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# MEASURE OF DEPARTURE FROM MARGINAL POINT-SYMMETRY FOR TWO-WAY CONTINGENCY TABLES

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## 1. INTRODUCTION

First, consider an $r \times r$ square contingency table. Let $p_{ij}$ denote the probability that an observation will fall in the $i$ th row and $j$ th column of the table ($i = 1,\ldots,r; j = 1,\ldots,r$). Wall and Lienert (1976) considered the point-symmetry model defined by

$$p_{ij} = \psi_{ij} \quad (i = 1,\ldots,r; j = 1,\ldots,r),$$

where $\psi_{ij} = \psi_{i^* j^*}$ and the symbol "$*$" denotes $i^* = r + 1 - i$.

Next, consider an $r \times c$ rectangular contingency table. Let $p_{ij}$ denote the probability that an observation will fall in the $i$ th row and $j$ th column of the table ($i = 1,\ldots,r; j = 1,\ldots,c$). Tomizawa (1985) extended the point-symmetry model for an $r \times c$ contingency table as follows:

$$p_{ij} = \psi_{ij} \quad (i = 1,\ldots,r; j = 1,\ldots,c),$$

where $\psi_{ij} = \psi_{i^* j^{**}}$, $i^* = r + 1 - i$ and $j^{**} = c + 1 - j$ (see also Tomizawa, Yamamoto and Tahata, 2007). In addition, Tomizawa (1985) considered the marginal point-symmetry defined by

$$p_{i\cdot} = p_{i^*\cdot} \quad (i = 1,\ldots,r),$$

and

$$p_{\cdot j} = p_{\cdot j^{**}} \quad (j = 1,\ldots,c),$$

where $p_{i\cdot} = \sum_{t=1}^{c} p_{it}$ and $p_{\cdot j} = \sum_{s=1}^{r} p_{sj}$.

Consider the data in Tables 1 and 2. Table 1 taken directly from Agresti (1984, p. 206) describes the cross-classification of father's and his son's social classes in British. Table 2 taken directly from Hashimoto (1999, p. 151) describes the cross-classification of father's and his son's social classes in Japan.

TABLE 1

*The data are cross-classification of father's and his son's social classes in British (Agresti, 1984, p. 206)*

| Father's status | Son's status | | | | | Total |
|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | |
| (1) | 50 | 45 | 8 | 18 | 8 | 129 |
| (2) | 28 | 174 | 84 | 154 | 55 | 495 |
| (3) | 11 | 78 | 110 | 223 | 96 | 518 |
| (4) | 14 | 150 | 185 | 714 | 447 | 1510 |
| (5) | 3 | 42 | 72 | 320 | 411 | 848 |
| Total | 106 | 489 | 459 | 1429 | 1017 | 3500 |

TABLE 2

*The data are cross-classification of father's and his son's social classes in Japan (Hashimoto, 1999, p. 151)*

| Father's status | Son's status | | | | | Total |
|---|---|---|---|---|---|---|
| | (1) | (2) | (3) | (4) | (5) | |
| (1) | 29 | 43 | 25 | 31 | 4 | 132 |
| (2) | 23 | 159 | 89 | 38 | 14 | 323 |
| (3) | 11 | 69 | 184 | 34 | 10 | 308 |
| (4) | 42 | 147 | 148 | 184 | 17 | 538 |
| (5) | 42 | 176 | 377 | 114 | 298 | 1007 |
| Total | 147 | 594 | 823 | 401 | 343 | 2308 |

For Table 1, we are interested in, for example, (i) whether the probability that both a father's and his son's social classes are "(1)" is equal to the probability that both a father's and his son's social classes are "(5)", the probability that a father's social class is "(1)" and his son's social class is "(5)" is equal to the probability that a father's social class is "(5)" and his son's social class is "(1)", and so on, (ii) whether the probability that a father's social class is "(1)" is equal to the probability that his social class is "(5)", the probability that a son's social class is "(2)" is equal to the probability that his social class is "(4)", and so on.

Namely, (i) means the point-symmetry of cell probabilities with respect to the center cell in the table, and (ii) means the point-symmetry of marginal probabilities (i.e., the marginal point-symmetry) in the table.

When the point-symmetry model does not hold, we are interested in measuring the degree of departure from point-symmetry. Tomizawa et al. (2007) proposed a measure to represent the degree of departure from point-symmetry for two-way contingency tables. However, when we want to see the degree of departure from marginal point-symmetry, we cannot use the measure given by Tomizawa et al. (2007), because the measure can only measure the degree of departure from point-symmetry. Therefore we are now interested in a measure to represent what degree the departure from marginal point-symmetry is.

The purpose of this article is to propose a measure which represents the degree of departure from marginal point-symmetry in an $r \times c$ contingency table.

Section 2 proposes such a measure. Section 3 gives an approximate confidence interval for the measure. Section 4 analyzes the occupational mobility data using the proposed measure. Section 5 describes the relationship between the proposed measure and bivariate normal distribution, in terms of the simulation studies.

