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EFFECTS OF THE TWO-COMPONENT MEASUREMENT ERROR MODEL ON $\bar{X}$ CONTROL CHARTS

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1. INTRODUCTION

It is widely acknowledged within the industrial context that measurement errors may significantly alter the performance of statistical process-control methodologies, as has been shown by the works of several authors, including Kanzuka (1986), Mittag and Stemann (1998), Linna and Woodall (2001), Linna *et al.* (2001), Maravelakis *et al.* (2004) and Maravelakis (2007).

In these studies, the usual statistical model relating the measured quantity to the true, albeit not observable, value is usually Gaussian and additive:

$$Y = X + \varepsilon \tag{1}$$

where Y is the measured quantity, X is the not observable value of the relevant quality characteristic, and ε is the measurement error. Both X and ε are assumed to be independent and normally distributed.

However, situations arise where the measurement error is not normally distributed, as has been pointed out in Burdick *et al.* (2003), or where a different error model needs to be considered. For example, Montgomery and Runger (1993) remark that the dependence of the measurements variance on the mean level of the product characteristic is a common phenomenon. Linna and Woodall (2001) and Maravelakis *et al.* (2004), in a statistical process control situation, examine a model where measurement error variance is a linearly increasing function of the process mean, while Wilson *et al.* (2004a), in assessing a manufacturing process's performance, stress the motivations for assuming a proportional error structure. Hence the need for studies concerning the performances of the control charts under realistic extensions of the most common error models.

A proposal in this direction comes from the practice of analytical chemistry and environmental monitoring, where experimental evidence shows that situations occur where two types of measurement errors ought to be considered (Rocke and Lorenzato (1995) and Rocke *et al.* (2003)): a measurement error that is constant over a range of measures close to zero, while, at higher measures, the measurement error is proportional to the amount of the chemical substance in question.

In this paper we propose to extend the additive and Gaussian error model, which is traditionally used in statistical process control literature, in a more general way so as to include the structure of the two-component error model. We study the effects of the proposed error model on the in-control and out-of control performances of the traditional 3-sigma Shewhart control chart for means. We also address the problem of designing the $\bar{X}$ control chart in the presence of this error model.

Since one of the effects of the proposed error model is the non-normality of the sample statistic, we examine several control chart design methods that take into account the asymmetry induced by measurement errors. We explore a method of constructing control charts for the process level using the weighted variance (WV) approach, a skewness correction (SC) method and a method based on the empirical reference distribution (ERD). These methods are compared by Monte Carlo simulation. Results show that the control charts designed with the SC and ERD methods are more robust with respect to the non normality caused by measurement errors.

Extremely summarising we find that the proposed error model causes an asymmetric behaviour and a great reduction in the power function of the monitoring algorithm. We discuss and compare control charts for facing the asymmetry. Our contribution is a step towards the proposal of a plausible physical model of measurement error, which includes the Gaussian additive model as a specific case, in the statistical process control framework.

The present paper is organized as follows. Section 2 describes the two-component measurement error model as proposed by Rocke and Lorenzato (1995), and then generalises model (1) using the two-component error model structure. Section 3 examines the effects of the proposed gauge imprecision model on the statistical properties of the Shewhart control chart for averages. Section 4 illustrates the different methods, which are compared in Section 5. Section 6 offers some concluding remarks. An Appendix summarizes some additional characteristics of the method based on the empirical reference distribution.

2. THE TWO-COMPONENT ERROR MODEL

Researches started in the field of analytical chemistry have established that the measurement error of an analytical method is of two types (Caulcutt and Boddy (1983), Massart *et al.* (1988)): (i) an additive error that always exists but is only noticeable for zero and near-zero concentrations; (ii) a proportional or multiplicative error that always exists but is noticeable at higher concentrations. This situation causes some difficulty in estimating the overall precision of an analytical method especially for data in the "gray area" where a transition occurs between near-zero and higher concentration levels. The two-component error model (Rocke and Lorenzato (1995)) overcomes these difficulties by incorporating both types of error in a single model, with the advantage of describing the analytical precision of measurements over the entire usable range.

The two-component error model, or Rocke and Lorenzato model, is:

$$Y_\mu^e = \alpha + \mu\beta e^\eta + \varepsilon \tag{2}$$

where μ is the true concentration, Y_μ^e is the response at concentration μ, and α and β and are the calibration curve parameters. The model contains two independent errors: $\eta \sim N(0,\sigma_\eta^2)$ represents the proportional error, which is always present but only noticeable at concentrations significantly above zero; $\varepsilon \sim N(0,\sigma_m^2)$ represents the additive error, which is also always present, but is only really noticeable at near zero concentrations.

In model (2) μ is assumed non random and the unknown parameters are α, β, σ_m and σ_η. In their original article, Rocke and Lorenzato (1995) discussed the use of the maximum likelihood estimation method for these parameters. Gibbons *et al.* (1997) suggested to estimate the model parameters using the weighted least squares (WLS) method, but Rocke *et al.* (2003) pointed out that the WLS method is often not very stable and can lead to nonconvergence and impossible estimates. More recently, within a Bayesian framework, Jones (2004) proposed a Markov Chain Monte Carlo method for estimating the parameters.

The Rocke and Lorenzato model (2) has proved to be of importance also in bioavailability analysis (Rocke *et al.* 2003), environmental monitoring (Wilson *et al.* 2004b), and in the analysis of gene expression data (Rocke and Durbin, 2001).

From the above considerations the measurement error model (2) is very realistic in connection with several typologies of measurement devices and in different empirical issues. It is suitable for being adopted within a statistical process-control framework, when monitored data are measured using a measurement technology for which the two component error model is suitable.

This last consideration is also supported by the fact that the additive Gaussian error model (1), traditionally considered in the statistical quality control literature, may prove inadequate in situations where measurement systems induce non standard variance structures. For instance, Wilson *et al.* (2004a) consider a manufacturing process where the quality characteristic is the iron concentration (in ppm) determined by an emission spectroscopy. According to these authors reasons exist for postulating a measurement error variance which is proportional to the true part value, due to the particular measurement technology used for gauging the quality characteristic.

