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BAYESIAN ANALYSIS FOR THE BURR TYPE XII DISTRIBUTION BASED ON RECORD VALUES

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1. INTRODUCTION

The theory and applications of record values is of great importance to scientists and engineers and has been studied extensively. For example, predicting the magnitude of an earthquake which has a greater magnitude than the previous ones, in a given region, is of importance to seismologists, also predicting the flood level of a river that is greater than the previous ones is of importance to climatologists. The theory of record values can be applied to sports records, estimating the strength of materials, etc. Record values are also important to ordinary people and the book "World Guinness Records" chronicles records of sports & games, natural disasters, science & technology, and of all kinds of strange and extreme talents, among others.

Let $X_1, X_2, \ldots$ be an infinite sequence of identical independent (iid) random variables having a common probability density function (pdf) $f(x,\theta)$, where θ could be a vector parameter. Intuitively an upper record value, X_j is the observation which is larger than all previous observations. Formally, X_j is an upper record value if $X_j > X_i$ for all $i < j.$ The sequence $\{T_m,\ m \geq 1\}$ of record times is defined as follows:

$T_1 = 1,$ with probability 1, and
$T_m = \min\{ j : j > T_{m-1},\ X_j > X_{T_{m-1}}\}$ for $m \geq 2$.

Based on the above, the sequence $R_m = X_{T_m},\ m \geq 1$ defines a sequence of upper record values. One can give a similar definition for lower record values.

Let $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ be a random vector of the upper record values for some pdf $f(x,\theta)$. One of the interests in the study of the theory of records values is to predict the s-th record value, $1 \leq m < s,$ when we have observed a realization of $\mathbf{R}$, namely $\mathbf{r} = (r_1, r_2, \ldots r_m)$. The theory of record values was first introduced by Chandler (1952) whereas Feller (1966) gave some examples of record values in gambling. The reader is referred to Arnold *et al.* (1998), Ahmadi (2000), Feller (1966), Gulati and Padgett (1994), and Nevzorov (2001) for more details on the applications of record values from a statistician's point of view.

When the underlying distribution is exponential the record values have been studied by Awad and Raqab (2000), they studied four procedures for obtaining prediction intervals for the future s-th record value and by means of computer simulation they compared these procedures. Based on the record values, Ali Mousa (2001) derived three types of estimators, maximum likelihood, minimum variance unbiased and Bayesian estimators for the one parameter Burr type X distribution. Based on record values from the two parameter Pareto distribution, Raqab *et al.* (2007) obtained maximum likelihood and Bayes estimators for the unknown parameters and point or interval prediction for the future record values. Statistical analysis of record values from the geometric distribution was done by Doostparast and Ahmadi (2006). Furthermore, they derived estimators for the unknown parameter and also considered the problem of predicting the future record values based on past record values from a non-Bayesian and Bayesian point of view. Based on upper record values, Hendi *et al.* (2007) obtained Bayes estimators, under SEL and LINEX loss functions, for the parameter, reliability function and hazard rate for the Rayleigh distribution. Finally, Ahmadi *et al.* (2009) studied the prediction of k-records from a general class of distributions under balanced type loss functions.

The purpose of this study is to review and extend some results that have been derived on record values from the two parameter Burr Type XII distribution which was first introduced in the literature by Burr (1942). Its probability density function and cumulative distribution are given below,

$$f(x;c,k) = c\ k\ x^{c-1}(1+x^{c})^{-(k+1)} \quad x,\ c,\ k > 0 \tag{1a}$$

$$F(x,c,k) = 1-(1+x^{c})^{-k} \tag{1b}$$

and was first proposed as a failure model by Dubey (1972, 1973). Furthermore, Papadopoulos (1978) derived Bayesian estimators, under a SEL function, for the parameter k and the reliability function under the assumption that the parameter c is known. When both of the parameters c and k are random variables and we have a type II censoring data, Al-Hussaini and Jaheen (1992, 1995) obtained Bayes estimates for the parameters c and k, the reliability function and the failure rate. Moore and Papadopoulos (2000) obtained Bayes estimates for the parameter k and reliability function, under various loss functions. When the parameter c is known, Ahmadi and Doostparast (2006) considered Bayesian estimation for the exponential, Weibull, Pareto and Burr Type XII distribution based on record values when both of the parameters are considered as random variables. Finally, Jaheen (2005) used the generalized order statistics to obtain estimators for the parameter of the Burr type XII distribution.