## 2. MEASURE OF DEPARTURE FROM MARGINAL POINT-SYMMETRY

Consider an $r \times c$ contingency table. Define

$$\left[\frac{r}{2}\right] = \begin{cases} \dfrac{r}{2} & (r : \text{even}), \\ \dfrac{r-1}{2} & (r : \text{odd}), \end{cases}$$

$$\left[\frac{c}{2}\right] = \begin{cases} \dfrac{c}{2} & (c : \text{even}), \\ \dfrac{c-1}{2} & (c : \text{odd}). \end{cases}$$

For instance, when $r = 4$, $\left[\frac{r}{2}\right] = 2$, and when $r = 5$, $\left[\frac{r}{2}\right] = 2$. Let

$$\delta_1 = \sum_{i=1}^{\left[\frac{r}{2}\right]} (p_{i\cdot} + p_{i^*\cdot}), \quad \delta_2 = \sum_{j=1}^{\left[\frac{c}{2}\right]} (p_{\cdot j} + p_{\cdot j^{**}}),$$

and let

$$q_{i\cdot} = \frac{p_{i\cdot}}{\delta_1}, \quad q_{i^*\cdot} = \frac{p_{i^*\cdot}}{\delta_1}, \quad q_{i\cdot}^{MPS} = q_{i^*\cdot}^{MPS} = \frac{q_{i\cdot} + q_{i^*\cdot}}{2}, \quad \left(i = 1, \ldots, \left[\frac{r}{2}\right]\right),$$

$$q_{\cdot j} = \frac{p_{\cdot j}}{\delta_2}, \quad q_{\cdot j^{**}} = \frac{p_{\cdot j^{**}}}{\delta_2}, \quad q_{\cdot j}^{MPS} = q_{\cdot j^{**}}^{MPS} = \frac{q_{\cdot j} + q_{\cdot j^{**}}}{2}, \quad \left(j = 1, \ldots, \left[\frac{c}{2}\right]\right).$$

Assuming that $\{p_{i\cdot} + p_{i^*\cdot} \neq 0\}$ and $\{p_{\cdot j} + p_{\cdot j^{**}} \neq 0\}$, we shall consider a measure to represent the degree of departure from marginal point-symmetry. For $\lambda > -1$, define the measure by

$$\Psi^{(\lambda)} = \frac{\delta_1 \psi_1^{(\lambda)} + \delta_2 \psi_2^{(\lambda)}}{\delta_1 + \delta_2},$$

where

$$\psi_1^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I_1^{(\lambda)}(\{q_{i\cdot}\};\{q_{i\cdot}^{MPS}\}),$$

with

$$I_1^{(\lambda)}(\cdot;\cdot) = \frac{1}{\lambda(\lambda+1)} \sum_{i=1}^{[\frac{r}{2}]} \left[ q_{i\cdot} \left\{ \left( \frac{q_{i\cdot}}{q_{i\cdot}^{MPS}} \right)^\lambda - 1 \right\} + q_{i^*\cdot} \left\{ \left( \frac{q_{i^*\cdot}}{q_{i\cdot}^{MPS}} \right)^\lambda - 1 \right\} \right],$$

and

$$\psi_2^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I_2^{(\lambda)}(\{q_{\cdot j}\};\{q_{\cdot j}^{MPS}\}),$$

with

$$I_2^{(\lambda)}(\cdot;\cdot) = \frac{1}{\lambda(\lambda+1)} \sum_{j=1}^{[\frac{c}{2}]} \left[ q_{\cdot j} \left\{ \left( \frac{q_{\cdot j}}{q_{\cdot j}^{MPS}} \right)^\lambda - 1 \right\} + q_{\cdot j^{**}} \left\{ \left( \frac{q_{\cdot j^{**}}}{q_{\cdot j}^{MPS}} \right)^\lambda - 1 \right\} \right],$$

and the value at $\lambda = 0$ is taken to be the continuous limit as $\lambda \to 0$. Thus,

$$\psi_1^{(0)} = \frac{1}{\log 2} I_1^{(0)}(\{q_{i\cdot}\};\{q_{i\cdot}^{MPS}\}),$$

where

$$I_1^{(0)}(\cdot;\cdot) = \sum_{i=1}^{[\frac{r}{2}]} \left[ q_{i\cdot} \log\left( \frac{q_{i\cdot}}{q_{i\cdot}^{MPS}} \right) + q_{i^*\cdot} \log\left( \frac{q_{i^*\cdot}}{q_{i\cdot}^{MPS}} \right) \right],$$

and

$$\psi_2^{(0)} = \frac{1}{\log 2} I_2^{(0)}(\{q_{\cdot j}\};\{q_{\cdot j}^{MPS}\}), \quad I_2^{(0)}(\cdot;\cdot) = \sum_{j=1}^{[\frac{c}{2}]} \left[ q_{\cdot j} \log\left( \frac{q_{\cdot j}}{q_{\cdot j}^{MPS}} \right) + q_{\cdot j^{**}} \log\left( \frac{q_{\cdot j^{**}}}{q_{\cdot j}^{MPS}} \right) \right];$$

therefore

$$\Psi^{(0)} = \frac{\delta_1 \psi_1^{(0)} + \delta_2 \psi_2^{(0)}}{\delta_1 + \delta_2}.$$

