STATISTICA, anno LXXI, n. 4, 2011

A METHOD OF MOMENTS TO ESTIMATE BIVARIATE SURVIVAL FUNCTIONS: THE COPULA APPROACH

S.A. Osmetti, P.M. Chiodini

1. INTRODUCTION

In this paper we study bivariate distributions used in reliability analysis by the meaning of copula. The copula is an instrument to generate bivariate and multivariate distributions (Nelsen, 2006 and Fisher, 1997). In particular we consider the survival copula of Marshall and Olkin. This copula comes from the bivariate Marshall-Olkin exponential distribution (Marshall and Olkin, 1967), proposed to study complex systems in which the two components are not independent. We generalize this model by the copula and different marginal distributions to construct several bivariate survival functions, i.e. bivariate Weibull distribution. These cumulative distribution functions are not absolutely continuous and their unknown parameters can not be obtained in explicit form by the maximum likelihood method. In order to estimate the parameters we propose an easy procedure based on the moments. This method consists in two steps: in the first step we estimate only the parameters of marginal distributions and in the second step we estimate only the copula parameter. The study of simulation is made either for complete or censored sample (II Type) in order to evaluate the performance of the proposed estimation procedure.

2. BACKGROUND TO THE MARSHALL-OLKIN COPULA

In multivariate distributions the dependence structure existing among the marginal random variables (r.v.s) is described by the means of copula, which is an helpful tool for handling multivariate probability law with given univariate marginals (Nelsen, 2006). Therefore, copula allows to generate new multivariate distributions with arbitrary marginal r.v.s and the same dependence structure between the marginals.

A bivariate copula is a function $C: I^2 \to I$, with $I^2 = [0,1] \times [0,1]$ and $I = [0,1]$, such that by an appropriate extension of its domain on $\Re^2$ has all the properties of a cumulative distribution function (c.d.f.). Therefore, it can be treated as a c.d.f. of a random variable (r.v.) (U,V) with uniform marginals in $[0,1]$

$$C(u,v) = P(U \leq u, V \leq v),$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

To better understand the copula model we introduce the Sklar's theorem (Sklar, 1959).

Theorem. Let (X,Y) a bivariate r.v. with c.d.f. $F_{X,Y}(x,y)$ and marginals $F_X(x)$ and $F_Y(y)$, there exists a copula function $C : I^2 \rightarrow I$ such that $\forall x, y \in \Re$

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)) \quad (1)$$

If $F_X(x)$ and $F_Y(y)$ are continuous then the copula C is unique. Conversely, if C is a copula and $F_X(x)$ and $F_Y(y)$ are marginal distribution functions, then the $F_{X,Y}(x,y)$ function in (1) is a joint distribution function with marginal cumulative distribution functions $F_X(x)$ and $F_Y(y)$.

This theorem tells us that, given a bivariate c.d.f. and continuous marginals, there is a function C such that the bivariate c.d.f. can be univocally expressed as a function of the marginals. Moreover, if the marginal distribution functions are continuous the copula can be found by the inverse of the equation (1):

$$C(u,v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) \quad (2)$$

with $u = F_X(x)$ and $v = F_Y(y)$.

In reliability analysis it is more convenient to express a joint survival function as a copula of its marginal survival functions. So let X and Y be r.v.s. with survival functions $\overline{F}_X(x)$ and $\overline{F}_Y(y)$, the expression

$$\overline{F}_{X,Y}(x,y) = \hat{C}(\overline{F}_X(x), \overline{F}_Y(y)) \quad (3)$$

yields a possible survival function for the pair (X,Y). The function $\hat{C}$ is a copula to which refer as the survival copula.

An example of survival copula used in reliability analysis is the Marshall and Olkin copula (see Embrecht et al., 2003).

Definition of Marshall-Olkin Copula. The Marshall-Olkin survival copula (MOC) is the function $\hat{C} : I^2 \rightarrow I$ such that

$$\hat{C}(u,v) = uv \min(u^{-\alpha}, v^{-\beta})$$

with $\alpha, \beta \in [0,1]$. This copula is the survival copula of the Marshall and Olkin distribution (see Nelsen, 2006). Hence this function yield a two parameter family of copulae. This copula class is known both as the Marshall and olkin family and the Generalized Cuadras-Augé family. If we consider the easy case of exchangeable marginal random variables, then $\alpha = \beta$ and the copula depends on an unique parameter θ

$$C(u,v) = uv \min(u^{-\theta}, v^{-\theta}) \tag{4}$$

with $\theta \in [0,1]$. The case $\alpha = \beta$ appeared first in Cuadras and Augé (1981).

The parameter θ reflexes the dependence structure between the marginals, that are positively dependent, from the stochastic independent situation $(\theta = 0)$ to the situation of co-monotonicity $(\theta = 1)$. For different values of θ we find several copulae in the Fréchet-Hoeffding class

$$\prod(u,v) = uv \le C(u,v) \le M(u,v) = min(u,v).$$

If $\theta = 0$ the marginals are independent and the copula is the independence copula $\prod(u,v)$; if $\theta = 1$ the variables are co-monotonic and the copula is $M(u,v)$.

By the copula we can calculate the association measures between the variables. For example the Kendall's τ is given by

$$\tau = \theta/(2-\theta) > 0 \tag{5}$$

so if $\theta = 0$ we have $\tau = 0$ and if $\theta = 1$ we have $\tau = 1$.

Moreover, the MOC is not absolutely continuous: $C(u,v)$ is absolutely continuous in $\{u,v : u < v \cup u > v\}$ and it has a sinular component in $u = v$, with a positive probability $P(u = v) = \theta/(2-\theta)$ equal to τ.

By the Sklar's theorem we can use the Marshall-Olkin copula and several marginal exchangeable random variables to construct several bivariate survival distributions for dependent r.v.s. For example, the celebrate bivariate exponential model of Marshall and Olkin is a special case of this construction. In section 2 we present also an other model, the bivariate Weibull distribution usually used to study the reliability of complex systems in which the two component are not independent. These distributions generated by the MOC are not absolutely continuous, except in the trivial case of $\theta = 0$.

In section 4 we discuss the problem on the estimation of their parameters. The problem is generally solved by the maximum likelihood methods. Beims et al. (Baims et al., 1972) discuss on the estimation of the Marshall and Olkin bivariate exponential distribution but the solution not always exist and it cannot be obtained in explicit form. An iterative procedure is also used: Kundu and Dey (Kundu and Day, 2009) propose an EM algorithm for the estimation of bivariate and multivariate Weibull distribution parameters.

In the copula literature, parametric and non parametric estimators are proposed. The maximum likelihood criteria, the Joe's IFM methods and the empirical copula are generally used, but they are not specific for the distributions generated by the Marshall-Olkin copula.

Since it is possible for some copula models to find the relationship between the copula parameter and the association measures, like Kendall's tau or Spearman's rho, it is possible to use these measures to estimate the copula parameter (see for example Ocana, 1990). For the Marshall-Olkin model the relationship between the Kendall's tau and the θ parameter is describe in (5). So we can estimate θ by the estimator $\hat{\theta} = 2\tau/(1+\tau)$.

