DISTRIBUTIONS OF PRODUCTS INVOLVING THE TYPE II BESSEL FUNCTION RANDOM VARIABLE

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1. INTRODUCTION

The distribution of product of independent random variables (r.v.s) is of interest in many areas of science and econometrics. The distribution of product of independent r.v.s X and Y has been studied by several authors especially when X and Y belong to the same family of distributions. In this context the names of Sakamoto (1943), Stuart (1962), Springer and Thompson (1970), Podolski (1972), Steece (1976), Wallgren (1980), Bhargava and Khatri (1981), Abu-Salih (1983), Tang and Gupta (1984), Malik and Trudel (1986) and most recently those of Glickman and Xu (2008) are of great importance.

In the present paper we shall study the distribution of XY as well as of |XY| when X and Y are independent r.v.s belonging to different families of distributions. Here we assume that the r.v. X follows the type II Bessel function distribution given as follows

Type II Bessel function distribution (Springer, 1979)

$$f(x) \;=\; D\,|x|^{\lambda} \exp\left\{-\frac{\theta x^2}{2}\right\} I_{\lambda-1}(\beta\,|x|) \quad ; \quad x \in (-\infty,\infty) \tag{1}$$

$$\textit{where} \quad D = \frac{\theta^{\lambda} e^{-(\beta^2/2\theta)}}{2\,\beta^{\lambda-1}} \quad ; \quad \theta > 0,\ \lambda > 0,\ \beta \geq 0 \tag{2}$$

$$\text{and} \;\; I_{\nu}(x) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu} \tag{3}$$

$(-\infty < \nu < \infty)$ is the modified Bessel function of the first kind.

This distribution generalizes the well known Rayleigh distribution, Chi distribution, noncentral Chi distribution and folded normal distribution (Springer, 1979).

The other r.v. Y belongs to one of the following families of distributions

Normal distribution

$$g(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(y-\mu)^2}{2\sigma^2} \right\} \quad ; \quad y \in (-\infty, \infty)\,,\ \sigma > 0 \tag{4}$$

Pearson VII distribution

$$g(y) = \frac{\Gamma\left(M - \frac{1}{2}\right)}{\sqrt{N\pi}\ \Gamma(M-1)} \left(1 + \frac{y^2}{N}\right)^{\left(\frac{1}{2} - M\right)} \quad ; \quad y \in (-\infty, \infty),\ M > 1,\ N > 0 \tag{5}$$

Maxwell Boltzmann distribution (Mathai, 1993)

$$g(y) = \frac{2}{\sqrt{\pi}} \alpha^{\frac{3}{2}}\, y^2 \exp(-\alpha y^2) \quad ; \quad \alpha > 0,\ y \in (-\infty, \infty) \tag{6}$$

We shall require the following definitions in the sequel

Modified Bessel function of the third kind (Gradshteyn and Rhyzik, 1994)

$$K_\nu(x) = \frac{\sqrt{\pi}\ x^\nu}{2^\nu \Gamma\left(\nu + \frac{1}{2}\right)} \int_1^\infty (t^2 - 1)^{\left(\nu - \frac{1}{2}\right)} \exp(-xt)\,dt \quad , \quad \nu + ½ > 0 \tag{7}$$

Tricomi's function (Srivastava and Manocha, 1984)

$$\psi(a, b; z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}\, t^{a-1} (1+t)^{b-a-1}\, dt \quad , \quad a > 0,\ z > 0 \tag{8}$$

$$\psi(a,\ b;\ z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\, {}_1F_1(a;\ b;\ z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{(1-b)}\, {}_1F_1(a-b+1;\ 2-b;\ z) \tag{9}$$

Kampé de Feriét function (Srivastava and Manocha, 1984)

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix} (a_p):(b_q);(c_k); \\ (\alpha_l):(\beta_m);(\gamma_n); \end{matrix}\ x, y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}, \tag{10}$$

where, for convergence,

(i) $p + q < l + m + 1, \quad p + k < l + n + 1, \quad |x| < \infty,\ |y| < \infty$, *or* (11)

(ii) $p + q = l + m + 1, \quad p + k = l + n + 1$, and

$$\begin{cases} |x|^{1/(p-l)} + |y|^{1/(p-l)} < 1, & \text{if } p > l, \\ \max\{|x|, |y|\} < 1, & \text{if } p \le l \end{cases} \tag{12}$$

2. THE DISTRIBUTION OF THE PRODUCT OF TWO INDEPENDENT RANDOM VARIABLES THAT ARE NOT EVERYWHERE POSITIVE

Let a r.v. X has the probability density function (p.d.f.) f(x), which can be written as

$$f(x) = f^{-}(x) + f^{+}(x) \quad , \quad -\infty < x < \infty \tag{13}$$

in which $f^{-}(x)$ vanishes identically except on the interval $-\infty < x < 0$, where $f^{-}(x) = f(x)$.

Similarly $f^{+}(x)$ is defined to be identically zero except over the interval $0 \le x < \infty$, where $f^{+}(x) = f(x)$.

Let X and Y be two independent r.v.s with p.d.f. f(x) and g(y), respectively. We know that the p.d.f. h(z) of the r.v. Z=XY is given by (Springer, 1979)

$$h(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f(x)\, g\left(\frac{z}{x}\right) dx = \int_{-\infty}^{\infty} \frac{1}{|y|} f\left(\frac{z}{y}\right) g(y)\, dy \tag{14}$$

Using the concept of partitioning as given by (13) for both $f(x)$ and $g(y)$, the above expression can be written as

$$\begin{aligned} h(z) = {} & \int_{-\infty}^{\infty} \frac{1}{|x|} f^{+}(x)\, g^{+}\left(\frac{z}{x}\right) dx + \int_{-\infty}^{\infty} \frac{1}{|x|} f^{+}(x)\, g^{-}\left(\frac{z}{x}\right) dx \\ & + \int_{-\infty}^{\infty} \frac{1}{|x|} f^{-}(x)\, g^{+}\left(\frac{z}{x}\right) dx + \int_{-\infty}^{\infty} \frac{1}{|x|} f^{-}(x)\, g^{-}\left(\frac{z}{x}\right) dx \end{aligned} \tag{15}$$

We shall write h(z) as

$$h(z) = h^{-}(z) + h^{+}(z) \quad , \quad -\infty < z < \infty \tag{16}$$

in which

$$h^{-}(z) = \begin{cases} h(z) & , \quad -\infty < z < 0 \\ 0 & , \quad 0 \le z < \infty \end{cases} \tag{17}$$

and

$$h^{+}(z)=\begin{cases} 0 & , \quad -\infty < z < 0 \\ h(z) & , \quad 0 \le z < \infty \end{cases} \tag{18}$$