We point out that the submeasure $\psi_1^{(\lambda)}$ represents the degree of departure from point-symmetry of row marginal distribution, the submeasure $\psi_2^{(\lambda)}$ represents the degree of departure from point-symmetry of column marginal distribution, and then the measure $\Psi^{(\lambda)}$, which represents the degree of departure from marginal point-symmetry, is expressed as the weighted sum of the $\psi_1^{(\lambda)}$ and $\psi_2^{(\lambda)}$.

We note that $I_1^{(\lambda)}(\{q_{i\cdot}\};\{q_{i\cdot}^{MPS}\})$ is the power-divergence (Cressie and Read, 1984) between two distributions $\{q_{i\cdot}, q_{i^*\cdot}\}$ and $\{q_{i\cdot}^{MPS}, q_{i^*\cdot}^{MPS}\}$, $i=1,\ldots,\left[\frac{r}{2}\right]$, especially $I_1^{(0)}(\{q_{i\cdot}\};\{q_{i\cdot}^{MPS}\})$ is the Kullback-Leibler (Kullback and Leibler, 1951) information between them. Similarly, $I_2^{(\lambda)}(\{q_{\cdot j}\};\{q_{\cdot j}^{MPS}\})$ is the power-divergence between two distributions $\{q_{\cdot j}, q_{\cdot j^{**}}\}$ and $\{q_{\cdot j}^{MPS}, q_{\cdot j^{**}}^{MPS}\}$, $j=1,\ldots,\left[\frac{c}{2}\right]$, especially $I_2^{(0)}(\{q_{\cdot j}\};\{q_{\cdot j}^{MPS}\})$ is the Kullback-Leibler information between them. [For more details of the power-divergence, see Cressie and Read (1984), and Read and Cressie (1988, p. 15).] Note that a real value $\lambda$ is chosen by the user.

Let

$$q_{i\cdot}^c = \frac{q_{i\cdot}}{q_{i\cdot}+q_{i^*\cdot}}, \quad q_{i^*\cdot}^c = \frac{q_{i^*\cdot}}{q_{i\cdot}+q_{i^*\cdot}} \quad \left(i=1,\ldots,\left[\frac{r}{2}\right]\right),$$

and

$$q_{\cdot j}^c = \frac{q_{\cdot j}}{q_{\cdot j}+q_{\cdot j^{**}}}, \quad q_{\cdot j^{**}}^c = \frac{q_{\cdot j^{**}}}{q_{\cdot j}+q_{\cdot j^{**}}} \quad \left(j=1,\ldots,\left[\frac{c}{2}\right]\right).$$

Note that the marginal point-symmetry model can be expressed as $\{q_{i\cdot}^c = q_{i^*\cdot}^c\}$ and $\{q_{\cdot j}^c = q_{\cdot j^{**}}^c\}$. Then, $\psi_1^{(\lambda)}$ and $\psi_2^{(\lambda)}$ may be expressed as

$$\psi_1^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1}\sum_{i=1}^{[\frac{r}{2}]}(q_{i\cdot}+q_{i^*\cdot})H_{1i}^{(\lambda)}(q_{i\cdot}^c, q_{i^*\cdot}^c),$$

and

$$\psi_2^{(\lambda)} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1}\sum_{j=1}^{[\frac{c}{2}]}(q_{\cdot j}+q_{\cdot j^{**}})H_{2j}^{(\lambda)}(q_{\cdot j}^c, q_{\cdot j^{**}}^c),$$

where

$$H_{1i}^{(\lambda)}(\cdot) = \frac{1}{\lambda}[1-(q_{i\cdot}^c)^{\lambda+1}-(q_{i^*\cdot}^c)^{\lambda+1}],$$

$$H_{2j}^{(\lambda)}(\cdot) = \frac{1}{\lambda}[1-(q_{\cdot j}^c)^{\lambda+1}-(q_{\cdot j^{**}}^c)^{\lambda+1}],$$

and the value at $\lambda = 0$ is taken to be the continuous limit as $\lambda \to 0$. Thus,

$$\psi_1^{(0)} = 1 - \frac{1}{\log 2} \sum_{i=1}^{[\frac{r}{2}]} (q_{i\cdot} + q_{i^*\cdot}) H_{1i}^{(0)}(q_{i\cdot}^c, q_{i^*\cdot}^c),$$

where

$$H_{1i}^{(0)}(\cdot) = -q_{i\cdot}^c \log q_{i\cdot}^c - q_{i^*\cdot}^c \log q_{i^*\cdot}^c,$$

and

$$\psi_2^{(0)} = 1 - \frac{1}{\log 2} \sum_{j=1}^{[\frac{c}{2}]} (q_{\cdot j} + q_{\cdot j^{**}}) H_{2j}^{(0)}(q_{\cdot j}^c, q_{\cdot j^{**}}^c),$$

where

$$H_{2j}^{(0)}(\cdot) = -q_{\cdot j}^c \log q_{\cdot j}^c - q_{\cdot j^{**}}^c \log q_{\cdot j^{**}}^c.$$

