TESTS AND ASYMPTOTIC NORMALITY FOR MIXED BIVARIATE MEASURE

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1. INTRODUCTION

Consider a pair of random variables (X,Y) whose joint probability measure is the sum of an absolutely continuous measure, a discrete measure and a finite number of absolutely continuous measures on some lines (called *jump lines*):

$$d\mu = f(x,y)dxdy + \sum_{j=1}^{q} a'_j \delta_{(\omega_{1j},\omega_{2j})} + \sum_{i=1}^{q'} \phi_i(u_1)\delta_{(u_1,a_i u_1+b_i)}, \tag{1}$$

The motivation for the choice of such model is illustrated through the concrete example that we study in the last section. This example concerns the study of structural fissure of the agricultural soil. On a homogenous soil, measures of the resistance variable X and the humidity variable Y are taken on several locations at a depth of 30cm. The measurement values are distributed according to a continuous law, except in certain locations where the experimentalist finds small galleries where measurement values of resistance and humidity decrease (the presence of jumps). When the measures are made in places where the passage of tractors is frequent, the variable Y becomes linear with respect to the variable X and their measures follow a new distribution noted ϕ_i (the presence of some measures continuous on the lines determined by the frequent passages of tractors).

In Sabre 2003, an asymptotically unbiased estimates of the continuous part density f is constructed from a finite number of observations of two-dimensional μ-distributed random variables (X,Y). Indeed, in the neighborhood of the *jump points* and on the *jump lines*, is chosen the double kernel method using four windows satisfying same conditions. The same technique is used to estimate the amplitude of the *jump points* a'_j and to estimate the densities ϕ_i of the *jump lines*.

For these estimates it is assumed that we know exactly the jump line and the *jump points* $(\omega_{1j},\omega_{2j})$ are unknown but can be localized in a block

$[\alpha_{1j}, \beta_{1j}] \times [\alpha_{2j}, \beta_{2j}]$. The block is assumed sufficiently small to contain only one *jump point*. The last assumption is not easy to satisfy in practice. Indeed, in order to determine these blocks, several samples must be taken which is impossible in some cases.

This work aims at finding a resolution to this problem. Its goal is to give a statistical test in order to check if any pair (x, y) is a *jump point* (ie. if $(x, y) = (\omega_{1j}, \omega_{2j})$). For that, we show the *limit theorems* for the amplitude estimate given in Sabre (2003). To achieve that, we first establish the optimal rates of the convergence for the variance of the amplitude estimate $\hat{a}'_j$ and for the variance of the density estimates on *jump lines* $\hat{\phi}_i$.

This paper is organized as following: the section 2 gives some preliminaries about estimations of f, a'_j and ϕ_i. The section 3 is devoted to the study of the rates of convergence for the variances of the estimates $\hat{a}'_j$ and $\hat{\phi}_i$ (theorems 3.1, 3.2). The section 4, presents the *limits theorems* for these estimates $\hat{a}'_j$ (theorem 4.1 and corollary 4.1) that we use for studying some tests on the existence of the *jump points*. The section 5 is reserved to prove the theorems. In the section 6 we study a concrete example where we apply the statistical tests proposed in section 4.

2. THE ESTIMATION AND THE OPTIMAL RATES OF CONVERGENCE

Suppose that we have n observations $(x_1, y_2), (x_2, y_2), \dots (x_n, y_n)$ independent identically distributed (iid) from the random variables (X, Y) for which the joint probability measure, μ, is defined in (1). The numbers q and q' are assumed nonnegative integers and known. f is the density of the continuous variable which is assumed to be a nonnegative uniformly continuous function. The real positive number a'_j is the amplitude of the *jump* at $(\omega_{1j}, \omega_{2j})$ and is assumed unknown. The densities ϕ_i are nonnegative uniformly continuous functions assumed unknown. The coefficients of the lines a_i, b_i are real numbers assumed unknown. δ is the Dirac measure. Suppose that the *jump points* don't belong to the *jump lines* (ie. $w_{2j} \neq a_i w_{1j} + b_i$ for all i and j).

