

MARSHALL-OLKIN GENERALIZED ASYMMETRIC LAPLACE DISTRIBUTIONS AND PROCESSES

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1. INTRODUCTION

Asymmetric Laplace distribution has received much attention in recent years. It can be used in modeling currency exchange rates, interest rates, stock price changes etc. With steeper peaks and heavier tails than normal distribution, Asymmetric Laplace laws reflect properties of empirical financial data sets much better than normal model. More recently several properties, generalizations and applications have been reported demonstrating that it is a natural and sometimes superior alternative to the conventional Gaussian distribution (Kotz *et al.*, 2001); The skew Laplace distribution has also gained an important role in Statistical analysis related to new emerging fields like Micro array modelling. (Bhowmick *et al.*, 2006); recently applied a Laplace and Asymmetric Laplace distribution for identification of differential expression in micro array experiments. In (Purdum and Holmes, 2005); an Asymmetric Laplace distribution is used to fit the gene expression distribution and its performance is compared to the Gaussian distribution. A general family of Asymmetric Probability density functions has been introduced by (Arellano *et al.*, 2004). (Julia and Vives-Rego, 2005); used Skew-Laplace distribution to model the bacterial sizes in axenic cultures. (Jayakumar and Kuttikrishnan, 2006); introduced and studied the properties of Marshall-Olkin Asymmetric Laplace distribution (Sim, 1994); discussed various issues such as diagnostic checking, inference etc with respect to non-normal time series modeling.

The Laplace random variable can be regarded as the difference of i.i.d. exponential random variables. Now we shall extend this to develop a generalized Laplace random variable which can be regarded as the difference of two gamma random variables. Let X_1 and X_2 be two independent random variables such that $X_1 \sim \text{Gamma}(\lambda_1, \beta_1)$ and $X_2 \sim \text{Gamma}(\lambda_2, \beta_2)$. Then $L = X_1 - X_2$ follows a generalized Asymmetric Laplace distribution with parameters $\lambda_1, \lambda_2, \beta_1, \beta_2$. When $\lambda_1 = \lambda_2 = \lambda, \beta_1 = \beta_2 = \beta$ we get symmetric generalized Laplace distribution introduced by (Mathai, 1993).

The characteristic function of the generalized Asymmetric Laplace distribution is

$$\phi(t) = \left(\frac{\lambda_1}{\lambda_1 - it} \right)^{\beta_1} \left(\frac{\lambda_2}{\lambda_2 + it} \right)^{\beta_2} \quad (1)$$

For $\beta_1 = \beta_2 = \beta$ we have,

$$\phi(t) = \frac{1}{(1 + t^2 \sigma^2 - i \mu t)^\beta}, \text{ where } \mu = \frac{(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}, \sigma^2 = \frac{1}{\lambda_1 \lambda_2}. \quad (2)$$

In section 2, we introduce the Marshall-Olkin generalised Asymmetric Laplace distribution $MOGAL(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ and discuss certain properties. In section 3, the approximated form of the new distribution is derived and its self-decomposability property is established. In section 4, we introduce two first order autoregressive (AR(1)) models with $MOGAL(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ as marginal distribution. Sample path properties are explored for the new model. In section 5, AR(1) model II is extended to k^{th} order. In section 6 the parameters are estimated by the method of m.l.e.'s and the distribution is fitted for a real data.