In the special case, when f(x) and g(y) are even functions i.e. $f^{+}(x) = f^{-}(-x)$ and $g^{+}(y) = g^{-}(-y)$, eq. (15) can be simplified as follows. For $-\infty < z < 0$,

$$\begin{aligned} h^{-}(z) \quad = h(z) = & \quad 0 + \int_0^\infty \frac{1}{x} f^{+}(x) g^{-}\left(\frac{z}{x}\right) dx - \int_{-\infty}^0 \frac{1}{x} f^{-}(x) g^{+}\left(\frac{z}{x}\right) dx + 0 \\ = & \quad \int_0^\infty \frac{1}{x} f^{+}(x) g^{+}\left(-\frac{z}{x}\right) dx + \int_0^\infty \frac{1}{x} f^{+}(x) g^{+}\left(-\frac{z}{x}\right) dx \\ = & \quad 2\int_0^\infty \frac{1}{x} f^{+}(x) g^{+}\left(-\frac{z}{x}\right) dx. \end{aligned} \tag{19}$$

For $0 \le z < \infty$,

$$\begin{aligned} h^{+}(z) \quad = h(z) = & \quad \int_0^\infty \frac{1}{x} f^{+}(x) g^{+}\left(\frac{z}{x}\right) dx + 0 + 0 - \int_{-\infty}^0 \frac{1}{x} f^{-}(x) g^{-}\left(\frac{z}{x}\right) dx \\ = & \quad \int_0^\infty \frac{1}{x} f^{+}(x) g^{+}\left(\frac{z}{x}\right) dx + \int_0^\infty \frac{1}{x} f^{+}(x) g^{+}\left(\frac{z}{x}\right) dx \\ = & \quad 2\int_0^\infty \frac{1}{x} f^{+}(x) g^{+}\left(\frac{z}{x}\right) dx. \end{aligned} \tag{20}$$

Thus in this case when f(x) and g(y) are even functions, we have

$$h^{-}(z) = \begin{cases} 2\int_0^\infty \frac{1}{x} f(x) g\left(-\frac{z}{x}\right) dx & ; \quad -\infty < z < 0 \\ 0 & ; \quad 0 \le z < \infty \end{cases} \tag{21}$$

and

$$h^{+}(z) = \begin{cases} 0 & ; \quad -\infty < z < 0 \\ 2\int_0^\infty \frac{1}{x} f(x) g\left(\frac{z}{x}\right) dx & ; \quad 0 \le z < \infty \end{cases} \tag{22}$$

Theorem 1 Let X be a r.v. following the type II Bessel function distribution given by (1) and Y be an independent r.v. following the normal distribution given by (4), then the p.d.f h(z) of Z=XY is given by

$$h(z)=\frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{\theta z^2}{2\sigma^2}\right)^{\frac{\lambda}{2}}\left(\frac{\theta}{\sigma^2 z^2}\right)^{\frac{1}{4}} e^{-(\beta^2/2\theta)}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(k+\lambda)}\left(\frac{\beta^4 z^2}{16\,\theta\sigma^2}\right)^{\frac{k}{2}} K_{k+\lambda-\frac{1}{2}}\left(\frac{\sqrt{\theta}}{\sigma}|z|\right)$$

$$-\infty < z < \infty \tag{23}$$

where $\sigma > 0,\ \theta > 0,\ \lambda > 0,\ \beta > 0,\ \lambda - (1/2) \neq 0, \pm1, \pm2, \ldots$ and $K_\nu(.)$ is given by (7). The cumulative density function (c.d.f.) H(z) of z is given by

$$H(z)=\frac{1}{2}+\frac{1}{2\sqrt{\pi}}e^{-(\beta^2/2\theta)}\left[\frac{\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma(\lambda)}\left(\frac{\theta z^2}{2\sigma^2}\right)A_1(z)-\frac{1}{3}\frac{\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\sqrt{\theta}}{\sigma}z\right)A_2(z)\right] \tag{24}$$

where

$$A_1(z)=F^{1:1;0}_{2:2;0}\left(\begin{matrix}\lambda; & \lambda+1; & -; \\ \lambda+\frac{1}{2},\lambda+\frac{1}{2}; & \lambda,\lambda; & -;\end{matrix}\ -\frac{\beta^2 z^2}{8\sigma^2},\ \frac{\theta z^2}{4\sigma^2}\right), \tag{25}$$

$$A_2(z)=\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(\lambda-\frac{1}{2}\right)_{k-r}\left(\frac{1}{2}\right)_r}{(\lambda)_k\left(\frac{3}{2}\right)_r}\frac{\left(\frac{\beta^2}{2\theta}\right)^k}{k!}\frac{\left(-\frac{\theta z^2}{4\sigma^2}\right)^r}{r!} \tag{26}$$

where F(.) is given by (10).

Proof : Since the p.d.f.s f(x) and g(y) as given by equations (1) and (4) are even, we shall use the results (21) and (22) to calculate the value of h(z). Substituting the values of *f(x)* and g(- z/x) from equations (1) and (4) in eq.(21) we get

$$h^-(z)=\frac{\theta^\lambda e^{-(\beta^2/2\theta)}}{\beta^{\lambda-1}}\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty x^{\lambda-1}\exp\left\{-\frac{\theta x^2}{2}\right\}I_{\lambda-1}(\beta x)\exp\left\{-\frac{1}{2\sigma^2}\left(\frac{z}{x}\right)^2\right\}dx, \tag{27}$$

$$-\infty < z < 0$$

Next, writing the modified Bessel function $I_\nu(\beta x)$ in its series form (3) and using the following known integral (Gradshteyn and Rhyzik, 1994)

$$\int_0^\infty x^{\nu-1}\exp(-\beta x^p-\gamma x^{-p})\,dx=\frac{2}{p}\left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}}K_{\frac{\nu}{p}}(2\sqrt{(\beta\gamma)})\qquad [\operatorname{Re}\beta>0,\ \operatorname{Re}\gamma>0] \tag{28}$$

we get

$$h^{-}(z) = \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{\theta z^2}{2\sigma^2} \right)^{\frac{\lambda}{2}} \left(\frac{\theta}{\sigma^2 z^2} \right)^{\frac{1}{4}} e^{-(\beta^2/2\theta)} \sum_{k=0}^{\infty} \frac{1}{k!\ \Gamma(k+\lambda)} \left(\frac{\beta^4 z^2}{16\ \theta\sigma^2} \right)^{\frac{k}{2}} K_{k+\lambda-\frac{1}{2}} \left(\frac{\sqrt{\theta}}{\sigma}(-z) \right) \tag{29}$$

$$-\infty < z < 0$$

The value of h^{+}(z) as given in (22) can be similarly obtained as

$$h^{+}(z) = \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{\theta z^2}{2\sigma^2} \right)^{\frac{\lambda}{2}} \left(\frac{\theta}{\sigma^2 z^2} \right)^{\frac{1}{4}} e^{-(\beta^2/2\theta)} \sum_{k=0}^{\infty} \frac{1}{k!\ \Gamma(k+\lambda)} \left(\frac{\beta^4 z^2}{16\ \theta\sigma^2} \right)^{\frac{k}{2}} K_{k+\lambda-\frac{1}{2}} \left(\frac{\sqrt{\theta}}{\sigma}(z) \right) \tag{30}$$

$$0 \le z < \infty$$

Now using the equation (16) and combining the results (29) and (30), we get the required p.d.f. $h(z)$ as given by equation (23).