2.1. *Estimation of the continuous part density*

In order to estimate the density f, Sabre (2003) has assumed, that any *jump point* $(\omega_{1j}, \omega_{2j})$ can be localized in a small block $[\alpha_{1j}, \beta_{1j}] \times [\alpha_{2j}, \beta_{2j}]$. Thus, the estimate of the density $f(x, y)$ purposed is different according to the position of (x, y):

$$\widehat{f}(x,y)=\begin{cases} g_n(x,y) \text{ if } (x,y)\in A \\ f_n(x,y) \text{ if } (x,y)\not\in A \end{cases}.$$

with

$$f_n(x,y)=\sum_{i=1}^{n}\frac{1}{nh_n^2}K\left(\frac{x-x_i}{h_n},\frac{y-y_i}{h_n}\right)$$
$$g_n(x,y)=\int_{R^2} S_n(x-u_1)R_n(y-u_2)f_n(u_1,u_2)du_1du_2 \tag{2}$$

where

$$A=\cup_{j=1}^{q}([\alpha_{1j},\beta_{1j}]\times R)\cup(R\times[\alpha_{2j},\beta_{2j}])\cup B$$
$$\textit{with } B=\{(x,y)\in R^2 \textit{ such as } \exists i\in\{1,...,q'\}: y=a_ix+b_i\}.$$

The kernel K is defined by $K(u,v)=K_1(u)K_2(v)$ with K_1 and K_2 two continuous, even, decreasing kernels such that: $\int\left\|y^2\right\|K_i(y)dy<\infty \quad i=1,2$. The smoothing parameter h_n, converges to zero and nh_n^2 converges to the infinite, where:

$$S_n(z)=\frac{W_n^{(2)}(z)-\dfrac{M_n^{(2)}}{M_n^{(1)}}W_n^{(1)}(z)}{1-\dfrac{M_n^{(2)}}{M_n^{(1)}}} \textit{ and } R_n(t)=\frac{W_n^{(3)}(t)-\dfrac{L_n^{(2)}}{L_n^{(1)}}W_n^{(4)}(t)}{1-\dfrac{L_n^{(2)}}{L_n^{(1)}}}$$

The windows functions are defined as follows:

$$W_n^{(1)}(t)=M_n^{(1)}W^{(1)}(tM_n^{(1)}) \ ; \ W_n^{(2)}(t)=M_n^{(2)}W^{(2)}(tM_n^{(2)})$$
$$W_n^{(3)}(t)=L_n^{(1)}W^{(3)}(tL_n^{(1)}) \text{ and } W_n^{(4)}(t)=L_n^{(2)}W^{(4)}(tL_n^{(2)})$$

where $M_n^{(1)},M_n^{(2)},L_n^{(1)}$ and $L_n^{(2)}$ are nonnegative real sequences satisfying:

$$M_n^{(r)}\to+\infty;\quad L_n^{(r)}\to+\infty;\ M_n^{(r)}h_n\to 0;\quad L_n^{(r)}h_n\to 0;\ \frac{M_n^{(2)}}{M_n^{(1)}}\to 0 \text{ and } \frac{L_n^{(2)}}{L_n^{(1)}}\to 0$$

The function $W^{(i)}$ is a nonnegative, even, integrable function vanishing outside the interval $[-1,1]$ such that $\int_{-1}^{1}W^{(i)}(x)dx=1$, $i=1,2,3,4$. Moreover $W^{(i)}$ satisfying the following equalities:

$$W^{(2)}(M_n^{(2)}\theta)-W^{(1)}(M_n^{(1)}\theta)=0 \qquad \forall\theta\in\left]\frac{-1}{M_n^{(1)}},\frac{1}{M_n^{(1)}}\right[. \tag{3}$$

$$W^{(4)}(L_n^{(2)}\theta)-W^{(3)}(L_n^{(1)}\theta)=0 \qquad \forall\theta\in\left]\frac{-1}{L_n^{(1)}},\frac{1}{L_n^{(1)}}\right[. \tag{4}$$

Assume that $\frac{1}{h_n}K_1\left(\frac{1}{h_nM_n^{(1)}}\right)$ and $\frac{1}{h_n}K_2\left(\frac{1}{h_nL_n^{(1)}}\right)$ converge to zero, for example $K_1(x)=\frac{1}{(2\pi)^{\frac{1}{2}}}\exp\left(\frac{-x^2}{2}\right)$.

2.2. *The amplitude jump estimate and the density on jump line estimate*

The estimator purposed is defined as follows: $\widehat{a'}_n(x,y)=\frac{1}{nK(0,0)}\sum_{i=1}^{n}K\left(\frac{x-x_i}{\beta_n},\frac{y-y_i}{\beta_n}\right)$, the smoothing parameter β_n satisfy $\beta_n\to 0; n\beta_n\to\infty$ and $n\beta_n^2\to 0$. It is shown that $\widehat{a'}_n(x,y)$ is an asymptotically unbiased and consistent estimate of $a'(x,y)$ defined by:

$$a'(x,y)=\begin{cases}0 & \text{if } (x,y)\neq(\omega_{1j},\omega_{2j}) \text{ for all } j=1,...,q\\ a'_j & \text{if } (x,y)=(\omega_{1j_0},\omega_{2j_0});1\le j_0\le q\end{cases}$$