2. MARSHALL OLKIN GENERALIZED ASYMMETRIC LAPLACE DISTRIBUTION

Now we introduce a new parameter α using Marshall-Olkin method (for details see Marshall-Olkin 1997); to obtain a Marshall-Olkin generalized Asymmetric Laplace distribution denoted by $MOGAL(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$. Its characteristic function is given by

$$\begin{aligned} \psi(t) &= \frac{\alpha \phi(t)}{1 - (1 - \alpha) \phi(t)}, \quad \alpha > 0 \\ &= \frac{\alpha}{\left(1 - \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 + \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1}. \end{aligned} \quad (3)$$

Mean and variance of the $MOGAL(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ distribution are respectively,

$$\begin{aligned} \mu_1 &= \frac{\lambda_1 \beta_2 - \beta_1 \lambda_2}{\alpha \lambda_1 \lambda_2} \\ \mu_2 &= \frac{(\lambda_1 \beta_2 - \beta_1 \lambda_2)^2 - \alpha [\beta_1 (\beta_1 - 1) \lambda_2^2 - 2 \beta_1 \beta_2 \lambda_1 \lambda_2 + \beta_2 (\beta_2 - 1) \lambda_1^2]}{(\alpha \lambda_1 \lambda_2)^2}. \end{aligned}$$

Definition - 1 A random variable Y is said to be geometrically infinitely divisible if for every $p \in (0,1)$ there exists a sequence of i.i.d random variables $X_1^{(p)}, X_2^{(p)}, \dots$ such that $Y = \sum_{j=1}^{N(p)} X_j^{(p)}$, and $P[N(p) = k] = p(1-p)^{k-1}$, $k = 1, 2, \dots$ where $Y, N(p), X_j^{(p)}, (j = 1, 2, \dots)$ are independent.

Theorem - 1 Suppose X_1, X_2, \dots are mutually independently and identically distributed as MOGAL($\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2$) distribution and N , independent of X_1, X_2, \dots , be a geometric random variable with probability of success $0 < p < 1$ then the geometric compound $X_1 + X_2 + \dots + X_N$ follows the MOGAL($p\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2$).

Proof. The geometric compound $X_1 + X_2 + \dots + X_N$ has its characteristic function

$$\begin{aligned} \sum_{k=1}^{\infty} [\psi_X(t)]^k p(1-p)^{k-1} &= \frac{p\psi_X(t)}{1-(1-p)\psi_X(t)} \\ &= \frac{p\alpha}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1} \\ &= \frac{1-(1-p)}{1-(1-p) \frac{\alpha}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1}} \\ &= \frac{p\alpha}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + p\alpha - 1}. \end{aligned}$$

3. APPROXIMATED ASYMMETRIC LAPLACE DISTRIBUTION

For $\beta_1 = \beta_2 = \beta$ (3) reduces to

$$\begin{aligned} \psi(t) &= \frac{\alpha}{\left[\left(1 - \frac{it}{\lambda_1}\right)\left(1 + \frac{it}{\lambda_2}\right)\right]^{\beta} + \alpha - 1} \\ &\approx \frac{\alpha}{\alpha + \beta \left(\frac{1}{\lambda_1 \lambda_2} t^2 - \left(\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}\right) it\right)} \\ &= \frac{1}{1 + \frac{\beta}{\alpha} (\sigma^2 t^2 - i\mu t)}, \text{ where } \mu = \frac{(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}, \sigma^2 = \frac{1}{\lambda_1 \lambda_2}. \end{aligned}$$

Now the p.d.f. of MOGAL $(\alpha, \beta, \mu, \sigma)$ is (see Jayakumar and Kuttikrishnan, 2006).

$$f(x) = \frac{\sqrt{\alpha}}{\sqrt{\beta\sigma}} \frac{\kappa}{1 + \kappa^2} \begin{cases} \exp\left(\frac{-\kappa\sqrt{\alpha}}{\sqrt{\beta\sigma}} x\right) & \text{if } x \geq 0 \\ \exp\left(\frac{\sqrt{\alpha}}{\kappa\sqrt{\beta\sigma}} x\right) & \text{if } x < 0 \end{cases} \quad (4)$$

where

$$\kappa = \frac{2\sigma\sqrt{\alpha}}{\sqrt{\beta\mu + \sqrt{4\sigma^2\alpha + \beta\mu^2}}}, \quad \kappa > 0$$

The graphs of the p.d.f for various parameter values are given in figure 1 and figure 2.