The c.d.f. of z is defined as

$$H(z) = \int_{-\infty}^{Z} h(z)\,dz = \int_{-\infty}^{0} h(z)\,dz + \int_{0}^{Z} h(z)\,dz = \frac{1}{2} + \int_{0}^{Z} h(z)\,dz \tag{31}$$

[since the p.d.f. $f(x)$ and $g(y)$ are even, $h(z)$ is also an even function and $\int_{-\infty}^{\infty} h(z)\,dz = 1$]

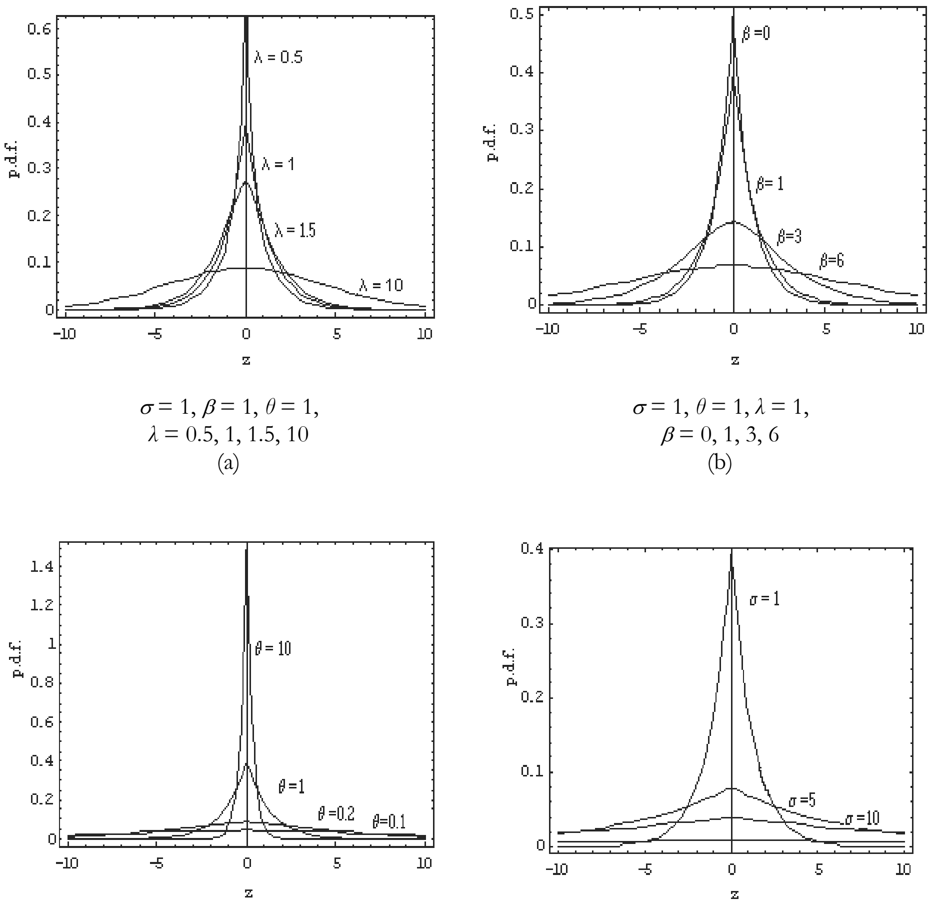
Writing the value of h(z) from eq. (23), using the known result (Prudnikov, Brychov and Marichev, 1986) as mentioned below

$$\int_0^x v^{\lambda} K_{\nu}(x)\,dx = \frac{2^{\nu-1}\Gamma(\nu)}{(\lambda-\nu+1)} x^{\lambda-\nu+1} {}_1F_2\left(\frac{\lambda-\nu+1}{2}; 1-\nu, \frac{\lambda-\nu+3}{2}; \frac{x^2}{4} \right)$$
$$+ \frac{2^{-\nu-1}\Gamma(-\nu)}{(\lambda+\nu+1)} x^{\lambda+\nu+1} {}_1F_2\left(\frac{\lambda+\nu+1}{2}; 1+\nu, \frac{\lambda+\nu+3}{2}; \frac{x^2}{4} \right) \tag{32}$$

$$(\operatorname{Re}\lambda > |\operatorname{Re}\nu| - 1)$$

and applying some useful properties of hypergeometric functions, we get the required result (24).

Fig. 1 illustrates possible shapes of the p.d.f. (23) for (a) $\sigma = 1, \beta = 1, \theta = 1$ and $\lambda = 0.5, 1, 1.5, 10$, (b) $\sigma = 1, \theta = 1, \lambda = 1$ and $\beta = 0, 1, 3, 6$, (c) $\sigma = 1, \beta = 1, \lambda = 1$ and $\theta = 0.1, 0.2, 1, 10$ and (d) $\theta = 1, \beta = 1, \lambda = 1$ and $\sigma = 1, 5, 10, 50$. The effect of the parameters is evident.



$\sigma = 1, \beta = 1, \theta = 1,$
$\lambda = 0.5, 1, 1.5, 10$
(a)

$\sigma = 1, \theta = 1, \lambda = 1,$
$\beta = 0, 1, 3, 6$
(b)

$\sigma = 1, \beta = 1, \lambda = 1,$
$\theta = 0.1, 0.2, 1, 10$
(c)

$\theta = 1, \beta = 1, \lambda = 1,$
$\sigma = 1, 5, 10, 50$
(d)

Fig. 1 – Different possible shapes of the p.d.f. (23) for specified values of parameters λ, β, θ and σ.