In order to estimate the density, ϕ_i, it is given that the following estimator: $\hat{\phi}_i(\lambda_1,\lambda_2)=\frac{1}{K_2(0)}\sum_{i=1}^{n}\frac{1}{nh'_nh''_n}K\left(\frac{\lambda_1-x_i}{h'_n},\frac{\lambda_2-y_i}{h''_n}\right)$ where $h''_n\to 0$; $h'_n\to 0$; $\frac{h'_n}{h''_n}\to 0$; $nh'_n\to\infty$; $nh''_n\to\infty$ and $nh_n'^2h''_n\to\infty$. Then it is shown that $\hat{\phi}_i(\lambda_1,\lambda_2)$ is an asymptotically unbiased and consistent estimate of $\phi_i(\lambda_1)$ if $\lambda_2=a_i\lambda_1+b_i$.

3. THE OPTIMAL RATES OF CONVERGENCE

In this section we establish that precise asymptotic expressions for the variances of the amplitude estimate $\widehat{a'}_n(x,y)$ and of the density estimate $\hat{\phi}_i(\lambda_1,\lambda_2)$. The following theorems give optimal rates of convergence that we will use in the sequel.

Theorem 3.1 *Let* (x, y) *be an element of* R^2.

1) If (x, y) is neither a *jump point* nor an element of *jump lines* (ie. if $(x, y) \neq (w_{1j}, w_{2j})$ and $y \neq a_i x + b_i \quad \forall i \quad \forall j$), then

$$Var(\hat{a}'(x, y)) = \frac{\beta_n^2}{n} \frac{1}{K^2(0,0)} f(x, y) \int K^2(t_1, t_2) dt_1 dt_2 + o\left(\frac{\beta_n^2}{n}\right).$$

2) If (x, y) is a *jump point* (ie. $(x, y) = (w_{1j_0}, w_{2j_0})$; $1 \leq j_0 \leq q$)

$$Var(\hat{a}'(x, y)) = \frac{a'_{j_0}}{n} + o\left(\frac{1}{n}\right).$$

3) If (x, y) belongs to one jump line (ie. $y = a_{i_0} x + b_{i_0}$ with $1 \leq i_0 \leq q'$), then

$$Var(\widehat{a'}_n(x, y)) = \frac{\beta_n}{n} \frac{\phi_{i_0}(x)}{K^2(0,0)} \int K_1^2(z) K_2(a_{i_0} z) dz + o\left(\frac{\beta_n}{n}\right).$$

Theorem 3.2 *Let* (x, y) *be an element of* R^2.

1) If (x, y) belongs to one jump line: $y = a_{i_0} x + b_{i_0}$, then

$$Var(\hat{\phi}_{i_0}(x, y)) = \frac{1}{nb'_n} \phi_{i_0}(x) + o\left(\frac{1}{nb'_n}\right).$$

2) If $y \neq a_i x + b_i$ and $(x, y) = (w_{1j_0}, w_{2j_0})$, then

$$Var(\hat{\phi}_i(x, y)) = \frac{1}{nb_n'^2} a_{j_0} K_1^2(0) + o\left(\frac{1}{nb_n'^2}\right).$$

3) If $y \neq a_i x + b_i$ and $(x = w_{1j_0} ; y \neq w_{2j_0})$, then

$$Var(\hat{\phi}_i(x, y)) = \frac{b'_n}{nb''_n K_1^2(0)} f(x, y) \int K^2(t_1, t_2) dt_1 dt_2 + o\left(\frac{b'_n}{nb''_n}\right).$$

4. LIMIT THEOREMS

Denote the rate of the convergence of the variance of $\hat{a}'_n(x, y)$ by:

$$U_n(x,y)=\begin{cases} \dfrac{\beta_n^2}{n}\dfrac{1}{K^2(0,0)}f(x,y) \times\int K^2(t_1,t_2)dt_1dt_2 & \text{if } \begin{cases}(x,y)\neq(w_{1j},w_{2j}) \text{ and} \\ y\neq a_ix+b_i\end{cases} \\ \dfrac{\beta_n}{n}\dfrac{\phi_{i_0(x)}}{K^2(0,0)} \times\int K_1^2(z_1)K_2^2(a_{i_0}z_2)dz_1dz_2 & \text{if } y=a_{i_0}x+b_{i_0} \\ \dfrac{a'_{j_0}}{n} & \text{if } (x,y)=(w_{1j_0},w_{2j_0}) \end{cases}$$

Theorem 4.1 *Let* (x,y) *be an element of* R^2*, then*

$$\frac{\hat{a}'_n(x,y)-E(\hat{a}'_n(x,y))}{(U_n(x,y))^{1/2}}\to N(0,1),$$

where $N(0,1)$ is the standard gaussian random variable.