In order to extend model (1) by incorporating the error structure depicted in (2) we propose the error model

$$Y^e = \alpha + X\beta e^\eta + \varepsilon \tag{3}$$

where $X \sim N(\mu, \sigma_p^2)$. This error model has been also proposed in a measurement system capability framework (Cocchi and Scagliarini (2010)).

The distribution of the observable response Y^e involves three random variables: the normal variable ε, and the product of a normal variable, X, and a log-normal variable e^{η}. It follows that one effect of the two-component error model may be a significant departure from normality.

The expected value and the standard deviation of Y^e can be written respectively as

$$E(Y^e) = \alpha + \beta\mu\sqrt{e^{\sigma_\eta^2}} \tag{4}$$

and

$$\sigma_{Y^e} = (\beta^2[\sigma_p^2 e^{\sigma_\eta^2} + \mu^2(e^{\sigma_\eta^2}(e^{\sigma_\eta^2}-1)) + \sigma_p^2(e^{\sigma_\eta^2}(e^{\sigma_\eta^2}-1))] + \sigma_m^2)^{1/2} \tag{5}$$

3. THE $\bar{Y}^e$ CONTROL CHART

In the two-component error case, assuming that all parameters are known, the usual Shewhart control chart (3-sigma limits) for the process mean is:

$$CL^e = \alpha + \beta\mu\sqrt{e^{\sigma_\eta^2}} \tag{6}$$

$$UCL^e = CL^e + 3\frac{\sigma_{Y^e}}{\sqrt{n}} \tag{7}$$

$$UCL^e = CL^e - 3\frac{\sigma_{Y^e}}{\sqrt{n}} \tag{8}$$

where σ_{Y^e} comes directly from (5).

Since non-normality may significantly affect the performance of control charts, we are first going to examine this particular feature.

We have created a simulated data set from a pseudo-population using an example from the literature reported in Rocke and Lorenzato (1995), where model (2) is assumed with: α=11.51, β=1.524, σ_m=5.698, σ_η=0.1032. The concentration μ is non-random with possible values ranging from 5 picograms to 15 nanograms in 100*ml*. We introduce the randomness of concentration into this model by considering several values of the coefficient of variation ($cv(X)$) and several values of the mean ($E(X)=\mu$) of the unobservable X. In this way, according to model (3), for a given error structure (σ_m and σ_η) we reproduce a set of different working conditions for the measurement device. We fixed the sample size n=5 and generated 10000 samples for each condition.

Table 1 shows the *p*-values of the Shapiro-Wilk normality test computed on the observed values of the sample mean $\bar{Y}^e$.

TABLE 1

p-values of the S-W normality test computed on the sample means

			***cv*(*X*)**				
***E*(*X*)**	**0.01**	**0.05**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
5	6.3E-01	6.4E-01	7.1E-01	9.7E-01	1.8E-01	4.7E-01	7.5E-01
10	9.2E-01	4.5E-01	9.2E-01	3.1E-01	5.6E-01	1.3E-01	2.7E-01
50	2.9E-01	1.3E-01	3.4E-04	2.9E-01	4.6E-01	5.3E-02	4.2E-01
100	8.8E-05	9.3E-05	3.7E-04	1.7E-04	8.4E-02	1.8E-01	4.6E-01
1000	2.9E-05	4.4E-04	2.9E-08	3.5E-02	4.2E-05	1.8E-01	4.2E-01
10000	4.9E-06	1.0E-04	2.2E-07	3.1E-03	2.6E-01	6.2E-01	7.9E-01
15000	6.1E-05	8.3E-04	5.1E-04	2.3E-01	1.8E-01	8.8E-01	2.5E-01

Non-normality appears evident when $E(X) \geq 100$ and $cv(X) \leq 0.2$ (the shaded part of Table 1). In order to appreciate the effects of error model (2) on the control chart, we have focused on the in-control and out-of-control situations.

In order to study the effects of error model (3) on the false alarm rates, the in-control conditions were simulated for $E(X) \geq 100$ and $cv(X) \leq 0.2$ of the unobservable X. With fixed sample size $n=5$, we generated 10^8 samples for each condition. Results are summarized in Figure 1 where the continuous line, denoted as "no errors", corresponds to 0.00135, *i.e.* the probability, for the undisturbed process and in the error-free case, of a signal below the *LCL* (or above the *UCL*). Continuous lines are used for the false alarm rates above the *UCL* and dotted lines are used for the false alarm rates below the *LCL*.

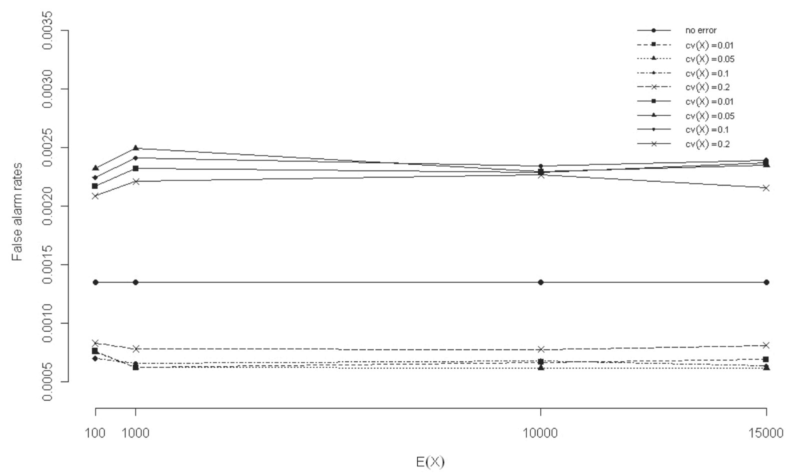

Figure 1 – False alarm rates from 10^8 replications (dotted lines for the values below the *LCL*, continuous lines for the values above the *UCL*).

Figure 1 shows a marked asymmetric behaviour: false alarm rates below the *LCL* are lower than the false alarm rate in the error-free case (0.00135), while the false alarm rates above the *UCL* are systematically greater than this value.