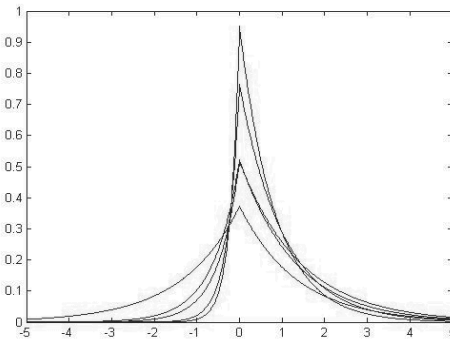


Figure 1 – Approximated Asymmetric Laplace densities, $\sigma=1$ and $\mu=0,1,1.5,3,4$ and $\alpha = 0.8,1.8,2.5,8$ which corresponds to $\kappa = 1,0.69,0.60,0.53,0.51$.

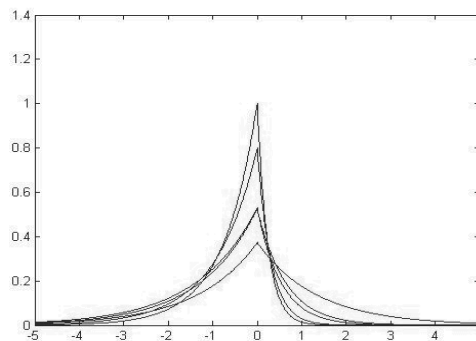


Figure 2 – Approximated Asymmetric Laplace densities, $\sigma=1$ and $\mu = 0,-1,-1.5,-3,-4$ and $\alpha = 0.8,1.8,2.5,8$ which corresponds to $\kappa = 1,1.44,1.66,1.88,1.93$.

For positive values of μ asymmetry is to the right of mode but for negative values of μ the asymmetry is to the left.

Definition 2 A distribution F with characteristic function $\phi(t)$ is called self-decomposable if and only if for every $\alpha \in (0, 1)$, there exists a characteristic function $\phi_\alpha(t)$ such that $\phi(t) = \phi(\alpha t)\phi_\alpha(t)$

Theorem 2 Let $X \sim \text{MOGAL}(\alpha, \beta, \mu, \sigma)$ and let X_1 and X_2 are exponential random variables with parameter $\frac{\kappa\sqrt{\alpha}}{\sqrt{\beta\sigma}}$ and $\frac{\sqrt{\alpha}}{\kappa\sqrt{\beta\sigma}}$ respectively. Then for each $c \in [0, 1]$

$$X \stackrel{d}{=} cX + B_1X_1 - B_2X_2$$

where B_1, B_2 are Bernoulli variables with probabilities

$$P(B_1 = 0, B_2 = 0) = c^2, P(B_1 = 1, B_2 = 1) = 0,$$

$$P(B_1 = 1, B_2 = 0) = (1-c) \left(c + \frac{(1-c)}{1+\kappa^2} \right),$$

$$P(B_1 = 0, B_2 = 1) = (1-c) \left(c + \frac{(1-c)\kappa^2}{1+\kappa^2} \right)$$

(see Kozubowski and Podgórski, 2000)

Proof Let $U = B_1X_1 - B_2X_2$. Then

$$\begin{aligned} \psi_u(t) &= c^2 + \\ &(1-c) \left[\left(c + \frac{(1-c)}{1+\kappa^2} \right) \left(\frac{\kappa\sqrt{\alpha}}{\kappa\sqrt{\alpha} - it\sqrt{\beta\sigma}} \right) + \left(c + \frac{(1-c)\kappa^2}{1+\kappa^2} \right) \left(\frac{\sqrt{\alpha}}{\sqrt{\alpha} + \kappa it\sqrt{\beta\sigma}} \right) \right] \end{aligned}$$