Corollary 4.1 *Let* (x,y) *be an element of* R^2*, then.*

1) If $(x,y)\neq(w_{1j},w_{2j})$ and $y\neq a_ix+b_i$, then

$$\left(\frac{n}{\beta_n^2}\right)^{1/2}\frac{\hat{a}'(x,y)-a'(x,y)}{(K^{-2}(0,0)\hat{f}(x,y)\int K^2(t_1,t_2)dt_1dt_2)^{1/2}}\to N(0,1)$$

2) If $(x,y)=(w_{1j_0},w_{2j_0})$, then

$$n^{1/2}\frac{\hat{a}'_n(x,y)-a'(x,y)}{(K^{-2}(0,0)\hat{a}'(x,y))^{1/2}}\to N(0,1).$$

4.1. *Statistical test*

Let (x,y) be an element of R^2 not belonging to any jump line (ie. $y\neq a_ix+b_i$ for all i). In order to test if (x,y) is a jump point, we consider the null hypothesis H_0: (x,y) is not a jump point (ie. $a'(x,y)=0$).

The alternative hypothesis H_1 is: (x,y) is a *jump point*. Under H_0, we have $(x,y)\neq(w_{1j},w_{2j})$ and $y\neq a_ix+b_i$. We calculate

$$L=\left(\frac{n}{\beta_n^2}\right)^{1/2}\frac{\hat{a}'(x,y)}{(K^{-2}(0,0)\hat{f}(x,y)\int K^2(t_1,t_2)dt_1dt_2)^{1/2}}.$$

If L belongs to $[-Z_{\alpha/2};Z_{\alpha/2}]$ we accept the hypothesis H_0 (ie. (x,y) is not a *jump point*). If not, we conclude that (x,y) a *jump point*.

5. PROOFS

5.1 *Proof of the theorem 3.1*

With the same arguments used to show the equality (16) in sabre (2003), we can show that:

$Var(\hat{a}'(x,y))=H_1+H_2+H_3-H_4$, *where*

$$H_1=\frac{1}{nK^2(0,0)}\int K^2\left(\frac{x-z_1}{\beta_n},\frac{y-z_2}{\beta_n}\right)f(z_1,z_2)dz_1dz_2$$

$$H_2=\frac{1}{nK^2(0,0)}\sum_{j=1}^{q}a_jK^2\left(\frac{x-w_{1j}}{\beta_n},\frac{x-w_{2j}}{\beta_n}\right)$$

$$H_3=\frac{1}{nK^2(0,0)}\sum_{j=1}^{q'}\int K^2\left(\frac{x-u}{\beta_n},\frac{x-a_iu-b_i}{\beta_n}\right)\phi_i(u)du$$

$$H_4=\frac{1}{nK^2(0,0)}E^2\left[K\left(\frac{x-x_i}{\beta_n},\frac{y-y_i}{\beta_n}\right)\right]$$

Write $H_1=\frac{\beta_n^2}{nK^2(0,0)}(\int K^2)K'_{\beta_n}*f(x,y)$ where $K'_{\beta_n}(t_1,t_2)=\frac{\frac{1}{\beta_n^2}K^2\left(\frac{t_1}{\beta_n},\frac{t_2}{\beta_n}\right)}{\int K^2(r_1,r_2)dr_1dr_2}$.

Since the function $(a,b)\to\frac{K^2(a,b)}{\int K^2(t_1,t_2)dt_1dt_2}$ is a Parzen kernel, $K'_{\beta_n}*f(x,y)$ converges to $f(x,y)$. Thus,

$$\lim_{n\to\infty}\frac{n}{\beta_n^2}H_1=\frac{1}{K(0,0)}f(x,y)\int K^2(t_1,t_2)dt_1dt_2 \qquad (5)$$

In order to bounding H_2, H_3 and H_4, we study the following cases:

a) First case: If $y = a_{i_0} x + b_{i_0}$

In this case, we have $x \neq w_{1j}$ or $y \neq w_{2j}$ for all $y \neq w_{2j}$. Therefore, we get

$$\frac{n}{\beta_n^4} H_2 = O\left(\sup\left(\frac{1}{\beta_n^2} K_1^2\left(\frac{1}{\beta_n}\right); \frac{1}{\beta_n^2} K_2^2\left(\frac{1}{\beta_n}\right)\right)\right)^2 . \tag{6}$$

On the other hand, the expression of H_3 can be written as the following sum:

$H_3 = A + B$, *where*

$$A = \frac{1}{nK_1^2(0)K_2^2(0)} \int K_1^2\left(\frac{x-u}{\beta_n}\right) K_2^2\left(\frac{a_{i_0}(x-u)}{\beta_n}\right) \phi_{i_0}(u)du$$

$$B = \sum_{i \neq i_0}^{q} \frac{1}{nK_1^2(0)K_2^2(0)} \int K_1^2\left(\frac{x-u}{\beta_n}\right) K_2^2\left(\frac{a_{i_0} x - b_{i_0} - a_i u - b_i}{\beta_n}\right) \phi_i(u)du$$

Since the function $z \to K'_{\beta_n}(z) = \dfrac{1}{\beta_n} \dfrac{K_1^2\left(\frac{z}{\beta_n}\right) K_2^2\left(\frac{a_{i_0} z}{\beta_n}\right)}{\int K_1^2(z) K_2^2(a_{i_0} z) dz}$ is a kernel, we get

$$\lim_{n\to\infty} \frac{n}{\beta_n} A = \phi_{i_0}(x) \frac{\int K_1^2(z) K_2^2(a_{i_0} z) dz}{K_1^2(0)K_2^2(0)} . \tag{7}$$

Showing now that $\lim_{n\to\infty} \dfrac{n}{\beta_n} B = 0$. Indeed, we split the integral of the expression of B as follows:

$$B = \sum_{i \neq i_0}^{q} \left[\int_{-\infty}^{x-\varepsilon} + \int_{x-\varepsilon}^{x+\varepsilon} + \int_{x+\varepsilon}^{\frac{a_{i_0}x - b_{i_0} - b_i}{a_i} - \eta} + \int_{\frac{a_{i_0}x - b_{i_0} - b_i}{a_i} - \eta}^{\frac{a_{i_0}x - b_{i_0} - b_i}{a_i} + \eta} + \int_{\frac{a_{i_0}x - b_{i_0} - b_i}{a_i} + \eta}^{+\infty} \right]$$

$$= \sum_{i \neq i_0}^{q} [Z_1 + Z_2 + Z_3 + Z_4 + Z_5].$$

$$\frac{n}{\beta_n} Z_1 = \frac{1}{K(0,0)} \int_{-\infty}^{x-\varepsilon} \frac{1}{\beta_n} K_1^2\left(\frac{x-u}{\beta_n}\right) K_2^2\left(\frac{a_{i_0} x - b_{i_0} - a_i u - b_i}{\beta_n}\right) \phi_i(u)du.$$

Since $x-u\neq 0$ and $a_{i_0}x-b_{i_0}-a_iu-b_i\neq 0 \quad \forall u\in]-\infty,x-\varepsilon[$, it is obvious that $\frac{1}{\beta_n}K_1^2\left(\frac{x-u}{\beta_n}\right)K_2^2\left(\frac{a_{i_0}x-b_{i_0}-a_iu-b_i}{\beta_n}\right)$ converges to zero. The kernels K_1 and K_2 are bounded, then we obtain $\frac{n}{\beta_n}Z_1$ converging to zero. Same arguments used to see that $\frac{n}{\beta_n}Z_3$ and $\frac{n}{\beta_n}Z_5$ converge to zero. We can bound $\frac{n}{\beta_n}Z_2$ as follows:

$$\frac{n}{\beta_n}Z_2\leq\frac{1}{K^2(0,0)}\sup_{t\in[x-\varepsilon,x+\varepsilon]}\left(K_2^2\left(\frac{a_{i_0}x-b_{i_0}-a_it-b_i}{\beta_n}\right)\right)$$
$$\times\int_{-\infty}^{+\infty}\frac{1}{\beta_n}K_1^2\left(\frac{x-u}{\beta_n}\right)\phi_i(u)du$$

K_2 being uniformly continuous on $[x-\varepsilon,x+\varepsilon]$, then there exists $t'\in[x-\varepsilon,x+\varepsilon]$ such that $\sup_{t\in[x-\varepsilon,x+\varepsilon]}\left(K_2^2\left(\frac{a_{i_0}x-b_{i_0}-a_it-b_i}{\beta_n}\right)\right)=K_2^2\left(\frac{a_{i_0}x-b_{i_0}-a_it'-b_i}{\beta_n}\right)$. Since the numerator of the last expression is not vanishing, it converges to zero. On the other hand, since $\int_{-\infty}^{+\infty}\frac{1}{\beta_n}K_1^2\left(\frac{x-u}{\beta_n}\right)\phi_i(u)du$ converges to $\phi_i(x)\int K_1^2(t)dt$, we obtain $\frac{n}{\beta_n}Z_2$ converging to zero. Same arguments used to prove that $\frac{n}{\beta_n}Z_4$ converges to zero. Consequently, $\lim_{n\to\infty}\frac{n}{\beta_n}B=0$. Thus from (7), we have

$$\lim_{n\to\infty}\frac{n}{\beta_n}H_3=\frac{\phi_{i_0}(u)\int K_1^2(z)K_2(a_{i_0}z)dz}{K^2(0,0)}.$$