In Section 2, we will derive estimators for the parameters c and k, based on record values, review the Bayes estimators that were derived under a SEL function for the parameters by Ahmadi (2000) and also we will derive Bayes estimators for c and k under a LINEX loss function. In Section 3, estimates for the future s-th record value will be derived using non Bayesian and Bayesian approaches. Finally, in Section 4, a numerical example will illustrate the findings of Sections 2 and 3.

2. PARAMETER ESTIMATION

The joint pdf of the first m upper record values according to Arnold *et al.* (1998) is

$$f(\mathbf{r};\theta)=\prod_{i=1}^{m-1} h(r_i;\theta)\, f(r_m;\theta), \quad -\infty<r_1<r_2<...<r_m<\infty \tag{2}$$

where $\mathbf{r}=(r_1,r_2,...r_m)$, $h(r_i;\theta)=\dfrac{f(r_i;\theta)}{1-F(r_i;\theta)}$, and $\theta\in\Theta$ may be a vector, where Θ is the parameter space. For the Burr Type XII pdf the joint distribution of the first m upper record values reduces to

$$f(\mathbf{r};c,k)=\frac{(ck)^m}{(1+r_m{}^c)^k}\prod_{i=1}^{m}\frac{r_i{}^{c-1}}{(1+r_i{}^c)}, \quad r_1<r_2<...<r_m. \tag{3}$$

2.1. *MLE Estimation*

Suppose we observed the first m upper record values $R_1=r_1,\ R_2=r_2,...,\ R_m=r_m$ from a Burr Type XII, with pdf and cdf given by equations (1a) and (1b). Then the joint density function is given by

$$f(\mathbf{r};c,k)=\frac{c^m\ \mathrm{k}^m}{(1+r_m{}^c)^k}\prod_{i=1}^{m}\frac{r_i{}^{c-1}}{(1+r_i{}^c)}, \qquad c>0,\ k>0;\ -\infty<\mathrm{r}_1<...<\mathrm{r}_\mathrm{m}<\infty \tag{4}$$

Then the log-likelihood function is

$$L_{\mathbf{r}}(c,k)=m\ln c+m\ln k-k\ln(1+r_m{}^c)+\sum_{i=1}^{m}(c-1)\ln r_i-\sum_{i=1}^{m}\ln(1+r_i{}^c) \tag{5}$$

The ML estimates of c and k can be obtained by maximizing the log-likelihood function (5). By taking derivatives with respect to c and k, the MLE are obtained by satisfying the following equations:

$$\frac{m}{c}-\sum_{i=1}^{m}\ln r_i-\frac{kr_m{}^c\ln r_m}{1+r_m{}^c}-\sum_{i=1}^{m}\frac{r_i^c\ln r_i}{1+r_i^c}=0 \tag{6}$$

$$\frac{m}{k}-\ln(1+r_m{}^c)=0 \tag{7}$$

The above system is nonlinear, but can be easily solved using numerical techniques.

2.2. *Bayes Estimation*

In this Subsection, we will assume that the parameters c and k of the Burr Type XII distribution are random variables with a joint bivariate density function that was first suggested by Al-Hussaini and Jaheen (1995) and is given by,

$$g(c,k)=g_1(k\,|\,c)g_2(c) \tag{8}$$

where

$$g_1(k\,|\,c)=\frac{c^{\alpha+1}}{\Gamma(\alpha+1)\gamma^{\alpha+1}}\;k^{\alpha}\;e^{-kc/\gamma},\quad \alpha>-1,\ \gamma>0 \tag{9}$$

is the gamma conjugate prior when c is know. This prior was first introduced by Papadopoulos (1978) and was also used later on by Al-Hussaini *et al.* (1992). The prior of c is

$$g_2(c)=\frac{c^{\delta-1}}{\Gamma(\delta)\beta^{\delta}}\;e^{-c/\beta},\quad \beta,\ \delta>0 \tag{10}$$

which is the gamma(β, δ) density. With the aid of equations (9) and (10) we obtain the bivariate prior of c and k, given as

$$\pi(c,k)\propto c^{\delta+\alpha}k^{\alpha}\exp(-c(1/\beta+k/\gamma)) \tag{11}$$

where $\alpha>-1$, β, δ and γ are positive real numbers.