Note that $H_{1i}^{(\lambda)}(q_{i\cdot}^c, q_{i^*\cdot}^c)$ and $H_{2j}^{(\lambda)}(q_{\cdot j}^c, q_{\cdot j^{**}}^c)$ are the Patil and Taillie's (1982) diversity indexes which include the Shannon entropy when $\lambda = 0$ in a special case.

We point out that when $\lambda = 0$, the submeasure $\psi_1^{(0)}$ in the measure $\Psi^{(0)}$ can be expressed as

$$\psi_1^{(0)} = \frac{1}{\log 2} \min_{\{\xi_{i\cdot}\}} I_1^{(0)}(\{q_{i\cdot}\};\{\xi_{i\cdot}\}), \tag{1}$$

where

$$I_1^{(0)}(\{q_{i\cdot}\};\{\xi_{i\cdot}\}) = \sum_{i=1}^{[\frac{r}{2}]} \left[ q_{i\cdot} \log\left(\frac{q_{i\cdot}}{\xi_{i\cdot}}\right) + q_{i^*\cdot} \log\left(\frac{q_{i^*\cdot}}{\xi_{i^*\cdot}}\right)\right],$$

$$\xi_{i\cdot} = \xi_{i^*\cdot}, \quad \xi_{i\cdot} > 0,$$

$$\sum_{i=1}^{[\frac{r}{2}]} (\xi_{i\cdot} + \xi_{i^*\cdot}) = 1.$$

Namely, $\psi_1^{(0)}$ indicates the minimum Kullback-Leibler information between $\{q_{i\cdot}, q_{i^*\cdot}\}$ and $\{\xi_{i\cdot}, \xi_{i^*\cdot}\}$ with the structure of marginal point-symmetry. We note that $\{\xi_{i\cdot}(=\xi_{i^*\cdot})\}$ minimize $I_1^{(0)}(\cdot;\cdot)$ in (1) when $\{\xi_{i\cdot} = (q_{i\cdot} + q_{i^*\cdot})/2 = q_{i\cdot}^{\mathrm{MPS}}\}$. In a similar way, the submeasure $\psi_2^{(0)}$ in the measure $\Psi^{(0)}$ is expressed. Note that the readers may also be interested in (1) with $I_1^{(0)}(\cdot;\cdot)$ replaced by the power-divergence $I_1^{(\lambda)}(\cdot;\cdot)$; however, it is difficult to obtain the value of $\{\xi_{i\cdot}\}$ such that $I_1^{(\lambda)}(\cdot;\cdot)$ is minimum when $\lambda \neq 0$.

The submeasures $\psi_1^{(\lambda)}$ and $\psi_2^{(\lambda)}$ must lie between 0 and 1. Therefore $\Psi^{(\lambda)}$ must lie between 0 and 1. We note that for each $\lambda$ $(>-1)$, (i) $\Psi^{(\lambda)}=0$ (i.e., $\psi_1^{(\lambda)}=\psi_2^{(\lambda)}=0$) if and only if the marginal point-symmetry model holds, and (ii) $\Psi^{(\lambda)}=1$ (i.e., $\psi_1^{(\lambda)}=\psi_2^{(\lambda)}=1$) if and only if the degree of departure from marginal point-symmetry is the largest in the sense that $q_{i\cdot}^c=0$ (then $q_{i^*\cdot}^c=1$) or $q_{i^*\cdot}^c=0$ (then $q_{i\cdot}^c=1$) for $i=1,\ldots,\left[\frac{r}{2}\right]$, and $q_{\cdot j}^c=0$ (then $q_{\cdot j^{**}}^c=1$) or $q_{\cdot j^{**}}^c=0$ (then $q_{\cdot j}^c=1$) for $j=1,\ldots,\left[\frac{c}{2}\right]$. We point out that this definition of maximum departure from marginal point-symmetry would be natural because the marginal point-symmetry $\{p_{i\cdot}=p_{i^*\cdot}\}$ and $\{p_{\cdot j}=p_{\cdot j^{**}}\}$ can be expressed as $\{q_{i\cdot}^c=q_{i^*\cdot}^c=1/2\}$ and $\{q_{\cdot j}^c=q_{\cdot j^{**}}^c=1/2\}$. We note that $\Psi^{(\lambda)}$ represents the degree of departure from marginal point-symmetry and the degree increases as the value of $\Psi^{(\lambda)}$ increases.