From (5) and (6), we obtain $\frac{n}{\beta_n}H_1\to 0$ and $\frac{n}{\beta_n}H_2\to 0$ Since $\frac{n}{\beta_n}H_4=\frac{\beta_n}{K^2(0,0)}E\left(\frac{1}{\beta_n^2}K^2\left(\frac{x-x_i}{\beta_n},\frac{y-y_i}{\beta_n}\right)\right)\to 0$, we get

$$Var(\hat{a}(x,y))=\frac{\beta_n}{n}\frac{\phi_{i_0}(x)\int K_1^2(z)K_2(a_{i_0}z)dz}{K_1^2(0)K_2^2(0)}+o\left(\frac{\beta_n}{n}\right). \tag{8}$$

b) Second case: If $(x, y) = (w_{1j_0}, w_{2j_0})$

$$H_2 = \frac{a'_{j_0}}{n} + \frac{\beta_n^4}{nK^2(0,0)} \sum_{j \neq j_0}^{q} a'_j \left(\frac{1}{\beta_n} K_1 \left(\frac{w_{1j_0} - w_{1j}}{\beta_n} \right) \right)^2 \left(\frac{1}{\beta_n} K_2 \left(\frac{w_{2j_0} - w_{2j}}{\beta_n} \right) \right)^2.$$

Therefore $\lim_{n\to\infty} nH_2 = a'_{j_0}$. (9)

Since $(x, y) = (w_{1j_0}, w_{2j_0})$, then $y \neq a_i x + b_i$ for all i.

$$H_3 = \frac{1}{nK^2(0,0)} \sum_{i=1}^{q'} \int K_1^2 \left(\frac{w_{1j_0} - u}{\beta_n} \right) K_2^2 \left(\frac{w_{1j_0} - a_i u - b_i}{\beta_n} \right) \phi_i(u) du$$

Using the same bounds shown to prove $\lim_{n\to\infty} \frac{n}{\beta_n} B = 0$, we get that nH_3 converges to zero. From (8) and (9), we obtain $Var(\hat{a}(x, y)) = \frac{a'_{j_0}}{n} + o\left(\frac{1}{n}\right)$.

c) Third case: If $y \neq a_i x + b_i$ and $(x, y) \neq (w_{1j}, w_{2j})$ $\forall i, j$

Using the same arguments that show $\lim_{n\to\infty} \frac{n}{\beta_n} B = 0$, we get $\frac{n}{\beta_n^2} H_3 \to 0$. From (6), we obtain

$$\lim_{n\to\infty} \frac{n}{\beta_n^2} H_2 = 0 \text{ and } \lim_{n\to\infty} \frac{n}{\beta_n^2} H_4 = \frac{1}{K^2(0,0)} f(x, y) \int K^2(t_1, t_2) dt_1 dt_2.$$

Thus, $Var(\hat{a}(x, y)) = \frac{\beta_n^2}{n} \frac{1}{K^2(0,0)} f(x, y) \int K^2(t_1, t_2) dt_1 dt_2 + o\left(\frac{\beta_n^2}{n}\right)$. From the asymptotic expressions of the variance, follows the result of the theorem.

5.2 *Proof of the theorem 3.2*

Consider the following estimate

$$\hat{\phi}_i(\lambda_1, \lambda_2) = \frac{1}{K_2(0)} \sum_{i=1}^{n} \frac{1}{nh'_n} K\left(\frac{\lambda_1 - x_i}{h'_n}, \frac{\lambda_2 - y_i}{h''_n} \right)$$

a) First case: If $y = a_{i_1} x + b_{i_1}$, then $(x, y) \neq (w_{1j}, w_{2j})$ for all *j*.

$$Var(\hat{\phi}(x,y)) = H_1'' + H_2'' + H_3'' + H_4''$$

$$H_1'' = \frac{1}{nK_2^2(0)h_n'^2}\int K^2\left(\frac{x-z_1}{h_n'}, \frac{y-z_2}{h_n''}\right) f(z_1,z_2)dz_1dz_2$$

$$\frac{nh_n'}{h_n''}H_1'' = \left(\frac{1}{K_2^2(0)}\int K_1^2(z_1)K_2^2(z_2)dz_1dz_2\right) K_{h_n',h_n''} * f(x,y)$$

where $K_{h_n',h_n''}(x,y) = \dfrac{\frac{1}{h_n'}K_1^2\left(\frac{x}{h_n'}\right)\frac{1}{h_n''}K_2^2\left(\frac{y}{h_n''}\right)}{\int K_1^2(z_1)K_2^2(z_2)dz_1dz_2}$. Therefore

$$\lim_{n\to\infty}\frac{nh_n'}{h_n''}H_1'' = \frac{1}{K_2^2(0)} f(x,y)\int K_1^2(z_1)K_2^2(z_2)dz_1dz_2 \qquad (10)$$