From (3) and (11), the joint posterior distribution is given by

$$\pi(k,c\,|\,\mathbf{r})=\frac{\prod_{i=1}^{m}\left(\frac{r_i^{c-1}}{1+r_i^c}\right)c^{\delta+m+\alpha}\;\frac{k^{m+\alpha}}{(1+r_m^c)^k}\;\exp(-c(1/\beta+k/\gamma))}{\Gamma(\mathrm{m}+\alpha+1)\displaystyle\int_0^{\infty}\prod_{i=1}^{m}\left(\frac{r_i^{c-1}}{1+r_i^c}\right)\frac{c^{\delta+m+\alpha}\exp(-c/\beta)}{\left[\frac{c}{\gamma}+\ln(1+r_m^c)\right]^{\mathrm{m}+\alpha+1}}} \tag{12}$$

If the loss function is the well known squared loss function, the Bayes estimators for the parameters c and k are the given by the posterior expectations. Ahmadi and Doostparast (2006) derived these estimators and are reproduced below,

$$\hat{k}_B = E(k \mid \mathbf{r}) = \int_0^\infty \int_0^\infty k\pi(k, c \mid \mathbf{r})\, dk\, dc$$

$$= (\alpha + m + 1) \frac{\int_0^\infty \prod_{i=1}^{m} \left(\frac{r_i^{c-1}}{1 + r_i^c} \right) \frac{c^{\alpha+\delta+m} \exp(-c/\beta)}{\left[\frac{c}{\gamma} + \ln(1 + r_m^c) \right]^{\alpha+m+2}} \, \mathrm{dc}}{\int_0^\infty \prod_{i=1}^{m} \left(\frac{r_i^{c-1}}{1 + r_i^c} \right) \frac{c^{\alpha+\delta+m} \exp(-c/\beta)}{\left[\frac{c}{\gamma} + \ln(1 + r_m^c) \right]^{\alpha+m+1}} \, \mathrm{dc}}. \tag{13}$$

Similarly,

$$\hat{c}_B = \frac{\int_0^\infty \prod_{i=1}^{m} \left(\frac{r_i^{c-1}}{1 + r_i^c} \right) \frac{c^{\alpha+\delta+m+1} \exp(-c/\beta)}{\left[\frac{c}{\gamma} + \ln(1 + r_m^c) \right]^{\alpha+m+1}} \, \mathrm{dc}}{\int_0^\infty \prod_{i=1}^{m} \left(\frac{r_i^{c-1}}{1 + r_i^c} \right) \frac{c^{\alpha+\delta+m} \exp(-c/\beta)}{\left[\frac{c}{\gamma} + \ln(1 + r_m^c) \right]^{\alpha+m+1}} \, \mathrm{dc}} \tag{14}$$

Instead of using the well known symmetric SEL function, one can use the asymmetric LINEX loss function which was first proposed by Varian (1975) and is given as

$$L(\theta, \delta) = e^{v(\delta-\theta)} - v(\delta - \theta) - 1 \tag{15}$$

where θ is a univariate parameter and $v \neq 0$. The parameter v is known and gives the degree of asymmetry. If $v > 0$ and the errors $\delta - \theta$ are positive, the LINEX loss function is almost exponential and for negative errors almost linear, in this situation overestimation is a more serious problem than underestimation. If $v < 0$, underestimation is more important than overestimation.

Let $M_{\Theta|\mathbf{r}}(t) = E_{\Theta|\mathbf{r}}(e^{t\theta})$ be the moment generating function of the Bayes predictive density function of Θ given $\mathbf{r}$. It can be easily verified that the value of $\delta(\Theta)$ that minimizes $E_{\Theta|\mathbf{r}}(L(\Theta, \delta(\Theta))$ in equation (15) is $\delta^*(\Theta) = -\frac{1}{v} \ln M_{\Theta|\mathbf{r}}(-v)$, provided that $M_{\Theta|\mathbf{r}}(.)$ exists and is finite, thus

$$E_{(k,c)|\mathbf{r}}(e^{-vk} \mid \mathbf{r}) = \int_0^\infty \int_0^\infty e^{-vk} \pi(k,c \mid \mathbf{r}) \, dc \, dk \tag{16}$$