When considering the out-of-control situation, a shift in the mean of the non-observable X, from μ to μ_1, corresponds to a standardized shift of magnitude

$$\delta = \frac{\mu_1 - \mu}{\sigma} \tag{9}$$

In the presence of shift (9) in X, the expected value of the response Y^e is

$$E(Y^e) = \alpha + \beta\mu_1\sqrt{e^{\sigma_\eta^2}} \tag{10}$$

and the corresponding standardized shift in the monitored Y^e is

$$\begin{aligned} \delta_{Y^e} &= \frac{\alpha + \beta\mu_1\sqrt{e^{\sigma_\eta^2}} - \alpha - \beta\mu_0\sqrt{e^{\sigma_\eta^2}}}{\sigma_{Y^e}} \\ &= \frac{\delta}{\left(1 + \frac{\mu^2}{\sigma_p^2}(e^{\sigma_\eta^2} - 1) + (e^{\sigma_\eta^2} - 1) + \frac{\sigma_m^2}{\beta^2\sigma_p^2 e^{\sigma_\eta^2}}\right)^{1/2}} \end{aligned} \tag{11}$$

For non-zero σ_η and σ_m, the denominator in (11) is greater than 1, and therefore $\left|\delta_{Y^e}\right| < |\delta|$. As a result, measurement errors lead to a smaller shift in the observed response Y^e, which means that the change is more difficult to detect. Table 2 shows the values of δ_{Y^e} corresponding to a shift $|\delta| = 0.5$ for those values of $E(X)$ and $cv(X)$ in question, where the reduction in the shift is evident for small values of $E(X)$ and $cv(X)$.

TABLE 2

Values of δ_{Y^e} corresponding to a standardized shift of magnitude $|\delta|=0.5$

	***cv*(*X*)**			
***E*(*X*)**	**0.01**	**0.05**	**0.1**	**0.2**
100	0.045	0.207	0.336	0.436
1000	0.048	0.217	0.346	0.442
≥10000	0.048	0.217	0.347	0.442

One of the most commonly-used measures for evaluating the statistical properties of control charts is the *ARL* (Average Run Length). For the $\bar{Y}^e$-chart, the theoretical *ARL* for shifts of magnitude δ_{Y^e} is:

$$ARL(\delta_{Y^e}) = (\Phi(-3+\delta_{Y^e}\sqrt{n}) + \Phi(-3-\delta_{Y^e}\sqrt{n}))^{-1} \quad (12)$$

The *ARL* values corresponding to the δ_{Y^e} in Table 2 are reported in Table 3

TABLE 3

Theoretical values of ARL for the δ_{Y^e} in Table 2

	cv(X)			
***E*(*X*)**	**0.01**	**0.05**	**0.1**	**0.2**
100	352.49	170.99	81.13	46.50
1000	350.34	161.06	76.27	45.09
≥10000	350.31	160.96	76.23	45.07

Expression (12) is based on the normality assumption of the sample statistic used in the chart, although we noticed that measurement errors lead to departures from the Normal distribution. Thus, in order to assess the effects of the two-component measurement error, we conducted a simulation study of the out-of-control situations. For each combination of *cv*(*X*) and *E*(*X*) in Table 1, we fixed shifts of magnitude $|\delta| = 0.5$ in the variable *X*, and estimated the off-target *ARLs* of the $\bar{Y}^e$-control-charts. Each condition was again replicated 10^8 times: results for negative and positive shifts are summarized in Table 4.

TABLE 4

Estimated ARLs (δ= -0.5 left values and δ= +0.5 right values within each parenthesis)

	cv(X)							
***E*(X)**	**0.01**		**0.05**		**0.1**		**0.2**	
100	(380.5	282.7)	(265.4	116.6)	(121.2	57.8)	(59.3	37.0)
1000	(392.3	279.9)	(279.2	108.1)	(117.2	54.6)	(57.5	36.1)
10000	(368.9	289.0)	(271.7	106.8)	(118.0	54.3)	(58.0	35.9)
15000	(384.8	271.0)	(269.8	104.9)	(118.3	53.7)	(57.0	36.1)

Table 4 confirms the asymmetric effect of measurement errors. The *ARLs* for positive shifts are smaller than the corresponding *ARLs* for negative shifts. As one would have expected, the values shown in the table tend to differ from the theoretical *ARL* values shown in Table 3, in particular as *E*(*X*) increases.

4. THE DESIGN OF THE CONTROL CHARTS UNDER THE TWO-COMPONENT ERROR MODEL

The results obtained so far show that error-model (3) leads to important modifications in the performances of the Shewhart control chart, which in turn result in problems in the practical use of such monitoring algorithms. In particular, we noticed departure from normality, a marked asymmetric behaviour, and a general difficulty in assessing the performances of the control chart itself.

The problem of designing and using control charts without assuming normality is something that frequently comes up in industrial practice.

When the distribution of the observable quality characteristic is known, exact methods can be used for analytically computing the control limits for the desired Type I risk. In our model (3) the observable quality characteristic Y^e is modelled as the sum of a normal variable, Y^e, and the product of a normal variable, *X*, by a log-normal variable e^{η}. The analytical derivation of this distribution is not immediate. Tools for computing the product of random variables have been proposed, for instance by Roahtgi (1976) and Springer (1979), the last mostly relying on Mellin transforms. The analytic derivation, and the further implementation and computation of these results, are, when possible, rather complicated, as Glen *et al.* (2004) pointed out. Moreover, once obtained the distribution of the product, say Y_p, the further difficulty of obtaining the distribution Y^e as the convolution of Y_p and ε has to be faced.

In general, the charts constructed by the exact method are not in a form familiar to practitioners and quality engineers who are used to conventional $\bar{X}$ charts (Bai and Choi, 1995, Chan and Cui, 2003). Therefore, other approaches will be examined.

4.1 *The weighted variance method*

For skewed distributions Choobineh and Ballard (1987) suggest a weighted variance (WV) method based on the semivariance approximation of Choobineh and Branting (1986). Following this mainstream, Bai and Choi (1995) propose an interesting heuristic WV method without distributional assumptions, which will be synthetically illustrated. This method provides asymmetric control limits that keep into account the direction and degree of skewness, that is estimated by using different variances in computing upper and lower control limits for skewed populations.