## 3. APPROXIMATE CONFIDENCE INTERVAL FOR MEASURE

Let $n_{ij}$ denote the observed frequency in the $i$th row and $j$th column of the table ($i=1,\ldots,r; j=1,\ldots,c$). Assuming that a multinomial distribution applies to the $r\times c$ table, we shall consider an approximate standard error and large-sample confidence interval for the measure $\Psi^{(\lambda)}$ using the delta method, descriptions of which are given by, for example, Bishop, Fienberg and Holland (1975, Sec. 14.6). The sample version of $\Psi^{(\lambda)}$, i.e., $\hat{\Psi}^{(\lambda)}$, is given by $\Psi^{(\lambda)}$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij}=n_{ij}/n$ and $n=\sum\sum n_{ij}$. Using the delta method, $\sqrt{n}(\hat{\Psi}^{(\lambda)}-\Psi^{(\lambda)})$ has asymptotically (as $n\to\infty$) a normal distribution with mean zero and variance

$$\sigma^2[\Psi^{(\lambda)}]=\frac{1}{(\delta_1+\delta_2)^2}\sum_{i=1}^{r}\sum_{j=1}^{c}(w_{ij}^{(\lambda)}-d_{ij}\Psi^{(\lambda)})^2 p_{ij},$$

where

$$w_{ij}^{(\lambda)} = d_{1(i)}\psi_1^{(\lambda)} + \delta_1\Delta_{1(i)}^{(\lambda)} + d_{2(j)}\psi_2^{(\lambda)} + \delta_2\Delta_{2(j)}^{(\lambda)},$$

$$d_{ij} = d_{1(i)} + d_{2(j)},$$

with

$$d_{1(i)} = \begin{cases} 0 & \left(r \text{ is odd and } i = \dfrac{r+1}{2}\right), \\ 1 & (\text{otherwise}), \end{cases}$$

$$d_{2(j)} = \begin{cases} 0 & \left(c \text{ is odd and } j = \dfrac{c+1}{2}\right), \\ 1 & (\text{otherwise}), \end{cases}$$

and for $\lambda > -1; \lambda \neq 0$,

$$\Delta_{1(i)}^{(\lambda)} = \begin{cases} 0 & \left(r \text{ is odd and } i = \dfrac{r+1}{2}\right), \\ \dfrac{1}{\delta_1}\Big[1 - \psi_1^{(\lambda)} - \dfrac{2^\lambda}{2^\lambda - 1}\{1 - (q_{i\cdot}^c)^\lambda & \\ \qquad - \lambda q_{i^*\cdot}^c((q_{i\cdot}^c)^\lambda - (q_{i^*\cdot}^c)^\lambda)\}\Big] & (\text{otherwise}), \end{cases}$$

$$\Delta_{2(j)}^{(\lambda)} = \begin{cases} 0 & \left(c \text{ is odd and } j = \dfrac{c+1}{2}\right), \\ \dfrac{1}{\delta_2}\Big[1 - \psi_2^{(\lambda)} - \dfrac{2^\lambda}{2^\lambda - 1}\{1 - (q_{\cdot j}^c)^\lambda & \\ \qquad - \lambda q_{\cdot j^{**}}^c((q_{\cdot j}^c)^\lambda - (q_{\cdot j^{**}}^c)^\lambda)\}\Big] & (\text{otherwise}), \end{cases}$$

for $\lambda = 0$,

$$\Delta_{1(i)}^{(0)} = \begin{cases} 0 & \left(r \text{ is odd and } i = \dfrac{r+1}{2}\right), \\ \dfrac{1}{\delta_1}\left(1 - \psi_1^{(0)} + \dfrac{1}{\log 2}\log q_{i\cdot}^c\right) & (\text{otherwise}), \end{cases}$$

$$\Delta_{2(j)}^{(0)} = \begin{cases} 0 & \left( c \text{ is odd and } j = \dfrac{c+1}{2} \right), \\ \dfrac{1}{\delta_2}\left(1 - \psi_2^{(0)} + \dfrac{1}{\log 2}\log q_{\cdot j}^{c}\right) & \text{(otherwise).} \end{cases}$$

Note that $\sigma^2[\Psi^{(0)}] = \lim_{\lambda \to 0} \sigma^2[\Psi^{(\lambda)}]$.

Let $\hat{\sigma}^2[\Psi^{(\lambda)}]$ denote $\sigma^2[\Psi^{(\lambda)}]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then, $\hat{\sigma}[\Psi^{(\lambda)}]/\sqrt{n}$ is an estimated standard error for $\hat{\Psi}^{(\lambda)}$, and $\hat{\Psi}^{(\lambda)} \pm z_{p/2}\hat{\sigma}[\Psi^{(\lambda)}]/\sqrt{n}$ is an approximate $100(1-p)\%$ confidence interval for $\Psi^{(\lambda)}$, where $z_{p/2}$ is the percentage point from the standard normal distribution that corresponds to two-tail probability equal to $p$.

## 4. EXAMPLES

Consider the data in Tables 1 and 2 again.

Since the confidence intervals for $\Psi^{(\lambda)}$ applied to the data in Tables 1 and 2 do not include zero for all $\lambda$ (see Table 3), these would indicate that there is not a structure of marginal point-symmetry in each table.