Corollary 1 If X and Y are independent r.v.s as given in Theorem 1, then the p.d.f. $h(w)$ and c.d.f. $H(w)$ of $W = |XY|$ are given by

$$h(w) = \frac{2\sqrt{2}}{\sqrt{\pi}} \left(\frac{\theta w^2}{2\sigma^2} \right)^{\frac{\lambda}{2}} \left(\frac{\theta}{\sigma^2 w^2} \right)^{\frac{1}{4}} e^{-(\beta^2/2\theta)} \sum_{k=0}^{\infty} \frac{1}{k!\ \Gamma(k+\lambda)} \left(\frac{\beta^4 w^2}{16\ \theta\sigma^2} \right)^{\frac{k}{2}} K_{k+\lambda-\frac{1}{2}}\left(\frac{\sqrt{\theta}}{\sigma} w \right) \tag{33}$$

and

$$H(w)=\frac{1}{\sqrt{\pi}}e^{-(\beta^2/2\theta)}\left[\frac{\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma(\lambda)}\left(\frac{\theta w^2}{2\sigma^2}\right)A_1(w)-\ \frac{1}{3}\frac{\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\sqrt{\theta}}{\sigma}w\right)A_2(w)\right] \quad (34)$$

$$0 \le w < \infty$$

where $\sigma > 0$, $\theta > 0$, $\lambda > 0$, $\beta \ge 0$ and $A_1(w)$ and $A_2(w)$ are given by eq. (26).

Proof: The result (33) for the p.d.f. of w is a direct consequence of (23) and the relation

$$h(w) = 2\, h(z) \quad ; \quad 0 \le z < \infty \tag{35}$$

and the result (34) for the c.d.f. of w follows from the relation $H(w) = 2\, H(z); 0 \le z < \infty$ where H(z) is given by (24). ■

Corollary 2 Let the r.v X follow the Rayleigh distribution (Springer, 1979) defined as follows

$$f(x) \;=\; \frac{\theta}{2}|\,x\,|\exp\left\{-\frac{\theta x^2}{2}\right\} \quad ; \quad \theta > 0,\ x \in (-\infty,\infty) \tag{36}$$

and the independent r.v Y follow the normal distribution defined by (4), then the p.d.f. $h(z)$ and the c.d.f. $H(z)$ of $Z = XY$ are given by

$$h(z) = \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{\theta^3 z^2}{4\sigma^6}\right)^{\frac{1}{4}}\ K_{\frac{1}{2}}\left(\frac{\sqrt{\theta}}{\sigma}|z|\right) \tag{37}$$

and

$$H(z) = \frac{1}{2}+\frac{1}{2}\left[\left(\frac{\sqrt{\theta}}{\sigma}z\right){}_0F_1\left(-;\tfrac{1}{2};\frac{\theta z^2}{4\sigma^2}\right)+\frac{1}{4}\left(\frac{\theta z^2}{\sigma^2}\right){}_1F_2\left(1;\frac{3}{2},2;\frac{\theta z^2}{4\sigma^2}\right)\right]$$

$$-\infty < z < \infty \tag{38}$$

where $\sigma > 0, \theta > 0$

Proof: On taking $\beta = 0$ and $\lambda = 1$ in Theorem 1, the type II Bessel function distribution reduces to the Rayleigh distribution and we easily arrive at the above result. ■

Remark 1 Note that the p.d.f.s given by Theorem 1 and Corollary 1 are infinite mixtures of type I Bessel function distributions. The p.d.f. given by Corollary 2 is precisely that of a type I Bessel function distribution.

Theorem 2 Let X be a r.v. following the type II Bessel function distribution given by (1) and Y be an independent r.v. following the Pearson VII distribution given by (5), then the p.d.f $h(z)$ of $Z=XY$ is given by

$$h(z)=\frac{1}{\sqrt{\pi}|z|}\left(\frac{\theta z^2}{2N}\right)^{\lambda} e^{-(\beta^2/2\theta)}$$
$$\left[\frac{\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(M+\lambda-1)}{\Gamma(\lambda)\Gamma(M-1)} F_{1:1;0}^{1:0;0}\begin{pmatrix}\lambda-1+M; & -; & -; & \dfrac{\beta^2 z^2}{4N}, & \dfrac{\theta z^2}{2N} \\ \lambda+\dfrac{1}{2}; & \lambda; & -; & & \end{pmatrix}\right.$$
$$\left. + \left(\frac{\theta z^2}{2N}\right)^{\frac{1}{2}-\lambda}\frac{\Gamma\left(M-\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(M-1)\Gamma(\lambda)} H_4\left(\lambda-\frac{1}{2}, M-\frac{1}{2}, \lambda;\ \frac{\beta^2}{2\theta}, -\frac{\theta z^2}{2N}\right)\right]$$
$$-\infty < z < \infty \tag{39}$$

where θ, λ, N > 0, M > 1, β ≥ 0, $\left|\dfrac{\beta^2}{2\theta}\right| < 1$, $\lambda-(1/2) \neq 0, \pm 1, \pm 2, \ldots$

F(.) is given by (10) and $H_4(.)$ (Erdelyi, 1953) is defined as follows

$$H_4(\alpha,\gamma,\delta,x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(\gamma)_n}{(\delta)_m\, m!\,n!}x^m x^n \tag{40}$$

Also the c.d.f of Z is given by

$$H(z)=\frac{1}{2}+\frac{1}{\sqrt{\pi}}\left(\frac{\theta}{2N}\right)^{\lambda}\frac{e^{-(\beta^2/2\theta)}}{\Gamma(M-1)\Gamma(\lambda)}\left[\frac{\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(M+\lambda-1)\Gamma(\lambda)}{2\Gamma(\lambda+1)}B_1(z) + z\,\Gamma\left(M-\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)\left(\frac{\theta}{2N}\right)^{\frac{1}{2}-\lambda}B_2(z)\right] \tag{41}$$

where

$$B_1(z)=F_{2:1;0}^{2:0;0}\begin{pmatrix}\lambda-1+M,\ \lambda; & -; & -; & -\dfrac{\beta^2 z^2}{4N}, & \dfrac{\theta z^2}{2N} \\ \lambda+\frac{1}{2}, \lambda+\dfrac{1}{2}; & \lambda; & -; & & \end{pmatrix} \tag{42}$$

$$B_2(z)=\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(\lambda-\frac{1}{2}\right)_{k-r}\left(M-\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_r}{(\lambda)_k\left(\frac{3}{2}\right)_r}\frac{\left(\frac{\beta^2}{2\theta}\right)^k}{k!}\frac{\left(-\frac{\theta z^2}{2N}\right)^r}{r!} \tag{43}$$

where F(.) is given by (10).