The WV method, like the Shewhart method, uses the standard deviation for setting the control limits. However, it differs from the Shewhart method since the standard deviation is multiplied by two different factors. One factor is used for the *UCL*, while the other is used for the *LCL*. Let P_X be the probability that the random variable *X* is less or equal to its mean μ_X. Then the *UCL* factor is $\sqrt{2P_X}$, and the *LCL* factor is $\sqrt{2(1-P_X)}$. The control limits of the $\bar{X}$ chart based on the WV method are

$$\begin{aligned} UCL &= \mu_X + 3\frac{\sigma_X}{\sqrt{n}}\sqrt{2P_X} \\ LCL &= \mu_X - 3\frac{\sigma_X}{\sqrt{n}}\sqrt{2(1-2P_X)} \end{aligned} \tag{13}$$

For using in practice the control chart (13) based on the WV method, both P_X and the process parameters must be estimated. Let $X_{i1}, X_{i2}, ..., X_{in}$, *i*=1,2,...,*r*, be *r* subgroups (samples) of size *n* from the (in-control) process. When the process distribution and parameters are unknown Bai and Choi (1995) propose the WV $\bar{X}$ Chart:

$$\begin{aligned} LCL_{WV} &= \bar{\bar{X}} - 3\frac{\bar{R}}{d_2^{'}\sqrt{n}}\sqrt{2(1-\hat{P}_X)} = \bar{\bar{X}} - W_L\bar{R} \\ CL_{WV} &= \bar{\bar{X}} \\ UCL_{WV} &= \bar{\bar{X}} + 3\frac{\bar{R}}{d_2^{'}\sqrt{n}}\sqrt{2\hat{P}_X} = \bar{\bar{X}} + W_U\bar{R} \end{aligned} \tag{14}$$

where $\bar{\bar{X}} = (1/(nr))\sum_{i=1}^{r}\sum_{j=1}^{n} X_{ij}$, $\bar{R} = (1/r)\sum_{i=1}^{r} R_i$, and R_i is the range of the *i*-th subgroup that is used to estimate the standard deviation. Further, the weight $\hat{P}_X$ estimates the probability that the random variable X is less than or equal the mean $E(X)$:

$$\hat{P}_X = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n}\lambda(\bar{\bar{X}} - X_{ij})}{nr} \tag{15}$$

with $\lambda(x) = 1$ for $x \geq 0$ and $\lambda(x) = 0$ for $x < 0$. The values of d_2' that appear in (14), for given n and P_X, can be computed using the method described by Bai and Choi (1995) whereas constants $W_L = 3\sqrt{2(1-P_X)}\big/ d_2'\sqrt{n}$ and $W_U = 3\sqrt{2P_X}\big/ d_2'\sqrt{n}$ for selected combinations of n and P_X are listed in Table 1 of Bai and Choi (1995).

The practical use of the WV method, with respect to the Shewhart method, doesn't imply an increase of the computational complexity since it requires only the estimation of the probability P_X from the preliminary r samples from the in-control process (*i.e.* Phase I data).

4.2 *The skewness correction method*

Also the skewness correction (SC) method (Chan and Cui, 2003 and Yazici and Kan, 2009)) correct the conventional Shewhart chart according to the skewness of the distribution. This approach is based on the Cornish-Fisher expansion reported by Johnson and Kotz (1970). For the standardized random variable X with mean 0, standard deviation 1 and skewness k_3, the LCL, CL and UCL of a standardized control chart for individual observation based on the SC method are:

$$LCL = 3 - \frac{\frac{4}{3}k_3}{1+0.2k_3^2}, CL = 0, UCL = 3 + \frac{\frac{4}{3}k_3}{1+0.2k_3^2}$$

respectively.

For the general case of subgroups of size n and unknown parameters, Chan and Cui (2003) propose the SC $\bar{X}$ Chart:

$$LCL_{SC} = \bar{\bar{X}} + \left(-3 + \frac{4\hat{k}_3/(3\sqrt{n})}{1+0.2\hat{k}_3^2/n} \right) \frac{\bar{R}}{d_2^*\sqrt{n}} = \bar{\bar{X}} - A_L^*\bar{R}$$

$$CL_{SC} = \bar{\bar{X}} \tag{16}$$

$$LCL_{SC} = \bar{\bar{X}} + \left(3 + \frac{4\hat{k}_3/(3\sqrt{n})}{1+0.2\hat{k}_3^2/n} \right) \frac{\bar{R}}{d_2^*\sqrt{n}} = \bar{\bar{X}} + A_U^*\bar{R}$$

where $\hat{k}_3$ is the estimated sample skewness

$$\hat{k}_3 = \frac{1}{nr-3}\sum_{i=1}^{r}\sum_{j=1}^{n}\left(\frac{X_{ij} - \bar{\bar{X}}}{\sqrt{\frac{1}{nr-1}\sum_{i=1}^{r}\sum_{j=1}^{n}(X_{ij} - \bar{\bar{X}})^2}} \right)^3 \tag{17}$$

Constants A_L^* and A_U^* are listed in Table 1 of Chan and Cui (2003). The SC method consists in a semi-parametric approach that requires the first three moments of the distribution, even if the distributional form has not to be explicited. Compared with the Shewhart approach, the SC method needs the estimation of the skewness parameter k_3 which represents only a negligible increase in the complexity of the method.

4.3 *The empirical reference distribution method*

Chakraborti *et al.* (2001) presented an extensive overview of the literature on univariate distribution-free control charts, and subsequent scientific works, among which we quote Vermaat *et al.* (2003), Chakraborti *et al.* (2004) and by Chakraborti and Van de Wiel (2008), have confirmed the growing interest in this issue.