TABLE 3

*Estimate of measure $\Psi^{(\lambda)}$, approximate standard error for $\hat{\Psi}^{(\lambda)}$ and approximate 95% confidence interval for $\Psi^{(\lambda)}$, applied to Tables 1 and 2*

| Values of $\lambda$ | Estimated measure | Standard error | Confidence interval |
|---|---|---|---|
| | (a) For Table 1 | | |
| –0.4 | 0.218 | 0.010 | (0.199, 0.238) |
| 0 | 0.295 | 0.012 | (0.271, 0.320) |
| 0.6 | 0.355 | 0.014 | (0.328, 0.383) |
| 1.0 | 0.372 | 0.014 | (0.344, 0.400) |
| 1.8 | 0.376 | 0.014 | (0.348, 0.404) |
| | (b) For Table 2 | | |
| –0.4 | 0.144 | 0.009 | (0.125, 0.162) |
| 0 | 0.193 | 0.012 | (0.170, 0.216) |
| 0.6 | 0.231 | 0.013 | (0.205, 0.257) |
| 1.0 | 0.242 | 0.013 | (0.215, 0.268) |
| 1.8 | 0.243 | 0.013 | (0.217, 0.270) |

We shall investigate the degree of departure from marginal point-symmetry in more details. For instance, when $\lambda = 1$, the estimated measure $\hat{\Psi}^{(1)}$ equals 0.372 for Table 1, and 0.242 for Table 2 (see Table 3). Thus, (i) for Table 1, the degree of departure from marginal point-symmetry is estimated to be 37.2 percent of the maximum degree of departure from marginal point-symmetry, (ii) for Table 2, it is estimated to be 24.2 percent of the maximum degree of departure from marginal point-symmetry.

When the degrees of departure from marginal point-symmetry in Tables 1 and 2 are compared using the confidence interval for $\Psi^{(\lambda)}$, the degree of departure in Table 1 is greater than that in Table 2.

## 5. SIMULATION STUDIES

We shall consider random variables $Z_1$ and $Z_2$ having a joint bivariate normal distribution with means $E(Z_1)=\mu_1$ and $E(Z_2)=\mu_2$, variances $Var(Z_1)=\sigma_1^2$, $Var(Z_2)=\sigma_2^2$, and correlation $Corr(Z_1,Z_2)=\rho$. When it is reasonable to assume underlying bivariate normal distribution, we shall consider the relationship between the degree of departure from marginal point-symmetry and the value of measure $\Psi^{(\lambda)}$.

Suppose that there is an underlying bivariate normal distribution and suppose that a $4\times 4$ table is formed using cutpoints for each variable at $-0.6$, $0$, and $0.6$. Then, in terms of simulation studies, we shall consider some $4\times 4$ tables of sample size 10000, formed from an underlying bivariate normal distribution with the conditions (i) $\sigma^2=1.0$, $\rho=0.6$ and $\mu_1=0,0.2,0.4,0.6$; $\mu_2=0,0.2,0.4,0.6$, (ii) $\mu_1=\mu_2=0.2$, $\rho=0.6$ and $\sigma_1^2=1.0,1.2,1.5$; $\sigma_2^2=1.0,1.2,1.5$, and (iii) $\mu_1=0$, $\mu_2=0.2$, $\sigma_1^2=\sigma_2^2=1.0$ and $\rho=0,0.3,0.6,0.9$.

First, we shall consider the case of (i). Table 4 gives the cases of $(\mu_1, \mu_2)$ = (0, 0), (0, 0.4), (0, 0.6), (0.2, 0.4), (0.2, 0.6), (0.4, 0.4) and (0.4, 0.6). Table 5 gives the corresponding values of $\hat{\Psi}^{(0)}$ and $\hat{\Psi}^{(1)}$ applied to $4\times 4$ tables of sample size 10000, formed from underlying bivariate normal distribution with various $(\mu_1, \mu_2)$.

We see from Table 4 that for a fixed $\mu_1$ (and fixed $\sigma_1^2$, $\sigma_2^2$ and $\rho$), the number of observations falling in fourth column increases as the value of $\mu_2$ increases. This is because each $4\times 4$ table is formed using cutpoints of each variable at $-0.6$, $0$, and $0.6$. Thus, for a fixed $\mu_1$, the degree of departure from marginal point-symmetry becomes greater as the value of $\mu_2$ increases. Similarly, for a fixed $\mu_2$, it becomes greater as the value of $\mu_1$ increases.

Indeed, we see from Table 5 that for a fixed $\lambda$ the measure $\hat{\Psi}^{(\lambda)}$ tends to increase in order of $(\mu_1,\mu_2)=(\mu_1,0)$, $(\mu_1,0.2)$, $(\mu_1,0.4)$, $(\mu_1,0.6)$ and in order of $(\mu_1,\mu_2)=(0,\mu_2)$, $(0.2,\mu_2)$, $(0.4,\mu_2)$, $(0.6,\mu_2)$. Therefore the measure $\Psi^{(\lambda)}$ may be appropriate for representing the degree of departure from marginal point-symmetry if it is reasonable to assume underlying bivariate normal distribution.