Proof: Substituting the values of $f(x)$ and $g(-z/x)$ from equations (1) and (5) in eq.(21), we obtain

$$h^{-}(z) =$$

$$\frac{\theta^{\lambda} e^{-(\beta^2/2\theta)}}{\beta^{\lambda-1}} \frac{\Gamma\left(M-\frac{1}{2}\right)}{\sqrt{N\pi}\ \Gamma(M-1)} \int_0^{\infty} x^{\lambda-1} \exp\left\{-\frac{\theta x^2}{2}\right\} I_{\lambda-1}(\beta x) \left(1+\frac{\left(-\frac{z}{x}\right)^2}{N}\right)^{\left(\frac{1}{2}-M\right)} dx \tag{44}$$

Writing the modified Bessel function in series form (3) and using the following known integral (Gradshteyn and Rhyzik, 1994)

$$\int_0^{\infty} e^{-px} x^{q-1} (1+ax)^{-\nu}\, dx = a^{-q}\,\Gamma(q)\psi(q, q+1-\nu) \quad [\operatorname{Re} q > 0,\ \operatorname{Re} p > 0, \operatorname{Re} a > 0] \tag{45}$$

we get

$$h^{-}(z) = \frac{1}{\sqrt{\pi}(-z)} \frac{\Gamma\left(M-\frac{1}{2}\right)}{\Gamma(M-1)} \left(\frac{\theta z^2}{2N}\right)^{\lambda} e^{-(\beta^2/2\theta)}$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(k+\lambda+M-1)}{k!\Gamma(\lambda+k)} \left(\frac{\beta^2 z^2}{4N}\right)^{k} \psi\left(k+\lambda+M-1;\ k+\lambda+\frac{1}{2};\ \frac{\theta z^2}{2N}\right) \tag{46}$$

where $\psi(a, b, z)$ is given by (8).

Writing the Tricomi's function $\psi(\alpha, \beta, z)$ in terms of ${}_1F_1(a, b, z)$ as given by (9), and simplifying the result using some properties of hypergeometric functions, we obtain

$$h^{-}(\mathrm{z}) = \frac{1}{\sqrt{\pi}(\text{-z})} \left(\frac{\theta \mathrm{z}^2}{2N}\right)^{\lambda} e^{-(\beta^2/2\theta)}$$
$$\left[\frac{\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(M+\lambda-1)}{\Gamma(\lambda)\Gamma(M-1)} F^{1:0;0}_{1:1;0}\left(\begin{matrix}\lambda-1+M; & -; & -; \\ \lambda+\frac{1}{2}; & \lambda; & -;\end{matrix}\ \frac{\beta^2 \mathrm{z}^2}{4N}, \frac{\theta \mathrm{z}^2}{2N}\right) + \left(\frac{\theta \mathrm{z}^2}{2N}\right)^{\frac{1}{2}-\lambda} \frac{\Gamma\left(M-\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(M-1)\Gamma(\lambda)} H_4\left(\lambda-\frac{1}{2}, M-\frac{1}{2}, \lambda;\ \frac{\beta^2}{2\theta}, -\frac{\theta \mathrm{z}^2}{2N}\right)\right] \tag{47}$$

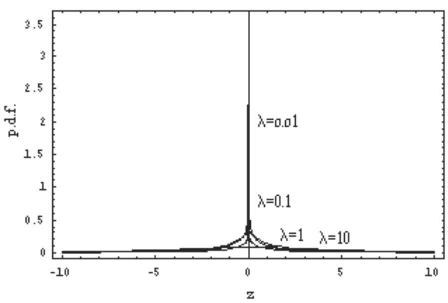
Similarly, $h^{+}(z)$ as given by (22) can be obtained as

$$
h^{+}(\mathrm{z})=\frac{1}{\sqrt{\pi}(\mathrm{z})}\left(\frac{\theta \mathrm{z}^{2}}{2N}\right)^{\lambda} e^{-(\beta^{2}/2\theta)}
$$

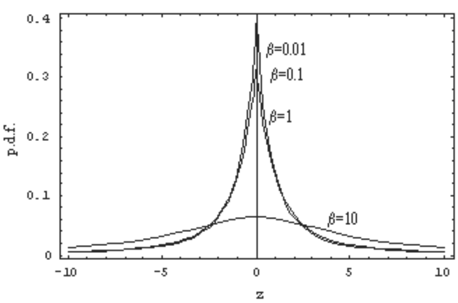
$$
\left[\frac{\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(M+\lambda-1)}{\Gamma(\lambda)\Gamma(M-1)} F_{1:1;0}^{1:0;0}\left(\begin{matrix}\lambda-1+M; & -; & -; \\ \lambda+\frac{1}{2}; & \lambda; & -;\end{matrix}\ \frac{\beta^{2}\mathrm{z}^{2}}{4N}, \quad \frac{\theta \mathrm{z}^{2}}{2N}\right) + \left(\frac{\theta \mathrm{z}^{2}}{2N}\right)^{\frac{1}{2}-\lambda} \frac{\Gamma\left(M-\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(M-1)\Gamma(\lambda)} H_{4}\left(\lambda-\frac{1}{2}, M-\frac{1}{2}, \lambda;\ \frac{\beta^{2}}{2\theta}, -\frac{\theta \mathrm{z}^{2}}{2N}\right)\right] \tag{48}
$$

Now using the equation (16) and combining the results (47) and (48) we get the required p.d.f. h(z) as given by eq. (39). The c.d.f. of z can easily be obtained on using the result (39) in (31) which after simplification yields the required result (41). ■

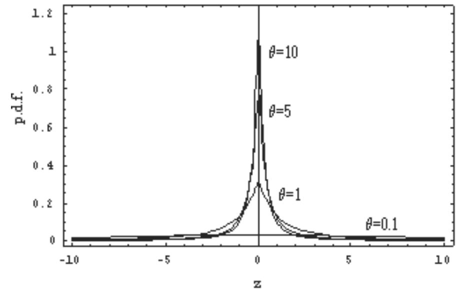
Fig. 2 illustrates possible shapes of the p.d.f. (39) for (a) N=1, M=1.5, $\beta = 1$, $\theta = 1$ and $\lambda = 0.01, 0.1, 1, 10$, (b) N=1, M=1.5, $\theta = 1$, $\lambda = 1$ and $\beta = 0.01, 0.1, 1, 5$, (c) N=1, M=1.5, $\beta = 1$, $\lambda = 1$ and $\theta = 0.1, 1, 5, 10$, (d) $\theta = 1$, $\beta = 0$, $\lambda = 1$, M=1.5 and N=0.1, 1, 2, 5 and (e) $\theta = 1$, $\beta = 0$, $\lambda = 1$, N=1 and M=1.01, 1.5, 2, 5. The effect of the parameters is evident.