For the parameter k the LINEX estimator is

$$\hat{k}_L = -\frac{1}{v} \ln \left(\frac{\displaystyle\int_0^\infty \prod_{i=1}^m \left(\frac{r_i^{c-1}}{1+r_i^c} \right) \frac{c^{\alpha+\delta+m} \exp(-c/\beta)}{\left[v + \dfrac{c}{\gamma} + \ln(1+r_m^c) \right]^{\alpha+m+1}} \mathrm{dc}}{\displaystyle\int_0^\infty \prod_{i=1}^m \left(\frac{r_i^{c-1}}{1+r_i^c} \right) \frac{c^{\alpha+\delta+m} \exp(-c/\beta)}{\left[\dfrac{c}{\gamma} + \ln(1+r_m^c) \right]^{\alpha+m+1}} \mathrm{dc}} \right) \tag{17}$$

Similarly, LINEX estimate for c is

$$E_{(k,c)|\mathbf{r}}(e^{-vc} \mid \mathbf{r}) = \int_0^\infty \int_0^\infty e^{-vc} \pi(k,c \mid \mathbf{r}) \, dc \, dk \tag{18}$$

$$\hat{c}_L = -\frac{1}{v} \ln \left(\frac{\displaystyle\int_0^\infty \prod_{i=1}^m \left(\frac{r_i^{c-1}}{1+r_i^c} \right) \frac{c^{\alpha+\delta+m} \exp(-c(v+1/\beta))}{\left[\dfrac{c}{\gamma} + \ln(1+r_m^c) \right]^{\alpha+m+2}} \mathrm{dc}}{\displaystyle\int_0^\infty \prod_{i=1}^m \left(\frac{r_i^{c-1}}{1+r_i^c} \right) \frac{c^{\alpha+\delta+m} \exp(-c/\beta)}{\left[\dfrac{c}{\gamma} + \ln(1+r_m^c) \right]^{\alpha+m+1}} \mathrm{dc}} \right) \tag{19}$$

It should be pointed out that equations (17) and (19) are not just simply the logarithmic transformation of equations (13) and (14), since argument of the logarithm involves the asymmetry parameter v. Furthermore, it should be mentioned that equations (17) and (19) are not in explicit form, but the practitioner should not be discouraged, there are several numerical methods that can be used to evaluate those expressions.

3. PREDICTION OF FUTURE RECORD VALUES

In this section we address the problem of estimating the s-th record value using non-Bayesian and Bayesian approaches.

3.1. *Non-Bayesian Prediction Approach*

Suppose that we observe the first m record values from a population with pdf $f(x;\theta)$. Our aim is to predict $Y = R_s$, $s > m$, having observed records

$\mathbf{r}=(r_1,r_2,...r_m)$. The joint predictive likelihood function of $Y=R_s$, and θ is given by Basak and Balakrishnan (2003).

$$L(y,\theta;\mathbf{r})=\prod_{i=1}^{m}h(r_i;\theta)\frac{[H(y;\theta)-H(r_i;\theta)]^{s-m-1}}{\Gamma(s-m)}f(y;\theta), \tag{20}$$

where

$$H(y;\theta)=-\ln(1-F(y)) \tag{21}$$

and

$$h(r_i;\theta)=\frac{f(r_i;\theta)}{1-F(r_i;\theta)} \tag{22}$$

The predictive likelihood function for the Burr Type XII pdf is,

$$L(y,c,k;\mathbf{r})=c^{m+1}k^s\prod_{i=1}^{m}\left(\frac{r_i^{c-1}}{1+r_i^c}\right)\frac{[\ln(1+y^c)-\ln(1+r_m^c)]^{s-m-1}}{\Gamma(s-m)}\frac{y^{c-1}}{(1+y^c)^{k+1}}, \tag{23}$$
$$y>r_m>r_{m-1}>...>r_1>0.$$