On simplification and using the fact that $\kappa - \frac{1}{\kappa} = \frac{\sqrt{\beta}\mu}{\sqrt{\alpha}\sigma}$ this reduces to

$$\begin{aligned} \psi_u(t) &= \frac{1 + \frac{\beta}{\alpha}((ct)^2\sigma^2 - i\mu(ct))}{1 + \frac{\beta}{\alpha}(t^2\sigma^2 - i\mu t)} \\ &= \frac{\psi_x(t)}{\psi_x(ct)} \end{aligned}$$

Definition 3 We define the geometric generalised Asymmetric Laplace distribution $GGAL(\beta, \mu, \sigma)$ as one with characteristic function

$$\psi_1(t) = \frac{1}{1 + \beta \log(1 + t^2 \sigma^2 - i \mu t)}. \quad (5)$$

It is called so since $\phi(t)$ in (2) can be written as,

$$\phi(t) = \exp\left(1 - \frac{1}{\psi_1(t)}\right)$$

Theorem 3 Geometric generalised Asymmetric Laplace distribution $GGAL(\beta/\alpha, \lambda_1, \lambda_2)$ is the limit of geometric sum of $MOGAL(\beta/n, \alpha, \lambda_1, \lambda_2)$ random variables where the geometric random variables have the probability of success $1/n$.

Proof. The characteristic function of $MOGAL(\beta/n, \alpha, \lambda_1, \lambda_2)$ is

$$\psi(t) = \frac{1}{1 + \frac{1}{\alpha} [(1 + t^2 \sigma^2 - i \mu t)^{\frac{\beta}{n}} - 1]}$$

Then the characteristic function of the geometric compound of independent and identical $MOGAL(\beta/n, \alpha, \lambda_1, \lambda_2)$ random variables with geometric random variable N having probability of success $p = 1/n$ is

$$\begin{aligned} \psi_n(t) &= \sum_{k=1}^{\infty} E[\exp(it[X_1 + X_2 + \dots + X_k]/N = k)] P(N = k) \\ &= \sum_{k=1}^{\infty} [\psi(t)]^k \frac{1}{n} \left(1 - \frac{1}{n}\right)^{k-1} \end{aligned}$$

$$= \frac{1}{n} \left\{ \frac{1}{1 + \frac{1}{\alpha} [(1 + t^2 \sigma^2 - i \mu t)^{\frac{\beta}{n}} - 1] - \left(1 - \frac{1}{n}\right)} \right\}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi_n(t) &= \frac{1}{1 + \lim_{n \rightarrow \infty} \frac{n}{\alpha} [(1 + t^2 \sigma^2 - i \mu t)^{\frac{\beta}{n}} - 1]} \\ &= \frac{1}{1 + \frac{\beta}{\alpha} \log\{1 + t^2 \sigma^2 - i \mu t\}}. \end{aligned}$$

4. AR(1) PROCESS WITH MOGAL $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ MARGINAL DISTRIBUTION

The past several decades witnessed the emergence of a number of autoregressive models constructed for the generation of non-Gaussian processes in discrete time because many naturally occurring time series are non-Gaussian. Since (Gaver and Lewis 1980); firstly built the fundamental frame work, the autoregressive model with non-Gaussian marginal distribution has received a tremendous attention in the recent two decades. The work by (Anderson and Arnold 1993); (Jayakumar and Pillai 1993); and (Seetha Lekshmi and Jose 2004a, 2004b, 2006); in this area can be referred.

4.1 AR(1) model I

Consider the model

$$X_n = aX_{n-1} + \varepsilon_n, |a| < 1 \quad (6)$$

where $\varepsilon_1, \varepsilon_2, \dots$ are i.i.d. random variables and ε_n is independent of X_1, X_2, \dots, X_{n-1} .

Theorem. 4 The AR(1) model $\{X_n\}$ in (6) is strictly stationary with MOGAL $(X_1, \beta_1, \beta_2, \lambda_1, \lambda_2)$ marginal distribution if and only if $\{\varepsilon_n\}$ has the characteristic function

$$\frac{\left(1 + \frac{iat}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{iat}{\lambda_2}\right)^{\beta_2} + \alpha - 1}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1} \quad (7)$$

and X_0 follows MOGAL $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$.