One of the distribution-free control charts discussed in Chakraborti *et al.* (2001) was previously proposed by Willemain and Runger (1996). This proposal makes it possible to design both one-sided and two-sided control charts from a sufficiently large in-control (or, a reference) sample, by selecting the control limits as specific order statistics of the variables to be charted. As stated by the authors the availability of a large preliminary data set from the undisturbed process (in-control) is a necessary condition for the nonparametric design of control charts, since estimation of extremes percentiles corresponding to a large on-target average run length (*ARL*) would be impossible without a large number of observations.

Suppose that Z is a sample statistic computed from a sample of size n. Let z_i (i=1,2,...,m) be the independently observed values of Z computed from m samples. Here Z is $\bar{Y}^e = E(Y^e)$ and the z_i values are the observed sample means $\bar{y}_i^e$. We assume that m preliminary samples are taken from the undisturbed process.

Let $z_{(k)}$ be the k-th order statistic in the sample, k=1, 2, ..., m, (for basic order statistics theory, see for example David and Nagaraja, 2004), and $z_{(m)}$ be the largest value of the $z_{(k)}$. By convention $z_{(0)} = -\infty$ and $z_{(m+1)} = \infty$. The m order statistics divide the theoretical range of Z into m+1 equally probable intervals.

If the control limits are defined as $LCL_{ERD} = z_{(k)}$ and $UCL_{ERD} = z_{(s)}$, with $0 \le k < s \le m$, then

$$P = \Pr[z_{(k)} \le Z \le z_{(s)}] \tag{18}$$

is the probability that a value of the sample statistic Z falls within the control limits LCL_{ERD} and UCL_{ERD}.

Probability P has a Beta distribution that depends only on m and on the number b=s-k of intervals within the control limits

$$P \sim Beta(b, m-b+1) \tag{19}$$

Assuming the independence of the plotted points, the number R of plotted points between alarms has a geometric distribution with parameter $1-P$. For a given P, the in-control ARL is

$$ARL = E(R \mid P) = (1-P)^{-1} \tag{20}$$

Since the $z_{(k)}$ are the observed values generated by the random variables $Z_{(k)}$, the ARL itself is a random variable with Inverted–Beta distribution

$$h(ARL) = \frac{m!}{(b-1)!(m-b)!}(ARL-1)^{b-1} \Big/ ARL^{m+1} \tag{21}$$

with expected value

$$E(ARL) = \frac{m}{m-b} \tag{22}$$

and standard deviation

$$s.d.(ARL) = \left(\frac{bm}{(m-b)^2(m-1-b)} \right)^{1/2} \tag{23}$$

Equation (22) is important for practice since, given the number m of preliminary samples and the number b of intervals within the control limits, it quantifies the on-target ARL performances of a ERD based control chart.

It is also possible to compute the quantiles of the *ARL* distribution using the relationship

$$\Pr[ARL \leq A] = \Pr[P \leq 1 - 1/A] \tag{24}$$

The ERD method allows also to assess the performances of the control chart in the out-of control-situations by evaluating the expected *ARL* using a shift procedure. Details are summarised in the Appendix.

5. COMPARISONS OF THE DIFFERENT METHODS

We compare the performances of the three methods when the measurement error model (3) is considered in designing control charts for the mean level of the observable quality characteristic Y^e. We select the situation where *cv*(*X*)=0.01 among the cases examined in Section 3 (the other cases can be treated similarly).

To build WV and SC control charts we simulate a preliminary data set of *r*=50 independent and "in-control" subgroups of size *n*=5 for the *E*(*X*) values of interest. The sample estimates are reported in Table 5.

Constants W_L, W_U A_L^*, and A_U^* obtained by interpolating the values of Table 1 in Bai and Choi (1995) and Table 1 in Chan and Cui (2003) respectively, are reported in Table 6, together with the control limits of the WV and SC control charts.

TABLE 5

Sample estimates from r=50 simulated samples for cv(X)=0.01

	E(*X*)			
	100	**1000**	**10000**	**15000**
$\bar{\bar{Y}}^e$	164.918	1534.168	15432.606	22910.358
$\bar{R}_{Y^e}$	40.970	373.927	3688.634	5648.898
$\hat{P}_{Y^e}$	0.524	0.492	0.492	0.528
$\hat{k}_3$	0.441	0.284	0.161	0.194

For building ERD based control charts with performances comparable with the 3σ limits charts just obtained, we choose *m*=10000 and *b*=9973, thus from equation (22) we have $E(ARL) = 370.4$

In order to obtain a reasonable chart performance, we set *k*=14 and consequently $LCL_{ERD} = z_{(14)}$ and $UCL_{ERD} = z_{(9987)}$, leaving 14 blocks below the *LCL* and 14 blocks above the *UCL*, allowing, in this way, a well-behaving *ARL*.

We created the undisturbed preliminary data set by generating the *m* samples of size *n*=5 from the observable response Y^e. After obtaining the sample means $\bar{y}_i^e$, as described in Section 4, we then computed the control limits reported in Table 6.

TABLE 6

WV and SC constants and WV, SC and ERD control limits for cv(X)=0.01

	***E*(*X*)**			
	100	**1000**	**10000**	**15000**
W_L	0.518	0.580	0.580	0.516
W_L	0.590	0.580	0.580	0.590
A_U^*	0.635	0.615	0.600	0.604
A_L^*	0.527	0.545	0.560	0.556
LCL_{WV}	143.696	1317.141	13291.723	19995.527
UCL_{WV}	189.099	1750.896	17570.538	26245.467
LCL_{SC}	143.331	1330.563	13367.369	19770.695
UCL_{SC}	190.940	1764.317	17646.184	26323.416
LCL_{ERD}	143.341	1336.275	13363.672	20027.123
UCL_{ERD}	189.062	1768.048	17589.563	26305.440

5.1. *Comparison of the in-control performances*

The in-control performances of the charts with the control limits reported in Table 6, are assessed by generating 10^8 samples (n=5) from the in-control process, for every combination of $E(X)$ and $cv(X)$=0.01. The observed false alarm rates, due to a signal above the upper control limits and below the lower control limits, are shown in Figures 2 and 3 respectively. In both figures, the "no-error" value equal to 0.00135 is denoted as a continuous line.