A recent proposal by Hering and Mai (2011) presents an estimation strategy (moment-based estimation) based on the minimization of the Euclidean distance between certain empirical and theoretical functionals of an extendible d-dimentional Marshall-Olkin distributions associated with a parametric family of Lévy subordinators.

One main problem with the method of moments for a bivariate distribution with a singular component is that it makes no use to the mass on the singular component. In this paper we propose an easy procedure based on the moments and on the copula function that overcame this problem. We find the parameter estimators, with a simpler anaclitic form, for a class of not-absolutely continuous bivariate distributions generated by Marshall-Olkin copula.

We analyze the performance of the procedure and the asymptotic properties of the estimators by several experiment simulations, considering complete and censoring sample. We conclude the paper comparing the proposed method with the one based on Kendall's tau.

3. TWO MODELS FROM MARSHALL-OLKIN COPULA

3.1 *The bivariate Marshall-Olkin exponential distribution*

Suppose to be interested in the evaluation of the reliability of a system with two dependent components that are subject to failure. Suppose that the lifetime marginal random variables X and Y related to the two components are distributed with same exponential law with rate of failure λ_1^*. Let

$$u = \overline{F}_X(x) = \exp(-\lambda_1^* x) \quad v = \overline{F}_Y(y) = \exp(-\lambda_1^* y)$$

and

$$\theta = \frac{\lambda_{1,2}}{\lambda_1 + \lambda_{1,2}} \quad \lambda_1^* = \lambda_1 + \lambda_{1,2} \tag{6}$$

Using the copula in (4) and the result in the Sklar's theorem, we obtain the Marshall-Olkin exponential distribution (Marshall and Olkin, 1969), in which both univariate marginals have the same mean. The reliability function could be expressed as

$$\overline{F}(x,y) = P(X > x, Y > y) = \exp\{-\lambda_1 x - \lambda_1 y - \lambda_{1,2}\max(x,y)\} \tag{7}$$

with $x \geq 0$, $y \geq 0$ and λ_1 and $\lambda_{1,2} > 0$. If X and Y are not identically distributed we have

$$\overline{F}(x,y) = P(X > x, Y > y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{1,2}\max(x,y)\} \tag{8}$$

The parameters λ_1 and λ_2 in (7) are the reliability parameters related to the failures of only the first and second components, respectively, while the parameter λ_{12} is related to the contemporary failures of both components.

The marginal random variables are exponential with cumulative distribution functions

$$F_X(x) = 1 - \exp(-\lambda_1^* x) \quad x \geq 0$$

$$F_Y(y) = 1 - \exp(-\lambda_2^* y) \quad y \geq 0$$

where $\lambda_1^* = \lambda_1 + \lambda_{1,2}$ and $\lambda_2^* = \lambda_2 + \lambda_{1,2}$ are the rate of failure parameters of the two components. If $\lambda_{1,2} = 0$ the marginal random variables are independent and so the failure of one of the components does not imply the failure of the other.

An interesting feature of the distribution in (7) is that the bivariate variable (X,Y) in is not absolutely continuous in $\Re^2$, as the corresponding copula. It is absolutely continuous in $\{(x,y): x > y \cup y > x\}$ with probabilities

$$P(Y < X) = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_{1,2}} = \frac{\lambda_2}{\lambda} \tag{9}$$

$$P(X < Y) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_{1,2}} = \frac{\lambda_1}{\lambda}, \tag{10}$$

with $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, and it has a singular component in the region defined by the condition $x = y$, with non null probability

$$P(Y = X) = \frac{\lambda_{1,2}}{\lambda_1 + \lambda_2 + \lambda_{1,2}} = \frac{\lambda_{1,2}}{\lambda}. \tag{11}$$

The event $X=Y$ occurs when the failure is caused by a simultaneous shock felt of both components. This event has a positive probability in the case $\lambda_{12} > 0$. Therefore, the survival function can be written as a linear combination of the absolutely continuous part $\overline{F}_c$ and the singular one $\overline{F}_s$:

$$\overline{F}(x,y) = \frac{\lambda_1 + \lambda_2}{\lambda}\overline{F}_a(x,y) + \frac{\lambda_{12}}{\lambda}\overline{F}_s(x,y). \tag{12}$$

where $\overline{F}_s(x) = \exp\{-\lambda x\}$ if $x = y$ and $\overline{F}_c$ is obtained by subtraction if $x \neq y$.

Treating the absolutely continuous and singular part of $\overline{F}$ separately the moment generating function is given by

$$\begin{aligned}\psi(s,t) = E(e^{sX+tY}) = \iint_{x>y} e^{-sx-ty}\lambda_2(\lambda_1+\lambda_{12})\overline{F}(x,y)dxdy \\ +\iint_{x<y} e^{-sx-ty}\lambda_1(\lambda_2+\lambda_{12})\overline{F}(x,y)dxdy + \int_0^\infty \lambda_{12}\overline{F}_s(x,y)dx\end{aligned} \tag{13}$$

In according to Marshall and Olkin (Marshall and Olkin, 1969), we obtain

$$\psi(s,t) = \frac{(\lambda+s+t)(\lambda_1+\lambda_{1,2})(\lambda_2+\lambda_{1,2}) + st\lambda_{1,2}}{(\lambda+s+t)(\lambda_1+\lambda_{1,2}+s)(\lambda_2+\lambda_{1,2}+t)}.$$

So the first two moments and the mixed moment are

$$\mu_x = \frac{\vartheta\psi(s,t)}{\vartheta s}\Big|_{s=0,t=0} = \frac{1}{\lambda_1^*} = \frac{1}{\lambda_1+\lambda_{1,2}}$$

$$\mu_y = \frac{\vartheta\psi(s,t)}{\vartheta t}\Big|_{s=0,t=0} = \frac{1}{\lambda_2^*} = \frac{1}{\lambda_2+\lambda_{1,2}}$$

$${}_2\mu_x = \frac{\vartheta^2\psi(s,t)}{\vartheta s^2}\Big|_{s=0,t=0} = \frac{2}{(\lambda_1^*)^2} = \frac{2}{(\lambda_1+\lambda_{1,2})^2}$$

$${}_2\mu_y = \frac{\vartheta^2\psi(s,t)}{\vartheta t^2}\Big|_{s=0,t=0} = \frac{2}{(\lambda_2^*)^2} = \frac{2}{(\lambda_2+\lambda_{1,2})^2}$$

$$\mu_{x,y} = \frac{\vartheta\psi(s,t)}{\vartheta s\vartheta t}\Big|_{s=0,t=0} = \frac{1}{\lambda}\left(\frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}\right)$$

The correlation coefficient is $\rho = \frac{\lambda_{1,2}}{\lambda}$ and it is positive. Note that the linear correlation coefficient corresponds to the probability in (10).

3.2. *The bivariate Weibull distribution*

Suppose that the lifetime random variables X and Y related to two components are identically distributed as a Weibull with scalar parameter λ_1^* and shape parameter ν_1. Consider the copula in (4) and the result in the Sklar's theorem. By the quantile transformation $u = \overline{F}_X(x) = \exp(-\lambda_1^* x^{\nu_1})$ and $v = \overline{F}_Y(y) = \exp(-\lambda_1^* y^{\nu_1})$ and the equations in (5), we obtain the reliability function

$$\overline{F}(x, y) = P(X > x, Y > y) = \exp\{-\lambda_1 x^{\nu_1} - \lambda_1 y^{\nu_1} - \lambda_{1,2} \max(x^{\nu_1}, y^{\nu_1})\} \tag{14}$$

with $x \geq 0$, $y \geq 0$, $\lambda_1, \lambda_{1,2} > 0$ and $\nu_1 > 0$.