Proof It is clear that

$$\psi_{X_n}(t) = \psi_{\varepsilon_n}(t) \psi_{X_{n-1}}(at).$$

Then by stationarity property of (6), we have for all t

$$\psi_{\varepsilon_n}(t) = \frac{\psi_{X_n}(t)}{\psi_{X_{n-1}}(at)} = \frac{\left(1 + \frac{iat}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{iat}{\lambda_2}\right)^{\beta_2} + \alpha - 1}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1}. \quad (8)$$

The converse can be proved as follows. If X_0 follows MOGAL $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ and $\{\varepsilon_n\}$ has characteristic function (7) it can be verified by induction that X_n follows MOGAL $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$.

Assuming that X_{n-1} is MOGAL we have,

$$\begin{aligned}\psi_{X_n}(t) &= \frac{\alpha}{\left(1 + \frac{iat}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{iat}{\lambda_2}\right)^{\beta_2} + \alpha - 1} \times \frac{\left(1 + \frac{iat}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{iat}{\lambda_2}\right)^{\beta_2} + \alpha - 1}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1} \\ &= \frac{\alpha}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1}.\end{aligned}$$

Hence $X_n \sim \text{MOGAL}(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$.

Hence the process is strictly stationary.

Remark. Suppose X_0 follows any arbitrary distribution. Then we have,

$$X_n = aX_{n-1} + \varepsilon_n = a^n X_0 + \sum_{k=1}^{n-1} a^k \varepsilon_{n-k}, \quad |a| < 1$$

$$\begin{aligned}\psi_{X_n}(t) &= \psi_{X_0}(a^n t) \prod_{k=0}^{n-1} \psi_{\varepsilon}(a^k t), \quad |a| < 1 \\ &= \psi_{X_0}(a^n t) \prod_{k=0}^{n-1} \frac{\left(1 + \frac{ia^{k+1}t}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{ia^{k+1}t}{\lambda_2}\right)^{\beta_2} + \alpha - 1}{\left(1 + \frac{iat}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{iat}{\lambda_2}\right)^{\beta_2} + \alpha - 1} \\ &= \psi_{X_0}(a^n t) \frac{\left(1 + \frac{ia^n t}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{ia^n t}{\lambda_2}\right)^{\beta_2} + \alpha - 1}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1} \\ &\rightarrow \frac{\alpha}{\left(1 + \frac{it}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it}{\lambda_2}\right)^{\beta_2} + \alpha - 1}, \text{ as } n \rightarrow \infty.\end{aligned}$$

Hence the process in (6) is asymptotically stationary with MOGAL marginal distribution.

4.2 Distribution of T_k and Joint Characteristic function

Now we address two important aspects of the autoregressive model in (6) with MOGAL $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ marginal distribution. For any positive integer k , the sum $T_k = X_n + X_{n-1} + \dots + X_{n+k-1}$ can be written as,

$$\begin{aligned} T_k &= X_n + X_{n-1} + \dots + X_{n+k-1} \\ &= \sum_{j=1}^{k-1} (a^j X_n + a^{j-1} \varepsilon_{n+1} + a^{j-2} \varepsilon_{n+2} + \dots + \varepsilon_{n+j}) \\ &= \frac{1-a^k}{1-a} X_n + \sum_{j=1}^{k-1} \frac{1-a^{k-j}}{1-a} \varepsilon_{n+j}. \end{aligned}$$