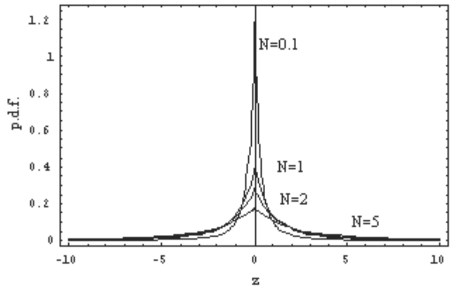
β = 1, θ = 1, N=1, M=1.5
λ = 0.01, 0.1, 1, 10
(a)



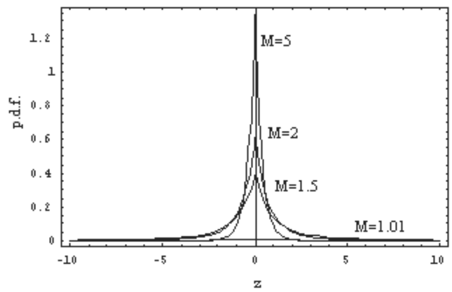
λ = 1, θ = 1, N=1, M=1.5
β = 0.01, 0.1, 1, 5
(b)



λ = 1, β = 1, N=1, M=1.5
θ = 0.1, 1, 5, 10
(c)



λ = 1, θ = 1, β=0, M=1.5
N=0.1, 1, 2, 5
(d)



λ = 1, θ = 1, β=0, N=1
M=1.01, 1.5, 2, 5
(e)

Fig. 2 – Different possible shapes of the p.d.f. (39) for specified values of parameters *λ, β, θ, N* and *M*.

Corollary 3 If X and Y are independent r.v.s as given in Theorem 2, then the p.d.f. h(w) and the c.d.f. *H(w)* of $W=|XY|$ are given by

$$h(w)=\frac{2}{\sqrt{\pi}w}\left(\frac{\theta w^2}{2N}\right)^{\lambda} e^{-\left(\frac{\beta^2}{2\theta}\right)}$$
$$\left[\frac{\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(M+\lambda-1)}{\Gamma(\lambda)\Gamma(M-1)} F^{1:0;0}_{1:1;0}\left(\begin{matrix}\lambda-1+M; & -; & -; & \frac{\beta^2 w^2}{4N}, & \frac{\theta w^2}{2N}\\ \lambda+\frac{1}{2}; & \lambda; & -; & & \end{matrix}\right)\right.$$
$$\left.+\left(\frac{\theta w^2}{2N}\right)^{\frac{1}{2}-\lambda}\frac{\Gamma\left(M-\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(M-1)\Gamma(\lambda)} H_4\left(\lambda-\tfrac{1}{2}, M-\frac{1}{2}, \lambda; \frac{\beta^2}{2\theta}, -\frac{\theta w^2}{2N}\right)\right] \tag{49}$$

and

$$H(w)=\frac{2}{\sqrt{\pi}}\left(\frac{\theta}{2N}\right)^{\lambda}\frac{e^{-(\beta^2/2\theta)}}{\Gamma(M-1)\Gamma(\lambda)}\left[\begin{array}{l}\dfrac{\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(M+\lambda-1)\Gamma(\lambda)}{2\Gamma(\lambda+1)}B_1(w)\\ \quad +w\,\Gamma\left(M-\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)\left(\dfrac{\theta}{2N}\right)^{\frac{1}{2}-\lambda}B_2(w)\end{array}\right] \tag{50}$$

$$0 \le w < \infty$$

where $\theta, \lambda, N > 0$, $M > 1$, $\beta \geq 0$, and $B_1(w)$and $B_2(w)$ as defined by (42) and (43) respecttively.

Proof: Result (49) for the p.d.f. of w is a direct consequence of (39) and the relation

$$h(w) = 2\,h(z) \quad ; \qquad 0 \le z < \infty \tag{51}$$

and the result (50) for the c.d.f. of w follows from the relation

$$H(w) = 2\,H(z) \quad ; \qquad 0 \le z < \infty \tag{52}$$

where H(z) is given by (41). ■

Corollary 4 Let the independent r.v. X follow the type II Bessel function distribution given by (1) and the r.v Y follow the student-t distribution (Johnson and Kotz, 1970) defined as follows

$$g(y)=\frac{1}{\sqrt{\nu}\;B\left(\frac{1}{2},\frac{\nu}{2}\right)}\left(1+\frac{y^2}{\nu}\right)^{\left(\frac{\nu+1}{2}-\right)} \quad ; \quad y \in (-\infty,\infty),\; \nu > 0 \tag{53}$$

then the p.d.f. h(z) and c.d.f. H(z) of Z = XY are given by

$$h(z)=\frac{1}{\sqrt{\pi}}\left(\frac{\theta}{2\nu}\right)^{\lambda}e^{-\left(\frac{\beta^2}{2\theta}\right)}$$
$$\left[\begin{array}{l}\dfrac{\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma\left(\lambda+\frac{\nu}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{\nu}{2}\right)}(z^2)^{\lambda-\frac{1}{2}}F^{1:0;0}_{1:1;0}\left(\begin{array}{llll}\lambda+\frac{\nu}{2}; & -; & -; & \dfrac{\beta^2 z^2}{4\nu},\;\dfrac{\theta z^2}{2\nu}\\ \lambda+\frac{1}{2}; & \lambda; & -; & \end{array}\right)\\ \quad +\left(\dfrac{\theta}{2\nu}\right)^{\frac{1}{2}-\lambda}\dfrac{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma(\lambda)}H_4\left(\lambda-\frac{1}{2},\frac{\nu}{2}+\frac{1}{2},\lambda;\;\frac{\beta^2}{2\theta},-\frac{\theta z^2}{2\nu}\right)\end{array}\right] \tag{54}$$

and

$$H(z)=\frac{1}{2}+\frac{1}{2\sqrt{\pi}}\left(\frac{\theta}{2\nu}\right)^{\lambda}e^{-(\beta^2/2\theta)}\left[\begin{array}{l}\frac{\Gamma\left(\frac{\nu}{2}+\lambda\right)\Gamma\left(\frac{1}{2}-\lambda\right)\lambda}{2\Gamma\left(\frac{\nu}{2}\right)\Gamma(\lambda+1)}D_1(z)+\\ \qquad z\,\frac{\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\theta}{2\nu}\right)^{\frac{1}{2}-\lambda}D_2(z)\end{array}\right]$$

$$-\infty < z < \infty \tag{55}$$

where $\theta > 0,\ \nu > 0, \lambda > 0$

$$D_1(z)=F^{2:0;0}_{2:1;0}\left(\begin{array}{llll}\lambda+\frac{\nu}{2},\ \lambda; & -; & -; & \\ & & & -\frac{\beta^2 z^2}{4\nu},\quad \frac{\theta z^2}{2\nu}\\ \lambda+\frac{1}{2},\lambda+1; & \lambda; & -; & \end{array}\right), \tag{56}$$

$$D_2(z)=\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{\left(\lambda-\frac{1}{2}\right)_{k-r}\left(\frac{\nu}{2}+\frac{1}{2}\right)_r\left(\frac{1}{2}\right)_r}{(\lambda)_k\left(\frac{3}{2}\right)_r}\frac{\left(\frac{\beta^2}{2\theta}\right)^k}{k!}\frac{\left(-\frac{\theta z^2}{2\nu}\right)^r}{r!} \tag{57}$$