If the marginal random variables X and Y have different scalar and shape parameters, (λ_1^*, ν_1) and (λ_2^*, ν_2) respectively, the survival function becomes

$$\overline{F}(x, y) = P(X > x, Y > y) = \exp\{-\lambda_1 x^{\nu_1} - \lambda_2 y^{\nu_2} - \lambda_{1,2} \max(x^{\nu_1}, y^{\nu_2})\} \tag{15}$$

with $x \geq 0$, $y \geq 0$, $\lambda_1, \lambda_2 \lambda_{1,2} > 0$ and $\nu_1, \nu_2 > 0$. This distribution has the same properties of the Marshall-Olkin one: if $\nu_1 = \nu_2 = 1$ we find the cumulative distribution function in (7).

The marginal random variables X and Y have survival functions

$$\overline{F}_X(x) = \exp\{-(\lambda_1 + \lambda_{12}) x^{\nu_1}\} = exp\{-\lambda_1^* x^{\nu_1}\}$$

$$\overline{F}_Y(y) = \exp\{-(\lambda_2 + \lambda_{12}) y^{\nu_2}\} = exp\{-\lambda_2^* y^{\nu_2}\}$$

with $\lambda_1^* = \lambda_1 + \lambda_{12}$, $\lambda_2^* = \lambda_2 + \lambda_{12}$; they are independent if $\lambda_{12} = 0$.

The distribution in (14) is not absolutely continuous in $\Re^2$. It is continuous for $x > y^{\nu_2/\nu_1}$ and $x < y^{\nu_2/\nu_1}$ and it has a singular component in the region defined by the condition $x = y^{\nu_2/\nu_1}$. The related probabilities are equal to the probabilities defined in (8), (9) and (10). Therefore, the survival function can be expressed as

$$\overline{F}_{X,Y}(x, y) = \frac{\lambda_1 + \lambda_2}{\lambda} \overline{F}_a(x, y) + \frac{\lambda_{12}}{\lambda} \overline{F}_s(x, y) \tag{16}$$

with $\overline{F}_s(x, y) = \exp\{-\lambda x^{\nu_1}\}$ if $x^{\nu_1} = y^{\nu_2}$ and $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.

By the (15) we find the characteristic function

$$\begin{aligned} \varphi(s, t) &= E(e^{sX + tY}) = \\ &\iint_{x^{\nu_1} > y^{\nu_2}} e^{-sx - ty} \lambda_2 (\lambda_1 + \lambda_{12}) \nu_1 x^{\nu_1 - 1} \nu_2 y^{\nu_2 - 1} \overline{F}(x, y) dy dx + \\ &+ \iint_{x^{\nu_1} < y^{\nu_2}} e^{-sx - ty} \lambda_1 (\lambda_2 + \lambda_{12}) \nu_1 x^{\nu_1 - 1} \nu_2 y^{\nu_2 - 1} \overline{F}(x, y) dx dy + \\ &+ \int_0^\infty x^2 \lambda_{12} \nu_1 x^{\nu_1 - 1} \overline{F}_s(x, y) dx \end{aligned} \tag{17}$$

The first and the second moments of the marginal random variable X are, respectively,

$$\mu_X = \frac{\Gamma\left(1+\frac{1}{\nu_1}\right)}{(\lambda_1^*)^{2/\nu_1}} \tag{18}$$

$$_2\mu_X = \frac{\Gamma\left(1+\frac{2}{\nu_1}\right)}{(\lambda_1^*)^{2/\nu_1}} \tag{19}$$

By using the (16) it is difficult to find an explicit form of the mixed moment (Chiodini, 1998).

4. ESTIMATING THE MARSHALL-OLKIN COPULA PARAMETER BY THE METHOD OF MOMENTS

In literature the problem of the estimation of multivariate distribution or copula parameters is usually solved by maximum likelihood methods (see for example Bhattacharyya and Johnson, 1973 and Proschan and Sullo, 1976). For the distributions generating by the Marshall-Olkin copula, the maximum likelihood estimators cannot be obtained in explicit form. In this work we propose a moment-based procedure which is preferred to other estimation methods for its simplex mathematic form.

By the copula approach we can estimate the parameters in two steps, separating the estimate of marginal distribution parameters and copula one (see Osmetti and Chiodini, 2008).

The proposed procedure is somehow related to Joe's two-step procedure (Joe, 1997, 2005). Joe's procedure is called inference functions of margins or IFM method. It consists in estimating the parametrs by the maximum likelihood criteria in two step. First the likelihod function of the marginal random variable are maximized in order to estimate the parameters of the marginal distributions. Than by the maximization of the copula likelihood function, the copula parameter is estimated. This method is generally used when the problem of maximization could be difficult, when the dimension is high and the number of parameters is large and it is necessary an iterative procedure. In this paper we propose e similar procedure based on the moments.

4.1. *Complete sampling*

Consider the model $\overline{F}(x,y) = C(\overline{F}_X(x), \overline{F}_Y(y))$ where $\overline{F}_X(x)$ and $\overline{F}_Y(y)$ are the survival functions of two exchangeable random variables that depend on a generic parameters vector $\underline{\lambda}$, and C is the Marshall-Olkin copula dependent on θ parameter. By the copula approach we can estimate the parameters in two steps, separating the estimate of marginal distribution parameter $\underline{\lambda}$ and copula one θ.

Step I - We use the sample observations to find the estimator $\hat{\underline{\lambda}}$ of the vector $\underline{\lambda}$ of the marginal random variables by the usually method of moments.
We consider for example the exponential and the Weibull distributions.

Example 4.1 (Exponential distribution). We consider two exchangeable exponential random variables with rate of failure parameter $\underline{\lambda} = (\lambda_1^*)$ and survival function

$$\overline{F}_{X_1}(x_1) = \exp\{-\lambda_1^* x_1\}$$

The parameter estimate is a function of the sample mean: $\hat{\lambda}_1^* = \dfrac{1}{\overline{x}_1}$.

Example 1.2 (Weibull distribution). We consider two exchangeable Weibull random variables with parameter vector $\underline{\lambda} = (\nu_1, \lambda_1^*)$ and survival function

$$\overline{F}_{X_2}(x_2) = \exp(-\lambda_1^* x_2^{\nu_1})$$

where ν_1 is the shape parameter and λ_1^* is the scalar parameter. The fraction between the moments in (17) and in (18) is a function of the shape parameter

$$\frac{(\mu_{X1})^2}{{}_2\mu_{X_2}} = \frac{\left(\Gamma\left(1+\frac{1}{\nu_1}\right)\right)^2}{\Gamma\left(1+\frac{2}{\nu_1}\right)} \tag{20}$$

Then, we estimate ν_1 by (19) with a iterative method, such that

$$\frac{(\overline{x}_2)^2}{{}_2\overline{x}_2} = \frac{\left(\Gamma\left(1+\frac{1}{\hat{\nu}_1}\right)\right)^2}{\Gamma\left(1+\frac{2}{\hat{\nu}_1}\right)},$$

where $\overline{x}_2$ and ${}_2\overline{x}_2$ are the first and the second sample moments, respectively.