TABLE 4

*The 4 × 4 tables of sample size 10000, formed by using cutpoints for each variable at* $-0.6$ *,* $0$ *and* $0.6$ *, from an underlying bivariate normal distribution with the conditions* $\sigma^2 = 1.0$ *,* $\rho = 0.6$ *with some (* $\mu_1$ *,* $\mu_2$ *)*

(a) $(\mu_1, \mu_2)$ = (0, 0)

| | | | | | Total |
|---|---|---|---|---|---|
| | 1515 | 692 | 359 | 152 | 2718 |
| | 695 | 627 | 532 | 373 | 2227 |
| | 360 | 611 | 642 | 657 | 2270 |
| | 176 | 362 | 687 | 1560 | 2785 |
| Total | 2746 | 2292 | 2220 | 2742 | 10000 |

(b) $(\mu_1, \mu_2)$ = (0, 0.4)

| | | | | | Total |
|---|---|---|---|---|---|
| | 1019 | 743 | 550 | 368 | 2680 |
| | 345 | 561 | 693 | 708 | 2307 |
| | 153 | 379 | 650 | 1114 | 2296 |
| | 44 | 202 | 493 | 1978 | 2717 |
| Total | 1561 | 1885 | 2386 | 4168 | 10000 |

(c) $(\mu_1, \mu_2)$ = (0, 0.6)

| | | | | | Total |
|---|---|---|---|---|---|
| | 712 | 742 | 649 | 541 | 2644 |
| | 239 | 470 | 662 | 909 | 2280 |
| | 102 | 296 | 535 | 1353 | 2286 |
| | 29 | 110 | 402 | 2249 | 2790 |
| Total | 1082 | 1618 | 2248 | 5052 | 10000 |

(d) $(\mu_1, \mu_2)$ = (0.2, 0.4)

| | | | | | Total |
|---|---|---|---|---|---|
| | 889 | 554 | 382 | 238 | 2063 |
| | 432 | 537 | 534 | 608 | 2111 |
| | 231 | 451 | 650 | 1040 | 2372 |
| | 93 | 283 | 691 | 2387 | 3454 |
| Total | 1645 | 1825 | 2257 | 4273 | 10000 |

(e) $(\mu_1, \mu_2)$ = (0.2, 0.6)

| | | | | | Total |
|---|---|---|---|---|---|
| | 692 | 570 | 489 | 339 | 2090 |
| | 248 | 445 | 588 | 796 | 2077 |
| | 136 | 358 | 621 | 1222 | 2337 |
| | 50 | 204 | 515 | 2727 | 3496 |
| Total | 1126 | 1577 | 2213 | 5084 | 10000 |

(f) $(\mu_1, \mu_2)$ = (0.4, 0.4)

| | | | | | Total |
|---|---|---|---|---|---|
| | 723 | 450 | 271 | 147 | 1591 |
| | 466 | 533 | 488 | 407 | 1894 |
| | 290 | 491 | 640 | 857 | 2278 |
| | 143 | 432 | 863 | 2799 | 4237 |
| Total | 1622 | 1906 | 2262 | 4210 | 10000 |

(g) $(\mu_1, \mu_2)$ = (0.4, 0.6)

| | | | | | Total |
|---|---|---|---|---|---|
| | 595 | 438 | 344 | 208 | 1585 |
| | 315 | 462 | 499 | 582 | 1858 |
| | 171 | 413 | 698 | 1067 | 2349 |
| | 86 | 322 | 774 | 3026 | 4208 |
| Total | 1167 | 1635 | 2315 | 4883 | 10000 |

TABLE 5

*The values of* $\hat{\Psi}^{(0)}$ *(upper value) and* $\hat{\Psi}^{(1)}$ *(lower value) applied to* $4\times 4$ *table, obtained from an underlying bivariate normal distribution with various (* $\mu_1$ *,* $\mu_2$ *)*

| $\mu_1$ | $\mu_2$ | | | |
|---|---|---|---|---|
| | 0 | 0.2 | 0.4 | 0.6 |
| 0 | 0.0001 | 0.0105 | 0.0465 | 0.1044 |
| | 0.0001 | 0.0144 | 0.0623 | 0.1338 |
| 0.2 | 0.0127 | 0.0245 | 0.0586 | 0.1158 |
| | 0.0174 | 0.0337 | 0.0789 | 0.1499 |
| 0.4 | 0.0445 | 0.0606 | 0.0902 | 0.1392 |
| | 0.0596 | 0.0815 | 0.1208 | 0.1822 |
| 0.6 | 0.0964 | 0.1116 | 0.1437 | 0.1997 |
| | 0.1245 | 0.1449 | 0.1876 | 0.2571 |

Next consider the case of (ii). For a fixed $\sigma_1^2$ (and fixed $\mu_1$, $\mu_2$ and $\rho$), even if the value of $\sigma_2^2$ increases, the degree of departure from marginal point-symmetry differs little. Indeed, the values of $\hat{\Psi}^{(\lambda)}$ applied to such $4\times 4$ tables, take near same value in most cases, although the details are omitted.