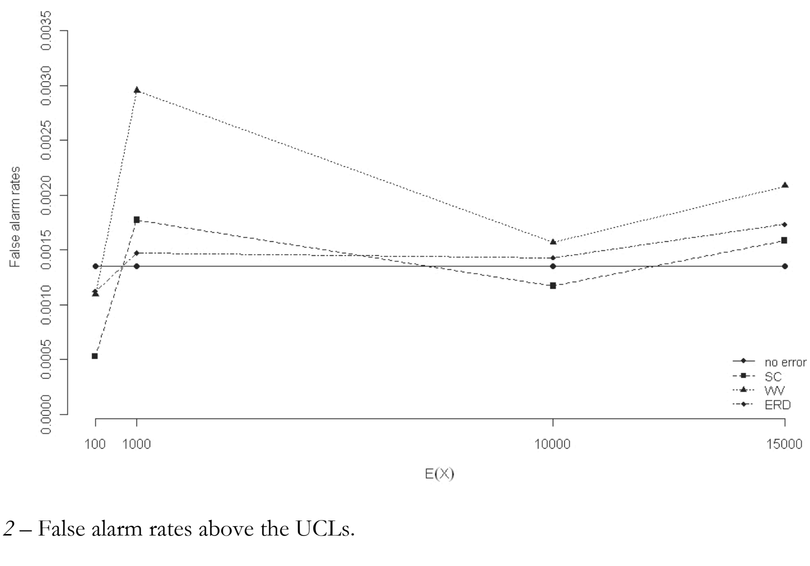

Figure 2 – False alarm rates above the UCLs.

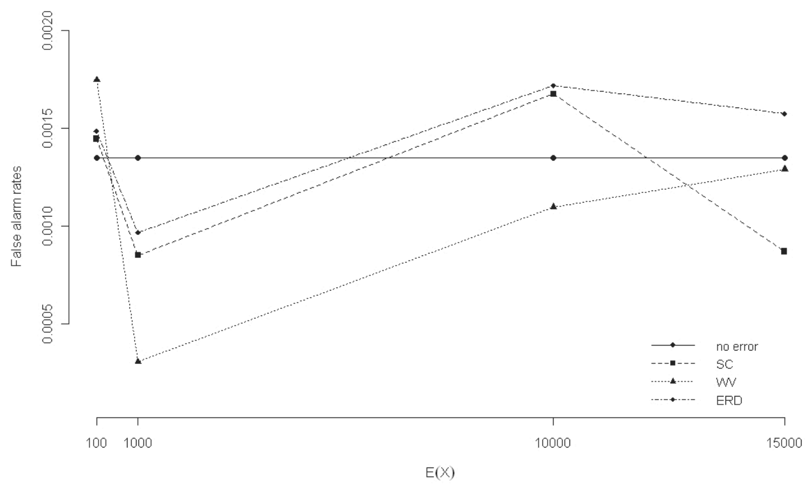

Figure 3 – False alarm rates below the LCLs.

Charts behaviour, when compared with Figure 1, shows a strong improvement: the false alarm rates now follow a random pattern around the theoretical value 0.00135, without evidence of asymmetry. Moreover, the false alarm rates of the SC and ERD charts are closer to the no-error value than the WV charts which show a less stable behaviour to the changes of *E(X)*.

5.2. *Comparison of the out-of-control performances*

For each value of $E(X)$, when $cv(X)$=0.01, we fixed shifts of magnitude $|\delta| = 0.5$ in the variable X. The off-target *ARLs* were estimated by performing 10^8 replications of the experiment. Results are reported in Figures 4 and 5.

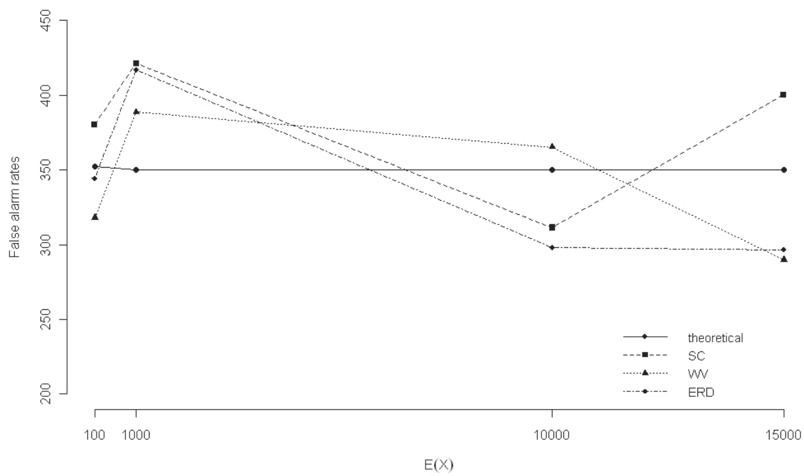

Figure 4 – Estimated *ARLs* for δ= -0.5.

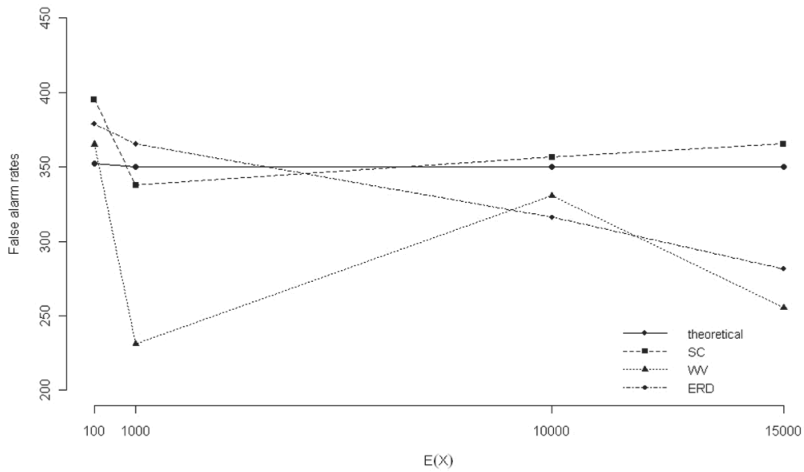

Figure 5 – Estimated *ARLs* δ=+0.5.