Subsequently, using $\hat{v}_1$ we transform the Weibull random variable in an exponential random variable with rate of failure λ_1^*: by the sample values $\{x_{2,i}\}$ we obtain the values $x_{1,i} = (x_{2,i})^{\hat{v}_1}$. Then, we estimate the λ_1^* parameter with the sample mean $\overline{x}_1$, as $\hat{\lambda}_1^* = \frac{1}{\overline{x}_1}$.

Step II - After estimating the parameter vector $\underline{\lambda}$ of the two marginal distributions, we estimate the copula parameter θ in (4) by the method of moments. We find the values $\hat{u}_i = [F_X(x_i, \hat{\underline{\lambda}})]^{-1}$ and $\hat{v}_i = [F_Y(y_i, \hat{\lambda})]^{-1}$. By the Sklar's theorem we make the transformations $\hat{u}_i = \exp\{-z_{1,i}\}$ and $\hat{v}_i = \exp\{-z_{2,i}\}$. Therefore, we change the copula in the Marshall and Olkin bivariate exponential survival function

$$\begin{aligned}\overline{F}(z_1, z_2) &= C_\theta(\overline{F}_{Z_1}(z_1), \overline{F}_{Z_2}(z_2)) \\ &= \exp\{-(1-\theta)z_1 - (1-\theta)z_2 - \theta \max(z_1, z_2)\}.\end{aligned} \tag{21}$$

in which the marginal random variables are exponential with the mean equal to one. Then, we obtain the based-moment estimator of θ. By the moment generating function (12), in according to Marshall and Olkin, we calculate the mixed moment of the random variables $Z_1 = -ln(u)$ and $Z_2 = -ln(v)$

$$E(Z_1 Z_2) = \frac{2}{(2-\theta)}, \tag{22}$$

that is a function of θ. Note that in (12) we consider also the singular component of the c.d.f.

Therefore, we estimate the copula parameter θ as a function of mixed moment

$$\hat{\theta} = 2 - \frac{2}{E(z_1 z_2)} = 2 - \frac{2}{E(-ln(\hat{u}), -ln(\hat{v}))} \tag{23}$$

4.2. *Censored sampling*

Consider now a case of II type censored sample from a bivariate random variable (X, Y) generated by the Marshall-Olkin copula with several marginal distributions. In the univariate case the II Type censoring consist to terminate the experiment after the observation of a fraction k/n of sample units at the k-th value of the ordered sample units $\{x_{1:n}, x_{2:n}, ..., x_{k:n}, ..., x_{n:n}\}$; the last observed value represents the length of the experiment and its value is assigned to the all $(n-k)$ not observed units. In the bivariate case, the II Type censoring consist to terminate the experiment once observed a fraction k/n of the system failures or of the bivariate sample units. Let be a sample of size n from the bivariate random variable (X,Y). We defined a variable W, with sample value $w_i = \max(x_i, y_i)$. By the order of the values $(w_{1:n},..,w_{k:n},...,w_{n:n})$, we order the bivariate observation $\{(x_i, y_i); i = 1,2,...,n\}$ obtaining the order statistics $\{(x_{i:n}, y_{i:n}); i = 1,2,...,n\}$. We terminate the experiment at the point $(x_{k:n}, y_{k:n})$ related to the k-th order value $w_{k:n}$. This value is the length of the experiment.

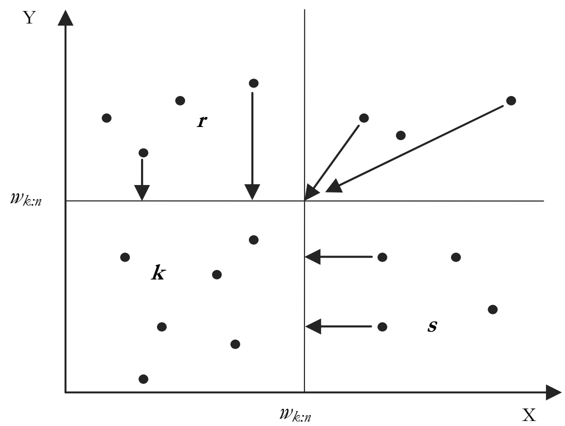

Figure 1 – Use of sample censored data.

In simulation procedure we proceed as shown in the Figure 1: consider the k values observed in both components, the r values observed for X only (such that $x_i \leq w_{k:n}$ and $y_i > w_{k:n}$), the s values observed for Y only (such that $y_i \leq w_{k:n}$ and $x_i > w_{k:n}$) and the other not observed values, in one or both components, that are set equal to $w_{k:n}$.

We note that the fractions of the observations number for the marginal random variables are grater than the fixed fraction k/n: equal to $p = \frac{k+r}{n}$ for X and $p = \frac{k+s}{n}$ for Y.

5. MONTE CARLO SIMULATIONS

The estimation procedure is verified by several simulations (Monte Carlo method) both for complete and censored sampling. We generate 2000 samples with a growing sample size n (200, 500 and 1000), from several bivariate survival distributions, in which the marginal random variables are exchangeable. These distributions are generated using the Marshall-Olkin copula with different values of copula parameter $\theta = 0.1, 0.7, 0.9$ and several marginal distributions, exponential and Weibull. The values of θ correspond to the values of Kendall's association measure τ equal to 0.81 (high positive association), 0.5 (mean association) and 0.1 (low association). The data are generted from the Marshall and Olkin copula by the following algorithm (see Devroye, 1987).

Algorithm

1. Generate three independent uniform in [0,1] variables *r*, *s* ad *t*.
2. Set $z_1 = \min\left(\frac{-\ln(r)}{(1-\theta)}; \frac{-\ln(t)}{\theta}\right)$ and $z_1 = \min\left(\frac{-\ln(s)}{(1-\theta)}; \frac{-\ln(t)}{\theta}\right)$.
3. Set $u = \exp(-z_1)$ and $v = \exp(-z_2)$
4. The desidered values of the copula variables are (*u,v*).

We estimate the parameters of these distributions by the proposed procedure, in two steps: estimating the marginal random variables parameters with the method of moments and estimating the copula parameter. The goodness of the estimates is valued calculating the bias (B) and the mean square error (MSE).