Estimates for c, k and y are obtained by minimizing the log-likelihood equation with respect to the above mentioned parameters. After some simplifications these equations are,

$$\frac{m+1}{c}+(s-m-1)\frac{\dfrac{y^c\ln y}{1+y^c}-\dfrac{r_m^c\ln r_m}{1+r_m^c}}{\ln(1+y^c)-\ln(1+r_m^c)}+ \tag{24}$$
$$+\ln y-(k+1)\frac{y^c\ln y}{1+y^c}+\sum_{i=1}^{m}\ln r_i-\sum_{i=1}^{m}\frac{r_i^c}{1+r_i^c}\ln r_i=0,$$

$$\frac{s}{k}-\ln(1+y^c)=0, \tag{25}$$

$$(s-m-1)\frac{\dfrac{cy^{c-1}}{1+y^c}}{\ln(1+y^c)-\ln(1+r_m^c)}+\frac{c-1}{y}-(k+1)\frac{cy^{c-1}}{1+y^c}=0. \tag{26}$$

We can reduce the above system of three equations into a system of two equations by replacing $k=\dfrac{s}{\ln(1+y^c)}$ (which is obtained from equation (25)) into the other equations (24), (26) and obtained

$$\frac{m+1}{c}+(s-m-1)\frac{\dfrac{y^c\ln y}{1+y^c}-\dfrac{r_m^c\ln r_m}{1+r_m^c}}{\ln(1+y^c)-\ln(1+r_m^c)}+\ln y-\left(\frac{s}{\ln(1+y^c)}+1\right)\frac{y^c\ln y}{1+y^c}+ \tag{27}$$
$$\sum_{i=1}^{m}\ln r_i-\sum_{i=1}^{m}\frac{r_i^c}{1+r_i^c}\ln r_i=0,$$

$$(s-m-1)\frac{\dfrac{cy^{c-1}}{1+y^c}}{\ln(1+y^c)-\ln(1+r_m^c)}+\frac{c-1}{y}-\left(\frac{s}{\ln(1+y^c)}+1\right)\frac{cy^{c-1}}{1+y^c}=0. \tag{28}$$

The above system can easily be solved numerically and an example in Section 4 will demonstrate this.

3.2. *Bayes Prediction Approach*

In this part, we consider the problem of prediction of future records based on a Bayesian approach using squared error and linear exponential loss functions. Assume that we have observed the first m upper records $R_1=r_1$, ..., $R_m=r_m$ from the Burr Type XII distribution. Based on this sample, we want to predict *s-th* upper record, $1<m<s$. Let $Y=R_s$ denote the *s*-th upper record value. The Bayes predictive density function of Y given $\mathbf{r}$ is,

$$h(y\,|\,\mathbf{r})=\int_{\Theta}f(y\,|\,\mathbf{r},\theta)\pi(\theta\,|\,\mathbf{r})d\theta \tag{29}$$

where the conditional pdf of $Y=R_s \mid R_m=r_s$ is,

$$f(y\,|\,\mathbf{r},\theta)=\frac{[H(y;\theta)-H(r_m;\theta)]^{s-m-1}}{\Gamma(s-m)}\frac{f(y\,|\,\theta)}{1-F(r_m\,|\,\theta)},\qquad r_m<y<\infty \tag{30}$$

for the Burr XII distribution with k and c as parameters.

$$f(y\,|\,\mathbf{r},\theta)=\frac{c\,k^{s-m}}{\Gamma(s-m)}[\ln\left(\frac{1+y^c}{1+r_m^c}\right)]^{s-m-1}\left(\frac{1+r_m^c}{1+y^c}\right)^k\frac{y^{c-1}}{1+y^c} \tag{31}$$

Using (12) and (31) and integrating out the parameters k and c in (30), one gets Bayesian predictive density function of $Y=R_s$, given the past m records, in the form

$$h(y\mid \mathbf{r})=\frac{\Gamma(s+\alpha+1)}{\Gamma(m+\alpha+1)\Gamma(s-m)}*$$

$$\frac{\int_0^\infty \prod_{i=1}^m\left(\frac{r_i^{c-1}}{1+r_i^c}\right)\frac{c^{\alpha+\delta+m+1}}{\left[\frac{c}{\gamma}+\ln(1+y^c)\right]^{s+\alpha+1}}\frac{y^{c-1}}{1+y^c}[\ln\left(\frac{1+y^c}{1+r_m^c}\right)]^{s-m-1}\exp(-c/\beta)dc}{\int_0^\infty \prod_{i=1}^m\left(\frac{r_i^{c-1}}{1+r_i^c}\right)\frac{c^{\alpha+\delta+m}}{\left[\frac{c}{\gamma}+\ln(1+r_m^c)\right]^{m+\alpha+1}}\exp(-c/\beta)dc} \tag{32}$$