In these figures the benchmark "theoretical" lines, corresponding to the values computed using formula (12) and appearing in Table 3 (first column), are reported. The estimated out-of-control *ARLs* do not show a systematic asymmetric behaviour, SC and ERD show comparable *ARLs* that seem to have more steady performances than the WV control charts.

As a conclusion, when the data are contaminated by the proposed two-component error model, the SC and ERD based $\bar{X}$ -control charts show similar performances being less sensitive to the presence of measurement errors.

6. CONCLUDING REMARKS

Studies of the effects of measurement errors on monitoring algorithms traditionally use the Gaussian additive error model. However, it has often been pointed out that, in connection with particular measurement devices, a more realistic error model ought to be considered.

In this paper we have proposed an extension of the usual error model (1) to cover a more general situation, by introducing the structure of the two-component error model (2). The two-component error model was proposed for uncertainty measurement in analytical chemistry and environmental monitoring. Model (2) jointly considers a constant error-component, which reveals its effects at low measurement levels, and a proportional error-component, which becomes significant at higher levels of measurement. The overall picture of measurement uncertainty given by the two-component error model is very realistic and responds to several among the improving requirements from the literature and the practice of statistical process control.

For these reasons we propose the error model (3) and assess its effects on the Shewhart $\bar{X}$ control chart. In this way, we unify two approaches, which to our knowledge have up until now been proposed separately. We have stressed the point that the proposed error model can lead to a significant departure from the normal distribution, and have an important impact on control chart performance. This happens because, while the additive error basically inflates the variance of the observable response Y^e, the proportional error component η leads to a remarkable asymmetry in the performance of the mean control chart.

Results indicate that when the process is in-control, the false alarm rate above the *UCL* is greater than the theoretical false alarm probability, while the false alarm rate below the *LCL* is smaller. This is an important issue, since false alarms may cause a series of expensive, unnecessary actions or regulations. Moreover, asymmetry can complicate monitoring management. The effects of the two-component error model are also evident in the out-of-control situation: the *ARL* for negative shifts is always greater than the corresponding *ARL* for positive shifts.

In order to take errors into account when designing the control chart, we have compared by simulation the use of several control chart design methods. Our results indicate that the SC and ERD methods of constructing $\bar{X}$ control chart improve the chart performance when the process is in-control, in the sense that the observed false alarm rates are symmetric and in agreement with theoretical false alarm probability. Also for the out-of-control cases performance of the SC and ERD charts do not show an asymmetric behaviour. As a conclusion we state that SC and ERD based $\bar{X}$-control charts are less sensitive to the presence of measurement errors.

However, the large number of observations from the so-called Phase I, necessary to ascertain the control limits in the ERD method, may be a potential limitation in practice. Therefore the solution based on the skewness correction (SC) method might appear the best choice at the price of assuming the existence of the first three moments of the distribution. In this case the design of the control chart is based on the estimation of the first three moments of the distribution.

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APPENDIX

A.1 *ARL* COMPUTATION FOR THE ERD METHOD IN THE OUT-OF CONTROL CASE

For the out-of-control situation, given the chosen control limits, the corresponding expected off-target *ARL* can be evaluated using a shift procedure. It is based on the idea that a shift in the process level will modify the number of observed order statistics falling within the control limits. Thus, the problem of assessing off-target *ARL* is transformed into the assessment of on-target *ARL* with a smaller number of intervals within the control limits.

In the two-sided chart, we have $LCL_{ERD} = z_{(k)}$ and $UCL_{ERD} = z_{(k+b)}$. If an assignable cause leads to an upward shift in the mean, then all order statistics will shift upwards in a way such that $z_{(k+b-k')}$ takes the position formerly held by $z_{(k+b)}$. The shift decreases the number of intervals below the UCL_{ERD}, from $b+k$ to $b+k-k'$, while it increases the number of intervals above the LCL_{ERD}. However, as Willemain and Runger (1996) pointed out, this interval-shifting is not symmetric since the decrease in the number of intervals in the upper part of the in-control region is only partially compensated by the increase in the number of intervals on the lower part by some fractional amount.

Thus, for a given upward shift, $b'=b+k-fk'$ intervals ($0<f<1$) remain, where k' is the decrease in the number of intervals within the control limits. If the observed spacings between the order statistics are used as values of the shifts, it is possible to estimate the off-target $E(ARL)$ at those particular values of shift, and thus the value k' is an integer.

A downward shift can be treated similarly: a negative shift reduces the number of intervals at the low end of the in-control region, and is partially compensated by an increase in the number of intervals on the high end by some fractional amount.

The effectiveness of the aforesaid method of evaluating the expected off-target *ARL* depends on the specification of a value for the fractional amount f. The choice of f may be critical, in the sense that a modification in the E(ARL) values for a given shift may be sensitive to this choice. Since there is no general criterion for a satisfactory choice of f, we conducted a preliminary set of simulations using several values of f. We found that small values of f lead to unreasonable and unstable results while large values of f give better results. Thus we decided to set the f value at 7/8.

Let us first consider the case of an upward shift. We denote with $z_{(j)}$ the j-th order statistics, and thus

$$\delta_1 = UCL_{ERD} - z_{(j)} \tag{A1}$$

is the observed spacing between the UCL_{ERD} and the order statistic $z_{(j)}$.

Since in our case the $z_{(j)}$ values correspond to the ordered values of $\overline{y}_j^e$, the estimated standard deviation of the sample means $\overline{y}_j^e$ (j=1,2,...,m) is

$$\hat{\sigma}_{\overline{Y}^e} = \left(\sum_{i=1}^{m} \frac{(z_{(j)} - \overline{z})^2}{m-1} \right)^{1/2} \tag{A2}$$

where $\overline{z} = m^{-1} \sum_{j=1}^{m} z_{(j)}$.