Proof: If $N = \nu$ and $M = 1 + \nu/2$ the Pearson VII distribution reduces to the student-t distribution, thus the above results can easily be obtained by setting $N = \nu$ and $M = 1 + \nu/2$ in Theorem 2. ■

Theorem 3 Let X be a r.v. following the type II Bessel function distribution given by (1) and Y be an independent r.v. following the Maxwell-Boltzmann distribution given by (6), then the p.d.f h(z) of Z=XY is given by

$$h(z)=\frac{4}{\sqrt{\pi}}\left(\frac{\alpha\theta}{2}\right)^{\frac{\lambda}{2}+\frac{1}{4}}(z^2)^{\frac{\lambda}{2}+\frac{3}{4}}e^{-(\beta^2/2\theta)}\sum_{k=0}^{\infty}\frac{1}{k!\ \Gamma(k+\lambda)}\left(\frac{\alpha\ \beta^4 z^2}{8\ \theta}\right)^{\frac{k}{2}}K_{k+\lambda-\frac{3}{2}}(\sqrt{2\alpha\theta}\,|z|) \tag{58}$$

and

$$H(z)=\frac{1}{2}+\frac{1}{\sqrt{\pi}}e^{-(\beta^2/2\theta)}\left[\frac{\Gamma\left(\frac{3}{2}-\lambda\right)}{\Gamma(\lambda+1)}\left(\frac{\alpha\theta z^2}{2}\right)^{\lambda}C_1(z)-\frac{\Gamma\left(\lambda-\frac{3}{2}\right)}{\Gamma(\lambda)}\left(\frac{\alpha\theta z^2}{2}\right)^{\frac{3}{2}}C_2(z)\right] \tag{59}$$

$$-\infty < z < \infty$$

where $\theta, \lambda, \alpha > 0, \beta \geq 0, k + \lambda - (1/2) \neq 0, \pm 1, \pm 2, \ldots$ and

$$C_1(z) = F^{1:0;0}_{2:1;0}\left(\begin{matrix} \lambda; & -; & -; \\ \lambda+1, \lambda-\frac{1}{2}; & \lambda; & -; \end{matrix} -\frac{\alpha\beta z^2}{4}, \quad \frac{\alpha\theta z^2}{2}\right), \tag{60}$$

$$C_2(z) = \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \frac{\left(\lambda-\frac{3}{2}\right)_{k-r}\left(\frac{3}{2}\right)_r}{(\lambda)_k\left(\frac{5}{2}\right)_r} \frac{\left(\frac{\beta^2}{2\theta}\right)^k}{k!} \frac{\left(-\frac{\alpha\theta z^2}{2}\right)^r}{r!} \tag{61}$$

Proof: Substituting the values of *f(x)* and *g(- z/x)* from equations (1) and (6) in eq. (21) as follows

$$h^-(z) = \frac{2}{\sqrt{\pi}}\alpha^{\frac{3}{2}} \frac{\theta^\lambda e^{-(\beta^2/2\theta)}}{\beta^{\lambda-1}} \int_0^\infty x^{\lambda-1} \exp\left\{-\frac{\theta x^2}{2}\right\} I_{\lambda-1}(\beta x)\left(-\frac{z}{x}\right)^2 \exp\left(-\alpha\left(-\frac{z}{x}\right)^2\right) dx \tag{62}$$

$$-\infty < z < 0$$

Next, writing the modified Bessel function in series form and using the known result (Gradshteyn and Rhyzik, 1994) given in Theorem 1, we get $h^-(z)$ as follows

$$h^-(z) = \frac{4}{\sqrt{\pi}}\left(\frac{\alpha\theta}{2}\right)^{\frac{\lambda}{2}+\frac{3}{4}} (z^2)^{\frac{\lambda}{2}+\frac{1}{4}} e^{-(\beta^2/2\theta)} \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+\lambda)} \left(\frac{\beta^4 \alpha z^2}{8\,\theta}\right)^{\frac{k}{2}} K_{k+\lambda-\frac{3}{2}}(\sqrt{2\alpha\theta}(-z)) \tag{63}$$

$$-\infty < z < 0$$

Similarly, $h^+(z)$ can be obtained as

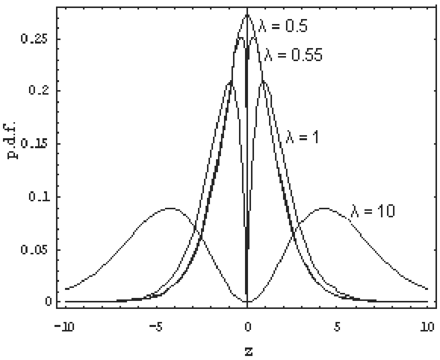
$$h^+(z) = \frac{4}{\sqrt{\pi}}\left(\frac{\alpha\theta}{2}\right)^{\frac{\lambda}{2}+\frac{3}{4}} (z^2)^{\frac{\lambda}{2}+\frac{1}{4}} e^{-(\beta^2/2\theta)} \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+\lambda)} \left(\frac{\beta^4 \alpha z^2}{8\,\theta}\right)^{\frac{k}{2}} K_{k+\lambda-\frac{3}{2}}(\sqrt{2\alpha\theta}z) \tag{64}$$

$$0 \leq z < \infty$$

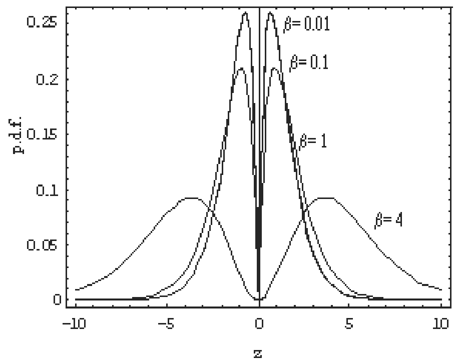
Now using the equation (16) and combining the results (63) and (64) we can get the required p.d.f. as given by eq. (58).