The second term of the right handside of expression of the variance is

$$H_2'' = \frac{1}{nK_2^2(0)h_n'^2}\sum_{j=1}^{q} a_j K^2\left(\frac{x-w_{1j}}{h_n'}, \frac{y-w_{2j}}{h_n''}\right)$$

Since $x \neq w_{1j}$ or $y \neq w_{2j}$, it is easy to obtain $\lim\limits_{n\to\infty} nh_n'^2 H_2'' = 0$ (11)

The third term of the right handside of expression of the variance is

$$H_3'' = \frac{1}{nK_2^2(0)h_n'^2}\int K_1^2\left(\frac{x-u}{h_n'}\right)K_2^2\left(\frac{a_{i_0}(x-u)}{h_n''}\right)\phi_i(u)du$$

$$+\sum_{i\neq i_0}\frac{1}{nK_2^2(0)h_n'^2}\int K_1^2\left(\frac{x-u}{h_n'}\right)K_2^2\left(\frac{a_{i_0}x + b_{i_0} - a_i u - b_i}{h_n''}\right)\phi_i(u)du$$

$$= S + \mathrm{R}$$

Putting $\dfrac{x-u}{h_n'} = v$ in the integral of the expression of S .

$$S = \frac{1}{nK_2^2(0)}\frac{h_n'}{h_n'^2}\int K_1^2(v)K_2^2\left(a_{i_0} v\frac{h_n'}{h_n''}\right)\phi_i(x - vh')dv \qquad (12)$$

$$nh'_n S = \frac{1}{K_2^2(0)} \int K_1^2(v) K_2^2\left(a_{i_0} v \frac{h'_n}{h''_n} \right) \phi_i(x - vh') dv. \quad (13)$$

The functions K_2 and ϕ_i are continuous and bounded, we have $\lim_{n\to\infty} nh'_n S = \phi_i(x) \int K_1^2(v) dv$. From (10), (11) and (13), we obtain $Var(\hat{\phi}(x, y)) = \frac{\phi_i(x)}{nh'_n} \int K_1^2(t) dt + o\left(\frac{1}{nh'_n} \right)$.

b) Second case: if $(x, y) = (w_{1j_0}, w_{2j_0})$

$$H''_2 = \frac{a_{j_0}}{nh'^2_n} K_1^2(0) + \frac{h''^2_n}{nK_2^2(0)} \sum_{j \neq j_0} \frac{a_j}{h'^2_n} K_1^2\left(\frac{x - w_{1j}}{h'_n} \right) \frac{1}{h''^2_n} K_1^2\left(\frac{x - w_{2j}}{h''_n} \right).$$

Therefore $nh'^2_n H''_2 \to a_{j_0} K_1^2(0)$. As above, we get that $nh'^2_n H''_3$ converges to zero. From (10), $nh'^2_n H''_1$ converges to zero. $Var(\hat{\phi}(x, y)) = \frac{a_{j_0} K_1^2(0)}{nh'^2_n} + o\left(\frac{1}{nh'^2_n} \right)$.

c) Third case: If $(x, y) \neq (w_{1j_0}, w_{2j_0})$ and $y \neq a_i x + b_i$

$$\frac{nh'_n}{h''_n} H'_1 \to \frac{1}{K_2^2(0)} f(x, y) \int K_1^2(z_1) K_2^2(z_2) dz_1 dz_2$$

$$\frac{nh'_n}{h''_n} H''_2 \le \frac{(h'_n h''_n)^3}{K_2^2(0)} \left(\sup\left(\frac{1}{h'^2_n} K_1^2\left(\frac{1}{h'_n} \right); \frac{1}{h''^2_n} K_2^2\left(\frac{1}{h''_n} \right) \right) \right)^2$$

Thus, $\lim_{n\to\infty} \frac{nh'_n}{h''_n} H''_2 = 0$. We show that $\lim_{n\to\infty} \frac{nh'_n}{h''_n} H''_3 = 0$. Then,

$$Var(\hat{\phi}_i(x, y)) = \frac{h''_n}{nh'_n K_2^2(0)} f(x, y) \int K^2(z_1, z_2) dz_1 dz_2 + o\left(\frac{h''_n}{nh'_n} \right).$$