The table 1 and table 2 describe the results obtained in the case of a complete sample from a distribution generated by the Marshall-Olkin copula in (4) and exponential marginal distributions with several values of the rate of failure parameter λ_1^* equal to 0,7, 1 and 1.5 (correspondent to the mean <1 and>1). We esti-

TABLE 1

Exponential distribution: complete sampling

θ	λ^*	n	λ_1^*		θ		Time
			B	MSE	B	MSE	
0.9	0.7	200	-0.0044	0.0026	0.0294	0.0110	8.9078
		500	-0.0018	0.0010	0.0115	0.0023	10.283
		1000	-0.0006	0.0005	0.0051	0.0012	11.283
	1	200	-0.0062	0.0052	0.0266	0.0060	6.2856
		500	-0.0020	0.0021	0.0106	0.0024	7.2352
		1000	-0.0014	0.0010	0.0077	0.0012	7.9642
	1.5	200	-0.0097	0.0125	0.0315	0.0061	4.1643
		500	-0.0017	0.0046	0.0072	0.0024	4.8064
		1000	-0.0031	0.0023	0.0072	0.0011	5.2769
0.7	0.7	200	-0.0039	0.0025	0.0260	0.0168	9.2972
		500	-0.0017	0.0010	0.0106	0.0033	10.671
		1000	-0.0011	0.0005	0.0091	0.0016	11.577
	1	200	-0.0051	0.0052	0.0256	0.0083	6.4964
		500	-0.0026	0.0020	0.0139	0.0032	7.4739
		1000	-0.0008	0.0010	0.0041	0.0017	8.1151
	1.5	200	-0.0049	0.0116	0.0236	0.0085	4.3464
		500	-0.0039	0.0046	0.0144	0.0034	4.9462
		1000	-0.0018	0.0023	0.0036	0.0017	5.4300
0.1	0.7	200	-0.0008	0.0025	0.0174	0.0359	6.5422
		500	-0.0020	0.0010	0.0147	0.0073	7.5290
		1000	-0.0011	0.0005	0.0084	0.0036	8.1849
	1	200	-0.0030	0.0053	0.0249	0.0183	6.5865
		500	-0.0020	0.0019	0.0112	0.0069	7.5002
		1000	-0.0010	0.0010	0.0049	0.0035	8.1857
	1.5	200	-0.0041	0.0093	0.0292	0.0181	6.4943
		500	-0.0021	0.0044	0.0099	0.0071	7.4750
		1000	-0.0033	0.0023	0.0083	0.0039	8.1740

TABLE 2

Estimates of the exponential reliability parameters: complete sampling

θ	λ_1^*	λ_1	λ_{12}	n	λ_1		λ_{12}	
					B	MSE	B	MSE
0.9	0.7	0.07	0.63	200	-0.0286	0.0178	0.0242	0.0087
				500	-0.0111	0.0064	0.0094	0.0030
				1000	-0.0051	0.0033	0.0045	0.0015
	1	0.1	0.9	200	-0.0378	0.0357	0.0317	0.0177
				500	-0.0150	0.0133	0.0130	0.0063
				1000	-0.0099	0.0062	0.0084	0.0030
	1.5	0.15	1.35	200	-0.0657	0.0888	0.0559	0.0440
				500	-0.0172	0.0297	0.0155	0.0143
				1000	-0.0142	0.0144	0.0111	0.0068
0.7	0.7	0.21	0.49	200	-0.0265	0.0222	0.0226	0.0127
				500	-0.0109	0.0087	0.0092	0.0049
				1000	-0.0081	0.0040	0.0070	0.0022
	1	0.3	0.7	200	-0.0379	0.0462	0.0328	0.0261
				500	-0.0188	0.0171	0.0162	0.0096
				1000	-0.0062	0.0081	0.0054	0.0046
	1.5	0.45	1.05	200	-0.0532	0.1111	0.0484	0.0642
				500	-0.0288	0.0394	0.0249	0.0224
				1000	-0.0089	0.0188	0.0071	0.0105
0.1	0.7	0.63	0.07	200	-0.0198	0.0408	0.0190	0.0289
				500	-0.0148	0.0158	0.0129	0.0110
				1000	-0.0081	0.0075	0.0071	0.0053
	1	0.9	0.1	200	-0.0383	0.0874	0.0353	0.0612
				500	-0.0168	0.0316	0.0148	0.0223
				1000	-0.0078	0.0162	0.0068	0.0113
	1.5	1.35	0.15	200	-0.0632	0.1966	0.0592	0.0790
				500	-0.0230	0.0724	0.0208	0.0498
				1000	-0.0186	0.0371	0.0153	0.0255

mate λ_1^*, θ and the reliability parameters λ_1 and λ_{12} described in section 3.1. We see a good stability of the estimates: the bias and the MSE are usually lower than the one tenth of the real parameter. Once *n* increase we see an improving of the parameter estimates due to a strong decrease of the bias and the MSE, that are inversely proportional reading *n*.

In the table 3 and table 4 we show the results obtained in the case of a bivariate distribution generating by a Marshall-Olkin copula and marginal exchangeable Weibull random variables. We consider several values of the Weibull parameters: $\nu_1 = 2$ for the shape parameter and λ_1^* equal to 0.7, 1 and 1.5 for the scalar one. In this case we calculate the bias and the MSE. We obtain good results: the bias and the MSE are lower than one tenth of the parameter real value.

The shape parameter estimation is obtained by the iterative method described in section 4.1; the efficiency of the estimator $\hat{\nu}_1$ is valued, either in the case of complete and censored sample, by comparing its variance $Var(\hat{\nu}_1)$ with the lower limit of the Rao-Cramér inequality (RCL). For marginal X the Rao-Cramér inequality is:

$$Var(\hat{\nu}_1) \geq \frac{1}{-n\left[-\frac{1}{\nu_1^2} - (\lambda_1 + \lambda_{12})E(X^{\nu_1}(lnX)^2\right]} \tag{24}$$

TABLE 3

Weibull distribution: complete sampling

θ	λ_1^*	n	ν_1				λ_1^*		θ	
			B	MSE	V	*Eff*	B	MSE	B	MSE
0,9	0,7	200	-0,012	0,013	0,013	0,707	-0,003	0,003	0,029	0,062
		500	-0,008	0,005	0,005	0,719	0,000	0,001	0,012	0,011
		1000	-0,004	0,003	0,003	0,674	0,000	0,001	0,005	0,006
	1	200	-0,013	0,012	0,012	0,904	-0,004	0,006	0,026	0,031
		500	-0,005	0,005	0,005	0,866	-0,001	0,002	0,008	0,011
		1000	-0,003	0,002	0,002	0,905	-0,001	0,001	0,008	0,006
	1,5	200	-0,016	0,013	0,013	0,939	-0,009	0,011	0,029	0,031
		500	-0,007	0,005	0,005	0,965	-0,003	0,005	0,012	0,011
		1000	-0,003	0,003	0,003	0,967	-0,002	0,002	0,006	0,006
0,7	0,7	200	-0,017	0,013	0,013	0,688	0,000	0,004	0,026	0,075
		500	-0,010	0,005	0,005	0,700	0,000	0,001	0,013	0,015
		1000	-0,002	0,002	0,002	0,734	0,000	0,001	0,000	0,007
	1	200	-0,015	0,013	0,013	0,856	-0,004	0,006	0,027	0,034
		500	-0,003	0,005	0,005	0,878	-0,003	0,002	0,014	0,015
		1000	-0,004	0,003	0,003	0,851	0,000	0,001	0,003	0,008
	1,5	200	-0,016	0,014	0,014	0,897	-0,010	0,012	0,029	0,038
		500	-0,004	0,005	0,005	0,922	-0,002	0,005	0,008	0,014
		1000	-0,005	0,002	0,002	0,986	-0,003	0,002	0,008	0,007
0,1	0,7	200	-0,013	0,013	0,013	0,714	-0,003	0,003	0,033	0,016
		500	-0,007	0,005	0,005	0,697	0,001	0,001	0,009	0,013
		1000	-0,003	0,003	0,003	0,701	-0,001	0,001	0,008	0,008
	1	200	-0,014	0,013	0,013	0,847	-0,002	0,006	0,023	0,016
		500	-0,009	0,005	0,005	0,901	0,000	0,002	0,008	0,014
		1000	-0,004	0,003	0,003	0,857	-0,001	0,001	0,007	0,011
	1,5	200	-0,014	0,013	0,012	0,984	-0,004	0,011	0,024	0,019
		500	-0,006	0,005	0,005	0,997	-0,004	0,005	0,009	0,011
		1000	-0,003	0,003	0,003	0,920	-0,001	0,002	0,007	0,009