Finally, consider the case of (iii). For fixed $\mu_1$, $\mu_2$, $\sigma_1^2$ and $\sigma_2^2$, it seems that the degree of departure from marginal point-symmetry does not change so much, even if the value of $\rho$ increases. Indeed, the values of $\hat{\Psi}^{(\lambda)}$ applied to such $4\times 4$ tables, differ little, although the details are omitted.

In conclusion, if it is reasonable to assume underlying bivariate normal distribution, especially the values of means $\mu_1$ and $\mu_2$ may influence the degree of departure from marginal point-symmetry.

## 6. CONCLUDING REMARKS

The measure $\Psi^{(\lambda)}$ would be useful for comparing the degrees of departure from marginal point-symmetry in several tables because the measure $\Psi^{(\lambda)}$ always ranges between 0 and 1 independent of the dimensions $r$ and $c$, and sample size $n$.

The measure $\Psi^{(\lambda)}$ would be useful when we want to see with a single summary measure how degree the departure from marginal point-symmetry is toward the maximum departure from marginal point-symmetry. We defined the maximum departure from marginal point-symmetry in Section 2, as $\{q^c_{i\cdot}=0$ (then $q^c_{i^*\cdot}=1)$ or $q^c_{i^*\cdot}=0$ (then $q^c_{i\cdot}=1)\}$ and $\{q^c_{\cdot j}=0$ (then $q^c_{\cdot j^{**}}=1)$ or $q^c_{\cdot j^{**}}=0$ (then $q^c_{\cdot j}=1)\}$. This seems natural as the definition of the maximum departure from marginal point-symmetry.

The readers may be interested in which value of $\lambda$ is preferred for a given table. However, in comparing tables, it seems different to discuss this. For example, consider the artificial data in Tables 6a and 6b. We see from Table 6c that the value of $\hat{\Psi}^{(0)}$ is greater for Table 6a than for Table 6b, but the value of $\hat{\Psi}^{(1)}$ is less for Table 6a than for Table 6b. So, for these cases, it may be impossible to decide (by using $\hat{\Psi}^{(\lambda)}$) whether the degree of departure from marginal point-symmetry is greater for Table 6a or for Table 6b. But generally, for the comparison between two tables, it would be possible to draw a conclusion if $\hat{\Psi}^{(\lambda)}$ for every $\lambda$ is always greater (or always less) for one table than for the other table. Thus, it may be dangerous if the analyst uses $\hat{\Psi}^{(\lambda)}$ for *only* one specified value of $\lambda$ for comparing the degrees of departure from marginal point-symmetry in several tables. It seems to be important and safe that the analyst calculates the value of $\hat{\Psi}^{(\lambda)}$ for *various* values of $\lambda$ and discusses the degree of departure from marginal point-symmetry in terms of $\hat{\Psi}^{(\lambda)}$ values.

TABLE 6

*(a), (b) Artificial data ($n$ is sample size) and (c) corresponding values of* $\hat{\Psi}^{(\lambda)}$

(a) $n = 1052$

| | | | |
|---|---|---|---|
| 17 | 89 | 125 | 128 |
| 14 | 83 | 139 | 138 |
| 9 | 78 | 117 | 115 |

(b) $n = 1217$

| | | | |
|---|---|---|---|
| 24 | 113 | 89 | 51 |
| 16 | 102 | 198 | 170 |
| 15 | 75 | 181 | 183 |

(c) Values of $\hat{\Psi}^{(\lambda)}$

| Values of $\lambda$ | For Table 6a | For Table 6b |
|---|---|---|
| –0.4 | 0.1107* | 0.1054 |
| 0 | 0.1455* | 0.1427 |
| 0.6 | 0.1702 | 0.1715 |
| 1.0 | 0.1767 | 0.1797 |
| 1.8 | 0.1779 | 0.1813 |

* indicates that $\hat{\Psi}^{(\lambda)}$ is greater for Table 6a than for Table 6b.

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## SUMMARY

*Measure of departure from marginal point-symmetry for two-way contingency tables*

For two-way contingency tables, Tomizawa (1985) considered the point-symmetry and marginal point-symmetry models, and Tomizawa, Yamamoto and Tahata (2007) proposed a measure to represent the degree of departure from point-symmetry. The present paper proposes a measure to represent the degree of departure from marginal point-symmetry for two-way tables. The proposed measure is expressed by using Cressie-Read power-divergence or Patil-Taillie diversity index. This measure would be useful for comparing the degrees of departure from marginal point-symmetry in several tables. The relationship between the degree of departure from marginal point-symmetry and the measure is shown when it is reasonable to assume underlying bivariate normal distribution. Examples are shown.