From the above equation, under SEL Ahmadi and Doostparast (2006) derived the Bayes point predictor of the s-th upper record value $(s\geq m+1)$ which is reproduced below

$$\hat{Y}_{SEL}=E(Y\mid \mathbf{r})=\int_{r_m}^\infty y\ h(y\mid \mathbf{r})\ dy=\frac{\Gamma(s+\alpha+1)}{\Gamma(m+\alpha+1)\Gamma(s-m)}*$$

$$\frac{\int_{r_m}^\infty\int_0^\infty \prod_{i=1}^m\left(\frac{r_i^{c-1}}{1+r_i^c}\right)\frac{c^{\alpha+\delta+m+1}}{\left[\frac{c}{\gamma}+\ln(1+y^c)\right]^{s+\alpha+1}}\frac{y^{c}}{1+y^c}\left[\ln\left(\frac{1+y^c}{1+r_m^c}\right)\right]^{s-m-1}\exp(-c/\beta)\ dc\ dy}{\int_0^\infty \prod_{i=1}^m\left(\frac{r_i^{c-1}}{1+r_i^c}\right)\frac{c^{\alpha+\delta+m}}{\left[\frac{c}{\gamma}+\ln(1+r_m^c)\right]^{m+\alpha+1}}\exp(-c/\beta)dc} \tag{33}$$

From the above equation, under LINEX loss function, the moment generating function of the Bayes predictive density function of Y given $\mathbf{r}$ is $M_{Y|\mathbf{r}}(t)=E_{Y|\mathbf{r}}(e^{tY})$. It can be easily verified that the value of $\delta(Y)$ that minimizes $E_{Y|\mathbf{r}}(L(Y,\delta(Y))$ is $\delta^*(Y)=-\frac{1}{v}\ln M_{Y|\mathbf{r}}(-v)$, provided that $M_{Y|\mathbf{r}}(.)$ exists and is finite.

$$E_{Y|\mathbf{r}}(e^{-vY})=\int_0^\infty e^{-vY}h(y\mid \mathbf{r})\ dy=\frac{\Gamma(s+\alpha+1)}{\Gamma(m+\alpha+1)\Gamma(s-m)}*$$

$$\frac{\int_{r_m}^\infty\int_0^\infty \prod_{i=1}^m(\frac{r_i^{c-1}}{1+r_i^c})\frac{c^{\alpha+\delta+m+1}}{\left[\frac{c}{\gamma}+\ln(1+y^c)\right]^{s+\alpha+1}}e^{-vy+(c-1)\ln y-\ln(1+y^c)}\left[\ln\left(\frac{1+y^c}{1+r_m^c}\right)\right]^{s-m-1}\exp(-c/\beta)dcdy}{\int_0^\infty \prod_{i=1}^m\left(\frac{r_i^{c-1}}{1+r_i^c}\right)\frac{c^{\alpha+\delta+m}}{\left[\frac{c}{\gamma}+\ln(1+r_m^c)\right]^{m+\alpha+1}}\exp(-c/\beta)dc} \tag{34}$$

Then, we obtain the Bayes point predictor of the s-th upper record value $(s \geq m+1)$ as

$$\hat{Y}_{Linex} = -\frac{1}{v}\ln(M_{Y|\mathbf{r}}(-v)). \tag{35}$$

4. EXAMPLE

In order to illustrate the findings of Sections 2 and 3 an example is given. Using the values of $\alpha = 1$, $\beta = 3$, γ=2 and δ=3, we generate $c = 5.988$ and $k = 0.6068$ from the priors given by equations (9) and (10). Based on these values, a random sample of 9 record values from the Burr Type II distribution are generated, which are given below

0.8946, 0.9873, 1.2239, 1.7232, 1.7803, 3.0018, 3.8968, 5.0832, 6.3071

The first 7 will be used to estimate the parameters k and c, and also to predict the 9th record value. Based on this sample and using MATLAB, we obtained the MLE for the parameters k and c using

i) the joint pdf of the record values given by equations (6) and (7),
ii) by using the joint predictive likelihood function of $\mathbf{Y} = \mathbf{R}_s$, k and c given by equations (24), (25) and (26). Furthermore, the predicted value of the 9th record value was also estimated.