TABLE 4

Estimates of the Weibull reliability parameters: complete sampling

θ	λ_1^*	λ_1	λ_{12}	n	λ_1		λ_{12}		Time
					B	MSE	B	MSE	
0,9	0,7	0,07	0,63	200	-0,026	0,017	0,023	0,011	2.984
				500	-0,011	0,006	0,011	0,004	3.182
				1000	-0,005	0,003	0,005	0,002	3.362
	1	0,1	0,9	200	-0,036	0,036	0,032	0,020	2.498
				500	-0,012	0,013	0,011	0,007	2.675
				1000	-0,010	0,006	0,008	0,004	2.807
	1,5	0,15	1,35	200	-0,060	0,079	0,051	0,040	2.030
				500	-0,024	0,029	0,021	0,014	2.181
				1000	-0,013	0,014	0,011	0,007	2.285
0,7	0,7	0,21	0,49	200	-0,025	0,024	0,025	0,015	3.043
				500	-0,011	0,008	0,011	0,005	3.243
				1000	-0,001	0,004	0,001	0,003	3.402
	1	0,3	0,7	200	-0,038	0,047	0,034	0,029	2.542
				500	-0,018	0,017	0,016	0,010	2.728
				1000	-0,005	0,008	0,005	0,005	2.846
	1,5	0,45	1,05	200	-0,064	0,111	0,053	0,063	2.070
				500	-0,019	0,038	0,017	0,021	2.218
				1000	-0,016	0,020	0,012	0,011	2.317
0,1	0,7	0,63	0,07	200	-0,033	0,042	0,030	0,029	3.042
				500	-0,009	0,016	0,009	0,011	3.250
				1000	-0,008	0,008	0,007	0,006	3.406
	1	0,9	0,1	200	-0,035	0,085	0,033	0,059	2.553
				500	-0,012	0,033	0,012	0,023	2.720
				1000	-0,010	0,016	0,009	0,011	2.847
	1,5	1,35	0.15	200	-0,055	0,194	0,051	0,137	2.086
				500	-0,023	0,068	0,019	0,047	2.220
				1000	-0,015	0,035	0,013	0,024	2,322

with $\hat{v}_1$ an unbiased estimator of ν_1. By $Eff = RCL/Var(\hat{v}_1)$ we show that the shape parameter estimate is close to the efficiency: the variance is close to the lower limit of the inequality (23).

The lower limit of Rao-Cramér inequality is calculated either for complete and censored sample using the sample size *n* of the complete sample in order to obtained in the two situation comparable results for Eff (see Chiodini, 1998).

We obtain satisfying results also for II type censored sampling. The censoring is done after the observation of 80% of the system failures. The use of the observations is described in figure1.

In the tables 5 and 6 and in the tables 7 and 8 are shown the results for exponential marginal distributions and Weibull distributions, respectively.

In the censored sampling we observe for both distributions an increase of the bias and the MSE compared with the complete sampling. However, we obtain a decrease of the values with an increase of the sample size. Moreover, we show a considerable reduction of the length of the experiment (Censored Time) equal to the expected value of $W_{k:n}$ compared with the one (Time) equal to the expected value of $W_{n:n}$ in the complete sampling; therefore, there are an obvious decrease of the cost of the experiment.

TABLE 5

Exponential distribution: censored sampling

θ	λ_1^*	n	λ_1^*		θ		Censored Time
			B	MSE	B	MSE	
0.9	0.7	200	-0.1533	0.0276	0.2759	0.0796	2.4888
		500	-0.1479	0.0236	0.2718	0.0752	2.5088
		1000	-0.1471	0.0225	0.2720	0.0746	2.5075
	1	200	-0.2140	0.0547	0.2750	0.0790	1.7528
		500	-0.2119	0.0485	0.2717	0.0752	1.7531
		1000	-0.2085	0.0452	0.2718	0.0746	1.7581
	1.5	200	-0.3290	0.1283	0.2748	0.0789	1.1607
		500	-0.3160	0.1077	0.2710	0.0747	1.1717
		1000	-0.3141	0.1027	0.2716	0.0745	1.1716
0.7	0.7	200	-0.1174	0.0173	0.5222	0.1778	2.8042
		500	-0.1138	0.0144	0.5260	0.1740	2.8241
		1000	-0.1132	0.0135	0.5251	0.1748	2.8232
	1	200	-0.1664	0.0353	0.5209	0.1791	1.9695
		500	-0.1640	0.0295	0.5226	0.1774	1.9708
		1000	-0.1604	0.0271	0.5257	0.1743	1.9797
	1.5	200	-0.2545	0.0812	0.5192	0.1808	1.3070
		500	-0.2459	0.0671	0.5250	0.1750	1.3150
		1000	-0.2411	0.0613	0.5245	0.1755	1.3184
0.1	0.7	200	-0.0860	0.0101	0.0335	0.0292	2.2241
		500	-0.0848	0.0083	0.0272	0.0118	2.2285
		1000	-0.0841	0.0076	0.0197	0.0063	2.2343
	1	200	-0.1254	0.0214	0.0276	0.0307	2.2241
		500	-0.1217	0.0170	0.0245	0.0118	2.2288
		1000	-0.1213	0.0159	0.0221	0.0066	2.2288
	1.5	200	-0.1889	0.0484	0.0293	0.0297	2.2229
		500	-0.1824	0.0381	0.0209	0.0120	2.2275
		1000	-0.1803	0.0350	0.0222	0.0060	2.2316

TABLE 6

Estimates of the exponential reliability parameters: censored sampling

θ	λ_1^*	λ_1	λ_{12}	n	λ_1		λ_{12}	
					B	MSE	B	MSE
0.9	0.7	0.07	0.63	200	-0.2502	0.0654	0.0969	0.0141
				500	-0.2450	0.0611	0.0971	0.0114
				1000	-0.2450	0.0606	0.0980	0.0105
	1	0.1	0.9	200	-0.3545	0.1314	0.1405	0.0294
				500	-0.3501	0.1248	0.1382	0.0230
				1000	-0.3493	0.1231	0.1407	0.0217
	1.5	0.15	1.35	200	-0.5345	0.2985	0.2055	0.0637
				500	-0.5234	0.2788	0.2073	0.0510
				1000	-0.5238	0.2768	0.2097	0.0485
0.7	0.7	0.21	0.49	200	-0.1788	0.0362	0.0614	0.0110
				500	-0.1751	0.0324	0.0613	0.0067
				1000	-0.1759	0.0319	0.0626	0.0053
	1	0.3	0.7	200	-0.2564	0.0750	0.0900	0.0235
				500	-0.2550	0.0688	0.0910	0.0140
				1000	-0.2498	0.0642	0.0894	0.0110
	1.5	0.45	1.05	200	-0.3899	0.1730	0.1354	0.0530
				500	-0.3778	0.1511	0.1319	0.0314
				1000	-0.3772	0.1461	0.1361	0.0251
0.1	0.7	0.63	0.07	200	-0.0991	0.0233	0.0131	0.0187
				500	-0.0958	0.0144	0.0110	0.0073
				1000	-0.0902	0.0109	0.0061	0.0037
	1	0.9	0.1	200	-0.1370	0.0485	0.0116	0.0405
				500	-0.1346	0.0291	0.0129	0.0149
				1000	-0.1326	0.0233	0.0113	0.0080
	1.5	1.35	0.15	200	-0.2100	0.1040	0.0211	0.0814
				500	-0.1956	0.0618	0.0132	0.0324
				1000	-0.1978	0.0521	0.0174	0.0177

