble 3: $ 100 if a black or yellow ball is drawn, with respect to
Gamble 4: $ 100 if a red or yellow ball is drawn

(such preference patterns violate Savage's sure thing principle).

For the sake of simplicity, we put 0 = status quo and $u(0) = 0$; then the utilities of the actions to be compared can be written as follows (by numbering the actions a_1-a_4 as the corresponding gambles):

$$u(a_1) = u(100)(1/3);\ u(a_2) = u(100)p_2;\ u(a_3) = u(100)(2/3);\ u(a_4) = u(100)(1/3 + p_3).$$

If we assume $\overline{p}_2 = \overline{p}_3 = 1/3$ (this could ensue e.g. from a uniform distribution between 0 and 2/3), and correspondingly $p_2^* = p_3^* = 1/3 - \gamma$ (with $\gamma > 0$), the following inequalities are immediately checked:

$$u(a_2) = u(100)(1/3 - \gamma) < u(a_1);\ u(a_4) = u(100)(2/3 - \gamma) < u(a_3)$$

thus explaining the paradox by a simple application of the EU model, however to probabilities p_i^* (case of $\lambda = 0$ for the u_g criterion). In the general case, formula (5) should be rewritten as follows:

$$u_g(d) = \sum u(x_i)p_i^* - \lambda\,\text{MA}^* = \overline{u}^* - \lambda\,\text{MA}^* \tag{17}$$

being $\text{MA}^* = \sum |u(x_i) - \overline{u}^*|\, p_i^*$. Letting for simplicity $u(100) = 1$, we obtain:

$\overline{u}(a_1) = 1/3$; $\overline{u}^*(a_2) = 1/3 - \gamma$; $\overline{u}(a_3) = 2/3$; $\overline{u}^*(a_4) = 2/3 - \gamma$;

$\text{MA}(a_1) = |0 - 1/3|(2/3) + |1 - 1/3|(1/3) = 4/9$, coinciding also with $\text{MA}(a_3)$;

$$\text{MA}^*(a_2) = |0 - (1/3 - \gamma)|\,(2/3 + \gamma) + |1 - (1/3 - \gamma)|\,(1/3 - \gamma)$$
$$= 4/9 - (2/3)\gamma - 2\gamma^2;$$

$$\text{MA}^*(a_4) = |0 - (2/3 - \gamma)|\,(1/3 + \gamma) + |1 - (2/3 - \gamma)|\,(2/3 - \gamma)$$
$$= 4/9 + (2/3)\gamma - 2\gamma^2;$$

$$u_g(a_1) - u_g(a_2) = [1/3 - (4/9)\lambda] - [(1/3 - \gamma) - \lambda(4/9 - (2/3)\gamma - 2\gamma^2)]$$
$$= \gamma(1 - (2/3)\lambda - 2\lambda\gamma)$$

with roots 0 and $1/(2\lambda) - 1/3$. As we are only interested in values $0 \leq \gamma < 1/3$, for λ positive (case of risk-averse behaviour) the above difference is greater than 0. Analogously, the difference

$$u_g(a_3) - u_g(a_4) = [2/3 - (4/9)\lambda] - [(2/3 - \gamma) - \lambda(4/9 + (2/3)\gamma - 2\gamma^2)]$$
$$= \gamma(1 + (2/3)\lambda - 2\lambda\gamma),$$

with roots 0 and $1/(2\lambda) + 1/3$, is greater than zero under the same condition for λ. Also in the general case for the criterion u_g the Ellsberg paradox is fully explained.

6. CONCORDANCE WITH QUIGGIN'S APPROACH

An interesting feature of the above model is that it is compatible with a totally different approach to modelling utility functions, which has gained wide popularity in recent years: the rank-dependent expected utility (RDEU) model, proposed by Quiggin (1982, 1993), already mentioned in Section 1. The discussion of several inconsistencies in the applications of N-M-S- theory led Quiggin to propose an evaluation function for a prospect d of monetary outcomes of type (2).