Its characteristic function is obtained as

$$\begin{aligned} \psi_{T_k}(t) &= \psi_{X_n} \left(\frac{1-a^k}{1-a} t \right) \prod_{j=1}^{k-1} \psi_{\varepsilon} \left(\frac{1-a^{k-j}}{1-a} t \right) \\ &= \frac{\alpha}{\left(1 + \frac{1-a^k}{\lambda_1} t \right)^{\beta_1} \left(1 - \frac{1-a^k}{\lambda_2} t \right)^{\beta_2} + \alpha - 1} \\ &\quad \times \prod_{j=1}^{k-1} \frac{\left(1 + \frac{i \left(\frac{a-a^{k-j+1}}{1-a} \right) t}{\lambda_1} \right)^{\beta_1} \left(1 - \frac{i \left(\frac{a-a^{k+j-1}}{1-a} \right) t}{\lambda_2} \right)^{\beta_2}}{\left(1 + \frac{i \left(\frac{1-a^k}{1-a} \right) t}{\lambda_1} \right)^{\beta_1} \left(1 - \frac{i \left(\frac{1-a^k}{1-a} \right) t}{\lambda_2} \right)^{\beta_2} + \alpha - 1}. \end{aligned}$$

The distribution of T_k can be obtained by inverting the above expression. The joint characteristic function of (X_n, X_{n+1}) is

$$\begin{aligned}
\psi_{X_n, X_{n+1}}(t) &= E(e^{it_1 X_n + it_2 X_{n+1}}) \\
&= E(e^{it_1 X_n + it_2 (aX_n + \varepsilon_{n+1})}) \\
&= E(e^{i(t_1 + at_2) X_n}) E(e^{it_2 \varepsilon_{n+1}}) \\
&= \psi_{X_n}(t_1 + at_2) \psi_{\varepsilon_{n+1}}(t_2) \\
&= \frac{\alpha}{\left(1 + \frac{i(t_1 + at_2)}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{i(t_1 + at_2)}{\lambda_2}\right)^{\beta_2} + \alpha - 1} \\
&\quad \times \frac{\left(1 + \frac{iat_2}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{iat_2}{\lambda_2}\right)^{\beta_2} + \alpha - 1}{\left(1 + \frac{it_2}{\lambda_1}\right)^{\beta_1} \left(1 - \frac{it_2}{\lambda_2}\right)^{\beta_2} + \alpha - 1}
\end{aligned}$$

$$\text{Hence } \psi_{X_n, X_{n+1}}(t_1, t_2) \neq \psi_{X_n, X_{n+1}}(t_2, t_1).$$

This reveals that the autoregressive process (6) with MOGAL $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ marginal distribution is not time reversible.

4.3 AR(1) Model II

Consider a first order autoregressive process with structure

$$X_n = \begin{cases} \varepsilon_n, & w.p. \ p \\ X_{n-1} + \varepsilon_n, & w.p. \ 1-p, \quad 0 \leq p \leq 1, n \geq 1, \end{cases} \quad (8)$$

where $\{\varepsilon_n\}$ is a sequence of i.i.d random variables independent of $\{X_n\}$.

Theorem 5 Let $\{X_n\}$ be an AR(1) process having structure given by (8) and $\{\varepsilon_n\}$ is a sequence of i.i.d random variables with Asymmetric Laplace distribution independent of $\{X_n\}$. Then $\{X_n\}$ is stationary Markovian with MOGAL $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ marginal distribution and conversely.

Proof First we shall prove the sufficiency part. Using the structure stated above

$$\psi_{X_n}(t) = p\psi_{\varepsilon}(t) + (1-p)\psi_{X_{n-1}}(t)\psi_{\varepsilon}(t).$$