Similarly, Bayes estimators for the parameters k and c and the predicted 9th record value were obtained under the SEL and LINEX loss functions under different values for v. Table 1A summarizes the results for ML and Bayes estimates when the SEL function is used. Table 1B, gives LINEX estimates for different values of v.

TABLE 1A

Bayes and Bayes SEL estimates

Parameter	MLE		SEL Bayes Estimates
	joint	pred.	
c	6.1927	5.9065	8.2094
k	0.9667	0.6031	0.7588
y(9)	4.4966		6.6063

It should be pointed out that, for the LINEX estimators as the value of the asymmetry parameter v increases, the value of the estimate decreases. Furthermore, for small values of v the LINEX estimates are close to the SEL estimates. Figures 1, 2 and 3 show the plots of v versus c, k and $y(9)$.

TABLE 1B

LINEX estimates

v	LINEX estimates		
	c	k	y(9)
0.5000	5.7902	0.7009	8.4669
0.6000	5.5122	0.6915	8.2384
0.7000	5.2688	0.6826	8.0719
0.8000	5.0535	0.6742	7.9453
0.9000	4.8611	0.6662	7.8458
1.0000	4.6881	0.6585	7.7655
1.5000	4.1807	0.6250	7.5215
2.0000	4.0531	0.5973	7.3982
2.5000	3.9765	0.5737	7.3240
3.0000	3.9254	0.5533	7.2745
3.5000	3.8889	0.5352	7.2392
4.0000	3.8616	0.5191	7.2126
4.5000	3.8403	0.5074	7.1920
5.0000	3.8233	0.4919	7.1755

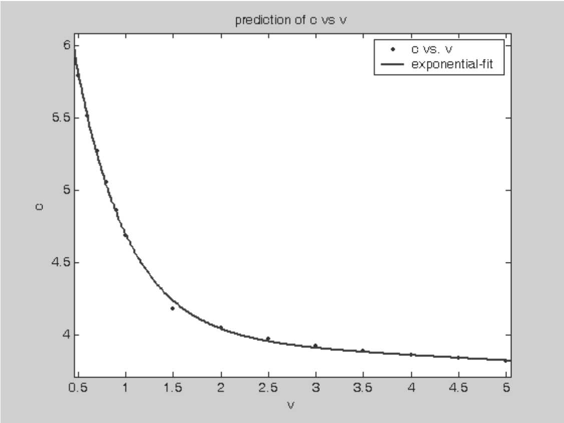

Figure 1 – Graph of prediction of c vs. v.

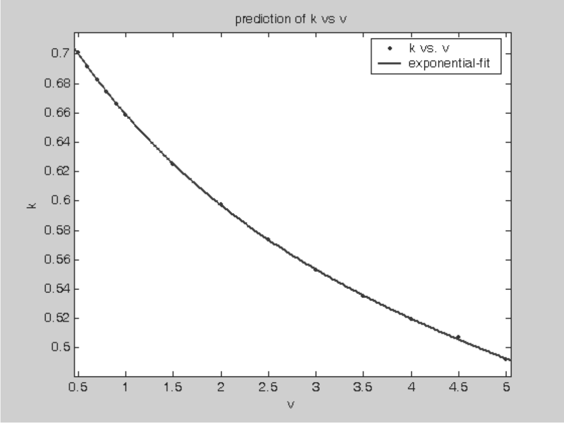

Figure 2 – Graph of prediction of k vs. v.

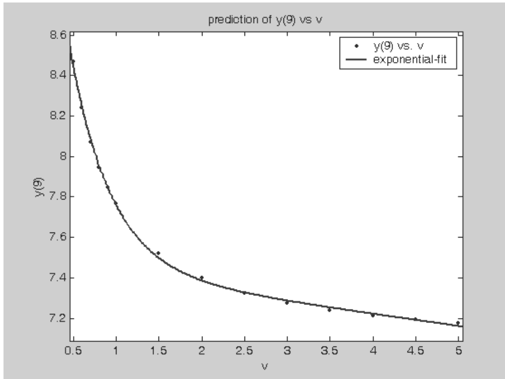

Figure 3 – Graph of prediction of $y(9)$ vs. v.