Therefore, the empirically estimated standardized shift in $\overline{Y}^e$ is

$$\hat{\delta}_{\overline{Y}^e} = \frac{\delta_1}{\hat{\sigma}_{\overline{Y}^e}} \tag{A3}$$

which corresponds to an upward shift that moves a $z_{(j)}$ into the position of UCL_{ERD}, while the empirically estimated standardized shift in Y^e, corresponding to a shift of size $\hat{\delta}_{\overline{Y}^e}$ in $\overline{Y}^e$, is

$$\hat{\delta}_{Y^e} = \frac{\hat{\delta}_{\overline{Y}^e}}{\sqrt{n}} \tag{A4}$$

We have to look for those values of $\hat{\delta}_{Y^e}$ that provide the best approximation of the values of δ_{Y^e} and estimate the off-target $E(ARL)$ at those particular values of shift.

Table A1 shows an extended version of the results for $E(X)$=100 and $cv(X)$=0.01. A summary of the results in all the other cases is given in Table A2. In Table A1 the column labelled as "j" shows the positions of the ordered sample means $z_{(j)}$. Column "δ_1" contains the values of the observed spacings between the UCL_{ERD} and the order statistic $z_{(j)}$, according to equation (A1), while the next two columns show the values of $\hat{\delta}_{\overline{Y}^e}$ and $\hat{\delta}_{Y^e}$ computed according to (A3) and (A4) respectively. For each value of δ_1 ($\hat{\delta}_{Y^e}$), we computed the number of intervals within the control limits $b'=b+k-fk'$, as described above. The values of $E(ARL)$ and $s.d.(ARL)$ conditional on shift δ_1 ($\hat{\delta}_{Y^e}$) are calculated by equations (22) and (23) respectively, with b' in the place of b, *i.e.*: $E(ARL) = m/(m-b')$ and $s.d.(ARL) = (b'm/((m-b')^2(m-1-b')))^{1/2}$. The columns "2.5%" and "97.5%" show the percentiles of the distribution of the ARL computed using equation (23), thus providing a 95% confidence interval for the unknown ARL.

For $E(X)$=100 and $cv(X)$=0.01 the standardized shift in Y^e is $\delta_{Y^e} = 0.045$ (Table 2), which is best approximated by $\hat{\delta}_{Y^e} = 0.052$, with a corresponding $E(ARL)$=337.6.

For a downward shift $\delta_1 = LCL - z_{(j)}$, $\hat{\delta}_{\bar{Y}^e}$ and $\hat{\delta}_{Y^e}$ are defined according to the usual equations (A3) and (A4). In this case, the best approximation to δ_{Y^e} is obtained by $\hat{\delta}_{Y^e} = -0.044$ with a corresponding $E(ARL)$=327.9.

TABLE A1

Off-target ARL computations for E(X)=100 and cv(X)=0.01 ($\hat{\sigma}_{\bar{Y}^e} = 7.51$).

j	$z_{(j)}$	δ_1	$\hat{\delta}_{\bar{Y}^e}$	$\hat{\delta}_{Y^e}$	***b+k-k'***	***b+k-fk'***	***E*(*ARL*)**	***s.d.*(*ARL*)**	**2.5%**	**97.5%**
					Upward shift					
9987	***UCL***$_{ERD}$**=189.062**	0.000	0.000	0.000	9973	9973.000	370.4	3.7	254.7	537.2
9986	188.887	0.175	0.023	0.010	9972	9972.125	358.7	3.5	248.2	517.4
9985	188.887	0.175	0.023	0.010	9971	9971.250	347.8	3.5	242.0	498.9
9984	188.186	0.876	0.117	**0.052**	9970	9970.375	**337.6**	3.4	**236.1**	**481.6**
					Down ward shift					
19	144.380	-1.038	-0.139	-0.062	9968	9968.625	318.7	3.2	225.2	450.3
18	144.078	-0.737	-0.098	**-0.044**	9969	9969.500	**327.9**	3.3	**230.5**	**465.4**
17	143.870	-0.528	-0.070	-0.032	9970	9970.375	337.6	3.4	236.1	481.6
16	143.784	-0.443	-0.059	-0.026	9971	9971.250	347.8	3.5	242.0	498.9
15	143.516	-0.175	-0.023	-0.010	9972	9972.125	358.7	3.6	248.2	517.4
14	**LCL**$_{ERD}$**=143.341**	0.000	0.000	0.000	9973	9973	370.4	3.7	254.7	537.2

TABLE A2

Off-target ARL computations for cv(X)=0.01

	j	$z_{(j)}$	$\hat{\delta}_{Y^e}$	*b+k-k'*	*b+k-fk'*	*E*(*ARL*)	*s.d.*(*ARL*)	2.5%	97.5%
				$E(X) = 1000$ $\hat{\sigma}_{\bar{Y}^e} = 71.13$					
upward shift	9982	1760.645	0.044	9969	9968.5	327.9	3.3	230.5	465.4
downward shift	20	1343.494	-0.045	9967	9967.75	310.1	3.3	220.1	436.0
				$E(X) = 10000$ $\hat{\sigma}_{\bar{Y}^e} = 698.52$					
upward shift	9982	17503.48	0.055	9968	9968.625	318.7	3.2	225.2	450.3
downward shift	19	13435.53	-0.046	9968	9968.625	318.7	3.2	225.2	450.3
				$E(X) = 15000$ $\hat{\sigma}_{\bar{Y}^e} = 1063.46$					
upward shift	9980	26192.59	0.047	9966	9966.875	301.9	3.0	215.2	422.6
downward shift	23	20152.75	-0.053	9964	9965.125	286.7	2.9	206.2	398.1

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SUMMARY

Effects of the two-component measurement error model on $\bar{X}$ control charts

The statistical properties of Shewhart control charts are known to be highly sensitive to measurement errors. The statistical model relating the measured value to the true, albeit not observable, value of a product characteristic, is usually Gaussian and additive. In this paper we propose to extend the said model to a more general formulation by introducing the two-component error model structure. We study the effects of the proposed error-model on the performance of the mean control charts, since gauge imprecision may seriously alter the statistical properties of the control charts. In order to take measurement errors into account in the design of the control charts we explore the use of different methods based on a weighted variance concept, a skewness correction method and an empirical reference distribution approach respectively. The different approaches are discussed and compared by Monte Carlo simulation. Results indicate that the last two methods produce the best results.