TABLE 7

Weibull distribution: censored sampling

θ	λ_1^*	n	ν_1				λ_1^*		θ	
			B	MSE	V	*Eff*	B	MSE	B	MSE
0.1	0.7	200	-0.276	0.092	0.016	0.557	-0.025	0,006	0,0130	0,0095
		500	-0.265	0.076	0.006	0.587	-0.025	0,002	0,0092	0,0036
		1000	-0.262	0.072	0.003	0.600	-0.025	0,006	0,0068	0,0185
	1	200	-0.274	0.090	0.015	0.742	-0.090	0,015	0,0087	0,0109
		500	-0.264	0.076	0.006	0.720	-0.086	0,010	0,0076	0,0038
		1000	-0.261	0.071	0.003	0.691	-0.085	0,009	0,0077	0,0021
	1.5	200	-0.274	0.090	0.015	0.799	-0.227	0,068	0,0119	0,0096
		500	-0.265	0.076	0.006	0.791	-0.219	0,054	0,0086	0,0036
		1000	-0.262	0.072	0.003	0.787	-0.219	0,051	0,0070	0,0020
0.7	0.7	200	-0.352	0.140	0.016	0.557	0.453	0,037	0,0917	0,0101
		500	-0.337	0.121	0.007	0.501	0.452	0,038	0,0919	0,0094
		1000	-0.337	0.117	0.003	0.526	0.451	0,039	0,0923	0,0090
	1	200	-0.346	0.137	0.018	0.626	0.682	0,018	0,0944	0,0118
		500	-0.338	0.121	0.007	0.666	0.684	0,016	0,0925	0,0094
		1000	-0.338	0.118	0.004	0.628	0.684	0,016	0,0925	0,0090
	1.5	200	-0.352	0.142	0.018	0.686	1.106	-0,055	0,0946	0,0116
		500	-0.337	0.121	0.007	0.678	1.100	-0,050	0,0924	0,0096
		1000	-0.336	0.116	0.003	0.732	1.097	-0,047	0,0921	0,0091
0.9	0.7	200	-0.430	0.205	0.020	0.450	-0.068	0,010	0,1642	0,0278
		500	-0.426	0.189	0.008	0.454	-0.064	0,006	0,1621	0,0266
		1000	-0.419	0.180	0.004	0.436	-0.064	0,005	0,1613	0,0262
	1	200	-0.435	0.210	0.021	0.539	-0.186	0,047	0,1619	0,0276
		500	-0.422	0.186	0.008	0.548	-0.181	0,037	0,1618	0,0264
		1000	-0.421	0.182	0.004	0.539	-0.441	0,229	0,1618	0,0267
	1.5	200	-0.433	0.208	0.020	0.611	-0.429	0,197	0,1625	0,0276
		500	-0.428	0.191	0.008	0.610	-0.420	0,183	0,1624	0,0264
		1000	-0,422	0,182	0,004	0,621	-0,219	0,051	0,1619	0,0262

TABLE 8

Estimates of the Weibull reliability parameters: censored sampling

θ	λ_1^*	λ_1	λ_{12}	n	λ_1		λ_{12}		Censored Time
					B	MSE	B	MSE	
0.1	0.7	0.07	0.63	200	-0,0318	0,0096	0,0069	0.005	1,7838
				500	-0,0293	0,0042	0,0040	0.002	1,7864
				1000	-0,0273	0,0024	0,0024	0.001	1,4889
	1	0.1	0.9	200	-0,0906	0,0257	0,0003	0.015	1,4935
				500	-0,0859	0,0142	-0,0005	0.010	1,4939
				1000	-0,0853	0,0108	-0,0001	0.009	1,2160
	1.5	0.15	1.35	200	-0,2239	0,0933	-0,0027	0.069	1,2187
				500	-0,2114	0,0602	-0,0073	0.054	1,2190
				1000	-0,2093	0,0522	-0,0101	0.051	1,6750
0.7	0.7	0.21	0.49	200	-0,0811	0,0083	0,0374	0.037	1,6791
				500	-0,0812	0,0073	0,0382	0.038	1,6803
				1000	-0,0808	0,0069	0,0394	0.039	1,4054
	1	0.3	0.7	200	-0,1437	0,0248	0,0181	0.018	1,4052
				500	-0,1419	0,0217	0,0160	0.016	1,4048
				1000	-0,1416	0,0208	0,0163	0.016	1,1426
	1.5	0.45	1.05	200	-0,2699	0,0831	-0,0558	-0.056	1,1470
				500	-0,2603	0,0716	-0,0499	-0.050	1,1481
				1000	-0,2578	0,0684	-0,0474	-0.047	1,5792
0.9	0.7	0.63	0.07	200	-0,1326	0,0184	0,0648	0.010	1,5824
				500	-0,1303	0,0173	0,0660	-0.006	1,5836
				1000	-0,1297	0,0170	0,0653	0.005	1,3202
	1	0.9	0.1	200	-0,2106	0,0464	0,0249	0.047	1,3233
				500	-0,2090	0,0445	0,0283	0.037	1,3265
				1000	-0,2078	0,0433	0,0339	0.229	1,0788
	1.5	1.35	0.15	200	-0,3593	0,1347	-0,0814	0.198	1,0799
				500	-0,3561	0,1291	-0,0724	0.183	1,0821
				1000	-0,3528	0,1255	-0,0671	0,051	1,7816

Finaly, in order to investigate the performance of the estimation algorithm, we compare the proposed procedure with other methods genarally used in lectera-ture. In particular we consider the copula parameter estimation via Kendall's tau.

We consider a Marshall-Olkin distribution with two known exponential marginal distributions with parameter λ_1^* equal to 0.1 and 0.7. We estimate the copula parameter by using the proposed estimator in (23) and the estimator $\hat{\theta} = 2\tau/(1+\tau)$ based on the Kendall's tau of the Marshall and Olkin copula defined in (5). Since the copula is simmetric and the region of the discontinuity is a line, the marginal random variables have a linear dependency (they are correlated). Therefore, the copula parameter is related to the linerar correlation coefficient. Moreover, the Kendall's tau in (5) corresponds to the linear correlation coefficient ρ of the Marshall Olkin bivariate exponential distributions (see section 3.1).

In order to evaluate the goodness of the estimation we compare the bias and the MSE obteined with the two methods. The results, shown in table 9, are obteined by considering 2000 samples and several sample size *n*.