The c.d.f. of z can easily be obtained by using the result (58) in the definition (31), which on using a known result (Prudnikov, Brychov and Marichev, 1986) as mentioned in the proof of Theorem 1 and applying some useful properties of hypergeometric functions, yields the required result (59). ■

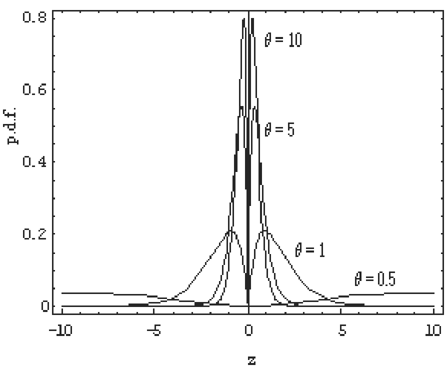
Fig. 3 illustrates possible shapes of the p.d.f. (58) for (a) $\alpha = 1$, $\beta = 1$, $\theta = 1$ and $\lambda = 0.5, 0.55, 1, 10$, (b) $\alpha = 1$, $\theta = 1$, $\lambda = 1$ and $\beta = 0.01, 0.1, 1, 4$, (c) $\alpha = 1$, $\beta = 1$, $\lambda = 1$ and $\theta = 0.1, 1, 5, 10$ and (d) $\theta = 1$, $\beta = 1$, $\lambda = 1$ and $\alpha = 0.1, 1, 5, 10$. The effect of the parameters is evident



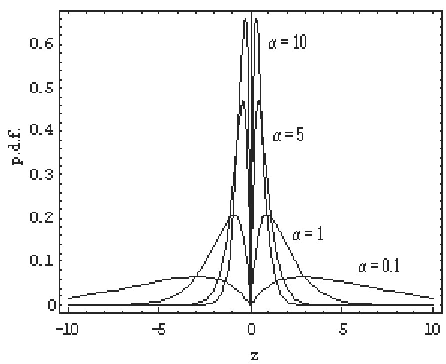
$\alpha = 1, \beta = 1, \theta = 1,$
$\lambda = 0.5, 0.55, 1, 10$



$\alpha = 1, \theta = 1, \lambda = 1,$
$\beta = 0.01, 0.1, 1, 4$



$\alpha = 1, \beta = 1, \lambda = 1,$
$\theta = 0.1, 1, 5, 10$



$\theta = 1, \beta = 1, \lambda = 1,$
$\alpha = 0.1, 1, 5, 10$

Fig. 3 – Different possible shapes of the p.d.f. (58) for specified values of parameters λ, β, θ and α.

Corollary 5 If X and Y are independent r.v.s as given in Theorem 3, then the p.d.f h(w) and the c.d.f H(w) of W= |XY| are given by

$$h(w) = \frac{8}{\sqrt{\pi}}\left(\frac{\alpha\theta}{2}\right)^{\frac{\lambda}{2}+\frac{1}{4}} (w^2)^{\frac{\lambda}{2}+\frac{3}{4}} e^{-(\beta^2/2\theta)} \sum_{k=0}^{\infty} \frac{1}{k!\ \Gamma(k+\lambda)} \left(\frac{\alpha\ \beta^4 w^2}{8\ \theta}\right)^{\frac{k}{2}} K_{k+\lambda-\frac{3}{2}}(\sqrt{2\alpha\theta}w) \tag{65}$$

and

$$H(w)=\frac{2}{\sqrt{\pi}}e^{-(\beta^2/2\theta)}\left[\frac{\Gamma\left(\frac{3}{2}-\lambda\right)}{\Gamma(\lambda+1)}\left(\frac{\alpha\theta w^2}{2}\right)^{\lambda}C_1(w)-\frac{\Gamma\left(\lambda-\frac{3}{2}\right)}{\Gamma(\lambda)}\left(\frac{\alpha\theta w^2}{2}\right)^{\frac{3}{2}}C_2(w)\right] \tag{66}$$

$$0 \le w < \infty$$

where θ, λ, $\alpha > 0$ $\beta \ge 0$ and $C_1(w)$ and $C_2(w)$ as defined by (60) and (61) respectively.

Proof: Result (65) for the p.d.f. of W is a direct consequence of (58) and the relation

$$h(w) = 2\, h(z) \quad ; \quad 0 \le z < \infty \tag{67}$$

and the result (66) for the c.d.f. of W follows from the relation

$$H(w) = 2\, H(z) \quad ; \quad 0 \le z < \infty \tag{68}$$

where $H(z)$ is given by (59). ■

Corollary 6 Let the independent r.v. X follow the Chi distribution (Springer, 1979) given as follows

$$f(x)=\frac{1}{\Gamma(\theta)}\left(\frac{\theta}{2\,\sigma^2}\right)^{\theta}|x|^{2\theta-1}\exp\left\{-\frac{\theta x^2}{2\sigma^2}\right\} \quad ;\ x\in(-\infty,\infty),\ \theta>0,\ \lambda>0 \tag{69}$$

and the independent r.v Y follow the Maxwell-Boltzmann distribution given by (6), then the p.d.f. h(z) and the c.d.f. H(z) of Z = XY are given by

$$h(z)=\frac{4}{\sqrt{\pi}}\left(\frac{\alpha\theta}{2\sigma^2}\right)^{\frac{\theta}{2}+\frac{1}{4}}(z^2)^{\frac{\theta}{2}+\frac{1}{4}}\frac{1}{\Gamma(\lambda)}K_{\theta-\frac{3}{2}}\left(\frac{\sqrt{2\alpha\theta}}{\sigma}|z|\right), \quad -\infty < z < \infty, \tag{70}$$

and

$$H(z)=\frac{1}{2}+\frac{1}{\sqrt{\pi}}\left[\frac{\Gamma\left(\frac{3}{2}-\theta\right)}{\Gamma(\theta+1)}\left(\frac{\alpha\theta z^2}{2\sigma^2}\right)^{\theta}{}_1F_2\left(\lambda;\lambda+1,\lambda-\frac{1}{2};\frac{\alpha\theta z^2}{2\sigma^2}\right)-\frac{\Gamma\left(\theta-\frac{3}{2}\right)}{\Gamma(\theta)}\left(\frac{\alpha\theta z^2}{2\sigma^2}\right)^{\frac{3}{2}}{}_1F_2\left(\frac{3}{2};\frac{5}{2},\frac{5}{2}-\theta;\frac{\alpha\theta z^2}{2\sigma^2}\right)\right]$$

$$-\infty < z < \infty, \tag{71}$$

where θ, $\alpha > 0$, $\sigma > 0$, θ - (3/2) $\neq 0, \pm1, \pm2,...$

Proof: On taking $\beta = 0$ and $\theta = \theta_1/\sigma^2$, $\lambda = \theta_1$ and replacing θ_1 by θ in Theorem 3, the type II Bessel function distribution reduces to the Chi distribution and we easily arrive at the above result. ■

Remark 2 Note that the p.d.f.s given by Theorem 3 and Corollary 5 are infinite mixtures of type I Bessel function distributions. The p.d.f. given by Corollary 6 is precisely that of a type I Bessel function distribution.

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SUMMARY

Distributions of products involving the type II Bessel function random variable

The aim of the present paper is to study the distributions of product of two independent random variables X and Y which are not everywhere positive. We have taken X to be a type II Bessel function random variate whereas Y belongs to one of normal, Pearson VII or Maxwell-Boltzmann families of distributions. Several special cases have also been obtained.