Under stationary equilibrium we have,

$$\begin{aligned} \psi_X(t) &= \frac{p\psi_\varepsilon(t)}{1-(1-p)\psi_\varepsilon(t)} \text{ and } \psi_\varepsilon(t) = \frac{\psi_X(t)}{p+(1-p)\psi_X(t)} \\ \psi_X(t) &= \frac{p \left(\frac{\lambda_1}{\lambda_1+it}\right)^{\beta_1} \left(\frac{\lambda_2}{\lambda_2-it}\right)^{\beta_2}}{1-(1-p) \left(\frac{\lambda_1}{\lambda_1+it}\right)^{\beta_1} \left(\frac{\lambda_2}{\lambda_2-it}\right)^{\beta_2}} \\ &= \frac{p}{\left(1+\frac{it}{\lambda_1}\right)^{\beta_1} \left(1-\frac{it}{\lambda_2}\right)^{\beta_2} + p-1}. \end{aligned}$$

which is MOGAL $(p, \beta_1, \beta_2, \lambda_1, \lambda_2)$. Conversely we have,

$$\begin{aligned} \psi_\varepsilon(t) &= \frac{\frac{p}{\left(1+\frac{it}{\lambda_1}\right)^{\beta_1} \left(1-\frac{it}{\lambda_2}\right)^{\beta_2} + p-1}}{p+(1-p) \frac{p}{\left(1+\frac{it}{\lambda_1}\right)^{\beta_1} \left(1-\frac{it}{\lambda_2}\right)^{\beta_2} + p-1}} \\ &= \left(\frac{\lambda_1}{\lambda_1+it}\right)^{\beta_1} \left(\frac{\lambda_2}{\lambda_2-it}\right)^{\beta_2} \end{aligned}$$

which is the characteristic function of Asymmetric Laplace distribution.

Now a simulation study of the sample path of the process is conducted. The corresponding sample path and histogram are given in figure 3 for various parameters combinations.

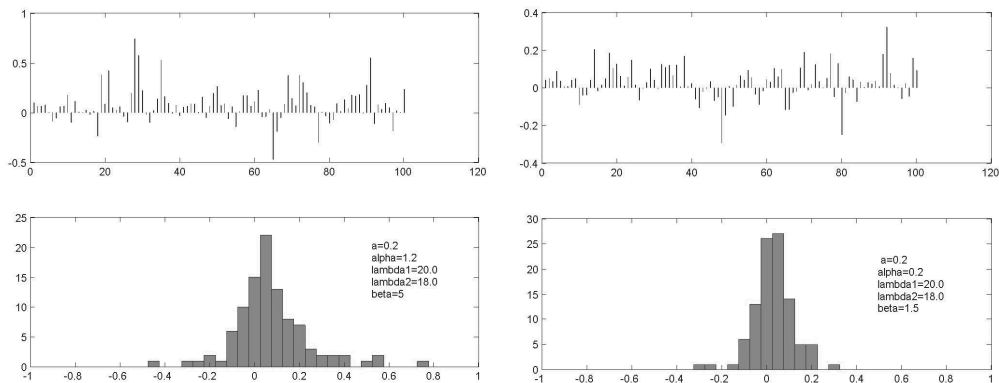


Figure 3

This shows that distribution gives rise to high peaks and can be used to model heavy-tailed data exhibiting high peaks.

5. GENERALISATION TO AR(K) MODEL

Now we generalize the Type II first-order MOGAL autoregressive process given by (8) to a k^{th} order MOGAL autoregressive model as follows.

The higher order autoregressive model constructed by Lawrance (1982) is

$$X_n = \begin{cases} \varepsilon_n & \text{with probability } p_0 \\ X_{n-1} + \varepsilon_n & \text{with probability } p_1 \\ \dots \\ X_{n-k} + \varepsilon_n & \text{with probability } p_k \end{cases} \quad (9)$$

where $0 < p_i < 1 (i = 1, 2, \dots, k)$ such that $p_1 + p_2 + \dots + p_k = 1$ and ε_n is independent of X_{n-1}, X_{n-2}, \dots . In terms of characteristic function (9) can be rewritten as,

$$\psi_{X_n}(t) = p_0 \psi_{\varepsilon_n}(t) + p_1 \psi_{X_{n-1}}(t) \psi_{\varepsilon}(t) + \dots + p_k \psi_{X_{n-k}}(t) \psi_{\varepsilon_n}(t).$$

Assuming the stationarity we get,

$$\psi_{\varepsilon}(t) = \frac{\psi_X(t)}{p + (1-p)\psi_X(t)}.$$

This shows that model (9) can be defined as in the case of model II given by (8).