We note that the results obteined with our method are appreciable like the ones obteined with Kendall's tau: the values of the bias and the MSE obtained by the proposed estimator in (23) are close with the ones obteined by $\hat{\theta} = 2\tau/(1+\tau)$.

TABLE 9

Comparison between our proposed estimators and the estimator based on Kenfdall's tau

θ	λ_1^*	n	Our proposal		Kendall's tau	
			B	MSE	B	MSE
0.1	0.7	200	0,009	0,017	0,008	0,018
		500	0,002	0,006	0,002	0,006
		1000	0,001	0,003	0,001	0,003
	0.1	200	0,006	0,017	0,005	0,018
		500	0,004	0,006	0,004	0,007
		1000	0,003	0,004	0,003	0,004
0.3	0.7	200	0,009	0,015	0,006	0,015
		500	0,002	0,006	0,002	0,006
		1000	0,004	0,003	0,003	0,003
	0.1	200	0,009	0,015	0,007	0,014
		500	0,005	0,006	0,004	0,006
		1000	0,004	0,003	0,004	0,003

6. CONCLUSION

We have proposed an easy procedure to estimate the parameters of bivariate survival distributions used in reliability analysis, generated by the Marshal-Olkin copula. The estimation procedure is based on the moments and the copula approach. Thanks to the copula we have estimated the parameters in two step, separating the estimate of the marginal random variable parameters and the copula one. We presented also the estimator of copula parameter by moment-based method. Finally, we verified the estimate procedure with Monte Carlo experiment simulations in the cases of complete and censored sampling, in order to analyze the asymptotic properties of the estimators. In the simulation we estimated the parameters of the bivariate Marshall-Olkin distribution (obtained with a Marshall-Olkin copula and marginal exponential random variables) and a bivariate Weibull distribution (obtained with the copula and marginal Weibull random variables). We shown good results in the cases of complete and censored sample. For complete sampling we shown for the estimates low values of the bias and the mean square error. For censored sampling we obtained an increase of the bias and the mean square error compared with the complete sampling. Moreover, we shown obviously a considerable reduction of the length of the experiment and so a decrease of the cost of the experiment. This result should justify the use of the proposed methodology. This easy procedure, based on the moments and on the copula instrument, can be used to estimate the parameters of complex survival functions in which it is difficult to find an explicit expression of the mixed moments, as in bivariate Weibull distribution. Moreover this method is preferred to the maximum likelihood one for its simplex mathematic form; in particular for distributions whose maximum likelihood parameter estimators can not be obtained in explicit form.

Department of Statistical Science
Università Cattolica del Sacro Cuore
Milan, Italy
SILVIA ANGELA OSMETTI

Department of Statistics
Università degli Studi di Milano Bicocca
Milan, Italy
PAOLA MADDALENA CHIODINI

ACNOWLEDGEMENT

The authors wish to thank the professor Umberto Magagnoli for his useful discussions and suggestions.

REFERENCES

G.K. BHATTACHARYYA, R. A. JOHNSON (1973), *Maximum Likelihood Estimation and Hypothesis Testing in the Bivariate Exponential Model of Marshall and Olki*, "Journal of the American Statistical Association", 68(343), pp. 704-706.

B. BEIMS, L.J. BAIN, J.J. HIGGINS (1972), *Estimation and hypotesis testing for the parameters of a bivariate exponential distribution*, "Journal of American Statistical Association", 67, pp. 927-929.

P.M. CHIODINI (1998), *Una procedura di stima dei parametri della distribuzione di Weibull bivariata su dati censurati*, "Istituto di Statistica, Università Cattolica del S.Cuore", Serie E.P.N., 92, pp. 1-17.

C.M. CUADRAS, J. AUGÈ (1981), *A continuous general multivariate distribution and its properties*, "Communications in Statistics A–Theory Methods" 10(4), pp. 339-353.

L. DEVROYE (1986), *Non uniform Random Variate Generation*, Springer, New York.

P. EMBRECHTS, F. LINDSKOG, AND A. MCNEIL (2003), *Modelling dependence with copulas and applications to risk managemen,*. In S. Rachev (Ed.), Handbook of Heavy Tailed Distributions in Finance Elsevier.

N.I. FISHER (1997), *Copulas*, in S. Kotz, C.B. Read, D.L. Banks (eds.), "Encyclopaedia of Statistical Sciences", Wiley, New York, pp. 159-163.

C. HERING J-F. MAI (2011). *Moment-based estimation of extendible Marshall-Olkin copulas,* "Metrika", Online First.

H. JOE (2005), *Asymptotic efficiency of th two-stage estimation method for copula-based models*, "Journal of Multivariate Analysis", 94, pp. 401-419.

H. JOE (1997), *Multivariate Models ad Dependence Concepts,* Chapman & Hall, Boca Raton.

D. KUNDU, A.K. DEY (2009), *Estimating the parameters of the Marshall-Olkin bivarate Weibull distribution by EM algorithm*, "Computational Statitics and Data Analysis", 53, pp. 956-965.

A.W. MARSHALL, I. OLKIN (1967), *A Multivariate Exponential Distribution*, "Journal of the American Statistical Association", 62(317), pp. 30-40.

R.B. NELSEN (2006), *An Introduction to Copulas*, Springer, New York.

J. OCANA, C. RUIZ-RIVAS (1990), *Computer generation and estimation in a one-parameter system of bivariate distributions with specified marginals*, "Communications in Statistics - Simulation and Computation", 19 (1), pp. 37-35.

K. OWZAR, P.K. SEN (2003), *Copulas: concepts and novel applications*, "Metron", LXI(3), pp. 323-353.

S.A. OSMETTI, P. CHIODINI, *Some Problems of the Estimation of Marshall-Olkin Copula Parameters*, "Atti XLIV Riunione Scientifica della Società Italiana di Statistica", Arcavacata di Rende.

F. PROSCHAN, P. SULLO (1976), *Estimating the Parameters of a Multivariate Exponential Distribution*, "Journal of the American Statistical Association", 7(354), pp. 465-472.

J.H. SHIH, T.A. LOUIS (1995), *Inferences on the Association Parameter in Copula Models for Bivariate Survival Data*, "Biometrics", 51(4), pp. 1384-1399.

A. W. SKLAR (1959), *Fonctions de répartition à n dimension et leurs marges*, Publ. Inst. Statist. Univ. Paris, 8, pp. 229-231.

SUMMARY

A method of moments to estimate bivariate survival functions: the copula approach

In this paper we discuss the problem on parametric and non parametric estimation of the distributions generated by the Marshall-Olkin copula. This copula comes from the Marshall-Olkin bivariate exponential distribution used in reliability analysis. We generalize this model by the copula and different marginal distributions to construct several bivariate survival functions. The cumulative distribution functions are not absolutely continuous and they unknown parameters are often not be obtained in explicit form. In order to estimate the parameters we propose an easy procedure based on the moments. This method consist in two steps: in the first step we estimate only the parameters of marginal distributions and in the second step we estimate only the copula parameter. This procedure can be used to estimate the parameters of complex survival functions in which it is difficult to find an explicit expression of the mixed moments. Moreover it is preferred to the maximum likelihood one for its simplex mathematic form; in particular for distributions whose maximum likelihood parameters estimators can not be obtained in explicit form.