Letting $u_i = u(x_i)$, and $P^* = \sum_{u_i \leq \overline{u}} p_i$, the mean absolute deviation of utilities can be written

$$\text{MA}(U) = \sum_{u_i \leq \overline{u}} (\overline{u} - u_i)\, p_i + \sum_{u_i > \overline{u}} (u_i - \overline{u})\, p_i$$
$$= \sum_{u_i \leq \overline{u}} 2\, u_i\, p_i\, (P^* - 1) + \sum_{u_i > \overline{u}} 2\, u_i\, p_i\, P^*;$$

thus, from (5):

$$u_g(d) = \bar{u} - \lambda\,\mathrm{MA}(U)$$
$$= \Sigma_{u_i \le \bar{u}} u_i\, p_i [1 + 2\lambda(1 - P^*)] + \Sigma_{u_i > \bar{u}} u_i\, p_i (1 - 2\lambda P^*) \qquad (18)$$

showing a special case of formula (2), with probabilities multiplied by $[1 + 2\lambda (1 - P^*)] > 1$ for $u_i \le \bar{u}$, and by $(1 - 2\lambda P^*) < 1$ for $u_i > \bar{u}$; thus the transformation of cumulative probabilities is concave, a property akin to risk aversion (Quiggin 1993, pp. 59, 78). It is immediately checked that the transformed probabilities sum to one:

$$= \Sigma_{u_i \le \bar{u}} p_i [1 + 2\lambda(1 - P^*)] + \Sigma_{u_i > \bar{u}} p_i (1 - 2\lambda P^*) = 1$$

Besides, we observe that the coefficient of p_i in the second term of formula (18) is positive only if $\lambda < 1/2P^*$; as sup $P^* = 1$, we find anew that λ is bound not to exceed 0.5.

As an example, for the random prospect

d = [0,0.05 ; 0.5,0.25 ; 0.8,0.60 ; 1,0.10]

we obtain $\bar{u} = 0.705$, $P^* = 0.3$, and the following k_i coefficients (cf formulae (2) and (18)), given $\lambda = 0.4$:

$k_1 = 0.05(1 + 2\times0.4\times0.7) = 0.078$

and analogously: $k_2 = 0.25\times1.56 = 0.39$, $k_3 = 0.6\times0.76 = 0.456$, $k_4 = 0.1\times0.76 = 0.076$ (note that $k_1 + ... + k_4 = 1$); hence $q(p_1) = 0.078$, $q(p_1 + p_2) = 0.468$, $q(p_1 + p_2 + p_3) = 0.924$, $q(p_1 + ... + p_4) = 1$. The concavity of the transformation function is clearly appreciated from Figure 3. From both formulae (18) $u_g(d) = 0.6358$.

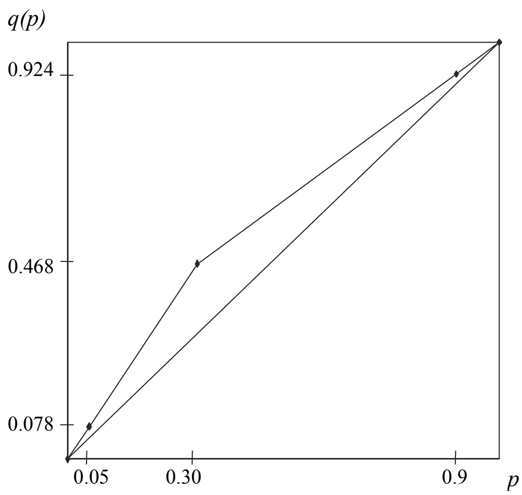

Figure 3 – Example of Quiggin's transformation function $q(p)$.

7. EXPERIMENT

An experiment of the usual kind has been devised, asking twenty people (some colleagues and students of Economics) to answer ten questions of a questionnaire, proposing games with only (limited) gains or nothing. The practical procedure was almost exactly borrowed from the one implemented by Tversky and Kahneman (1992, pp. 306-307).

The aim of the experiment (taking account of the small number of participants) was not about revealing actual behaviour of people, but only about the operational control of model (5). The outcome of the experiment proved wholly successful. A detailed exposition of the experiment, together with its main results, is included in Section 10 of Frosini (2010).

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SUMMARY

Realistic utility versus game utility: a proposal for dealing with the spread of uncertain prospects

The author develops the properties and implications of a proposal, concerning a summary statistic of the random prospect of utilities. Following a suggestion of Allais, such a statistic is increasing with expected utility, and decreasing – for most people, who are risk averse – with the mean absolute deviation of utilities; a parameter multiplying this disper-

sion measure allows for risk averse or risk prone behaviour, according to its sign, and also for more or less departure from a certain prospect. It is demonstrated that this statistic (a) satisfies the first stochastic dominance, (b) satisfies the independence condition, (c) satisfies the so called “problem of probabilistic insurance”, (d) resolves the paradoxes of Allais, Ellsberg and Kahneman-Tversky (paradox of the substitution axiom), (e) is compatible with Quiggin’s approach of rank-dependent expected utility models, (f) the mean absolute deviation cannot be replaced by the standard deviation.