6. ESTIMATION OF PARAMETERS AND DIAGNOSTIC CHECKING

In this section we address the problems of statistical inference as well as diagnostic checking of the new model, see Sim(1994). First we estimate the parameters of the approximated distribution given by (4) and then apply it to model a financial data. Now we make a re-parameterisation by taking $\kappa^2 = \frac{\gamma}{\delta}$ and $\beta\sigma^2 = \frac{1}{\gamma\delta}$

in (4). Then the p.d.f. becomes

$$f(x) = \sqrt{\alpha} \frac{\gamma\delta}{\gamma + \delta} \begin{cases} \exp(-\sqrt{\alpha} \gamma x) & \text{if } x \geq 0 \\ \exp(\sqrt{\alpha} \delta x) & \text{if } x < 0 \end{cases}$$

Fixing α , the maximum likelihood estimates of γ and δ are obtained. Then the estimate of α is obtained by maximising the log likelihood function.

The log likelihood is given by

$$\begin{aligned}
LL &= \log \prod_{i=1}^N f(x_i/\alpha, \gamma, \delta) \\
&= \sum_{i=1}^N \log f(x_i/\alpha, \gamma, \delta) \\
&= \sum_{x \leq 0} \log f(x_i/\alpha, \gamma, \delta) + \sum_{x > 0} \log f(x_i/\alpha, \gamma, \delta) \\
&= N \log \left(\sqrt{\alpha} \frac{\gamma \delta}{\gamma + \delta} \right) - \sqrt{\alpha} \gamma S_n + \sqrt{\alpha} \delta S_p
\end{aligned}$$

where $S_n = \sum_{x \leq 0} x_i$ and $S_p = \sum_{x > 0} x_i$.

By equating the partial derivatives to zero and solving for γ and δ we get

$$\hat{\gamma} = \frac{N}{\sqrt{\alpha}(S_p + \sqrt{-S_n S_p})}$$

and

$$\hat{\delta} = \frac{N}{\sqrt{\alpha}(-S_n + \sqrt{-S_n S_p})}$$

By iteration $\hat{\alpha}$ can be obtained.

6.1 Application to a financial data

As part of diagnostic checking, we apply the above results with respect to the data on weekly price of Gold/gm from 2006 February to 2010 March collected from District Statistical Office, Alappuzha, Kerala, India. Estimates of the parameters are respectively obtained as

$$\hat{\gamma} = 0.5898, \hat{\delta} = 0.4904, \hat{\alpha} = 1.2580$$

The histogram corresponding to actual data and the fitted frequency curve are superimposed and presented in figure 4.

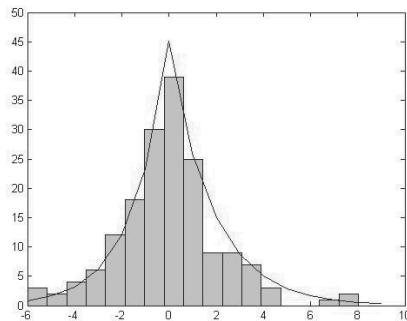


Figure 4

From the above figure it can be observed that they are very close to each other showing that the newly developed distribution is appropriate to model the above data.

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SUMMARY

Marshall-Olkin Generalized Asymmetric Laplace distributions and processes

The Marshall-Olkin Generalised Asymmetric Laplace distribution is introduced and studied. An approximation is made and various properties including self decomposability, geometric infinite divisibility, limit properties etc. are established. Two autoregressive processes namely model I and model II are developed and studied. The sample path properties are explored for various parameter combinations. The distribution of sums, joint distribution of contiguous observations of the process, etc. are obtained. The model is extended to k^{th} order also. Parameters are estimated by the method of maximum likelihood and a real data on gold prices is fitted to the new model.