Curve fitting reveals that there is an exponential relationship. Table 2, gives the exponential function and the R^2–adj.

TABLE 2

Relationship between the LINEX estimates and v

Parameter	function	R^2–adj
c	c = 4.599exp(-1.836v) + 4exp(-0.008864v)	0.9987
k	k = 0.158exp(-0.596v) + 0.5933exp(-0.04103v)	0.9999
y(9)	y(9) = 2.878exp(-2.092v) + 7.466exp(-0.00833v)	0.9991

The values of R^2–adj are very close to 1, which implies almost a perfect relationship between each of the parameter estimates and v. Furthermore, one can predict the value of the LINEX estimator if he has decided on the value of v.

It is also of interest to plot the graphs of c vs. k, c vs. $y(9)$ and k vs. $y(9)$ when the values of v are specified. These graphs are shown below in Figures 4, 5 and 6.

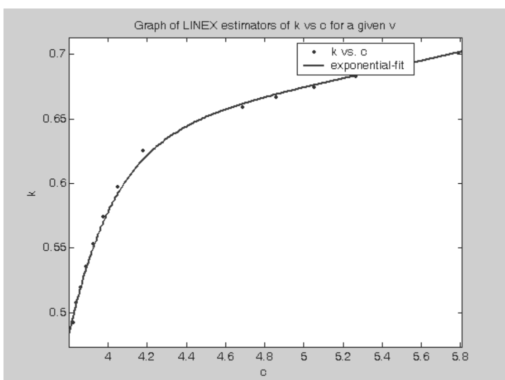

Figure 4 – Graph of LINEX estimators of k vs. c for a given v.

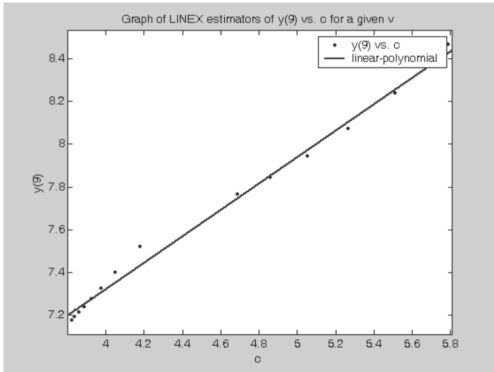

Figure 5 – Graph of LINEX estimators of $y(9)$ vs. c for a given v.

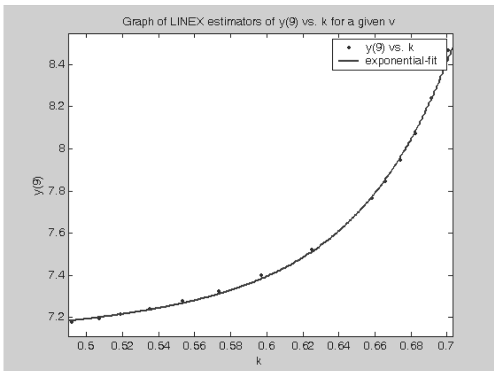

Figure 6 – Graph of LINEX estimators of $y(9)$ vs. k for a given v.

Again, from curve fitting it is evident that there is an exponential relationship in each of these figures. Table 3, gives the functions and R^2–adj.

TABLE 3

Relationship among LINEX estimates

Parameter	function	R^2–adj
k	k = 0.5292exp(0.04867c) – 2200000exp(-4.334c)	0.9977
y(9)	y(9) = 4.847+0.6188 c	0.9930
y(9)	y(9) = 6.904exp(0.07425k) + 0.00000exp(18.768)	0.9987

If one knows the estimate of one parameter, using the functions from Table 3 can estimate the remaining two.

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SUMMARY

Bayesian analysis for the Burr type XII distribution based on record values

In this paper we reviewed and extended some results that have been derived on record values from the two parameters Burr Type XII distribution. The two parameters were assumed to be random variables and Bayes estimates were derived on the basis of a linear exponential (LINEX) loss function. Estimates for future record values were derived using non Bayesian and Bayesian approaches. In the Bayesian approach we reviewed the estimators obtained by Ahmedi and Doostparast (2006) using the well known squared error loss (SEL) function and we derived estimate for the future record value under LINEX loss function. A numerical example with tables and figures illustrated the findings.