5.3 *Proof of the theorem 4.1*

Denote by $Z_n \stackrel{\Delta}{=} \frac{\hat{a}_n(x, y) - E(\hat{a}_n(x, y))}{(U_n(x, y))^{1/2}} = \sum_{i=1}^n Z_{ni}$ where

$$Z_{ni} = \left\{ \frac{1}{nK(0,0)} K\left(\frac{x - x_i}{\beta_n}, \frac{y - y_i}{\beta_n} \right) - \frac{1}{n} E[\hat{a}_n(x,y)] \right\} \frac{1}{\sqrt{U_n(x,y)}}$$

Showing that for some $\delta > 0$, $(Var(Z_n))^{-2-\delta} \sum_{i=1}^{n} E\left| Z_{n_i} \right|^{2+\delta}$ converges to zero. Indeed, due to the fact that the sample is iid, we obtain

$$E(Z_{n_i}) = \left(\frac{1}{nK(0,0)} EK\left(\frac{x - x_i}{\beta_n}, \frac{y - y_i}{\beta_n} \right) - \frac{1}{n} E\hat{a}_n(x,y) \right) (U_n(x,y))^{-1/2} = 0.$$

Thus, we have

$$Var(Z_{ni}) = \frac{1}{n^2} U_n^{-1}(x,y) \left(\frac{1}{K^2(0,0)} EK^2\left(\frac{x - x_i}{\beta_n}, \frac{y - y_i}{\beta_n} \right) - E^2 \hat{a}_n(x,y) \right)$$

$$Var(Z_{ni}) = \frac{U_n^{-1}(x,y)}{n} Var(\hat{a}_n(x,y)) = \frac{1}{n} + o\left(\frac{1}{n} \right).$$

On the other hand, since the sample is iid, we have $\sum_{i=1}^{n} E\left| Z_{ni} \right|^{2+\delta} = nE\left| Z_{n1} \right|^{2+\delta}$, then

$$(E\left| Z_{n1} \right|^{2+\delta})^{\frac{1}{2+\delta}} = U_n^{-1} \left(E\left| \frac{1}{K(0,0)} K\left(\frac{x - x_i}{\beta_n}, \frac{y - y_i}{\beta_n} \right) - E\hat{a}_n(x,y) \right|^{2+\delta} \right)^{\frac{1}{2+\delta}}$$

$nE(Z_{n1})^{2+\delta} \leq S_1 + S_2 + S_3$, *where*

$$S_1 = \frac{U_n^{-\frac{2+\delta}{2}}}{n^{\delta+1}} \int \left| \frac{1}{K(0,0)} K\left(\frac{x - t_1}{\beta_n}, \frac{y - t_2}{\beta_n} \right) - \frac{1}{n} E\hat{a}_n(x,y) \right|^{2+\delta} f(t_1, t_2) dt_1 dt_2$$

$$S_2 = \frac{U_n^{-\frac{2+\delta}{2}}}{n^{\delta+1}} \sum_{i=1}^{q} \frac{a'_j}{\left| K(0,0) \right|^{2+\delta}} \left| K\left(\frac{x - t_1}{\beta_n}, \frac{y - t_2}{\beta_n} \right) - \frac{1}{n} E\hat{a}_n(x,y) \right|^{2+\delta}$$

$$S_3 = \frac{U_n^{-\frac{2+\delta}{2}}}{n^{\delta+1}} \sum_{i=1}^{q'} \int \frac{\phi_i(u)}{\left| K(0,0) \right|^{2+\delta}} \left| K\left(\frac{x - u}{\beta_n}, \frac{y - a_i u - b_i}{\beta_n} \right) - \frac{1}{n} E\left| \hat{a}_n(x,y) \right| \right|^{2+\delta} du.$$

Putting $\dfrac{x-t_1}{\beta_n}=z_1$ and $\dfrac{y-t_2}{\beta_n}=z_2$ in the integral of S_1, we obtain

$$S_1=\frac{nU_n^{\frac{-2-\delta}{2}}}{\left|nK(0,0)\right|^{2+\delta}}\int\left|K(z_1,z_2)-\frac{1}{n}E\hat{a}_n(x,y)\right|^{2+\delta}$$
$$\times f(x-\beta_n z_1,y-\beta_n z_2)dz_1dz_2.\ \ Thus,$$

$$\frac{U_n^{\frac{2+\delta}{2}}n^{2+\delta}}{n\beta_n^2}S_1\to\frac{f(x,y)}{\left|K(0,0)\right|^{2+\delta}}\int\left|K(z_1,z_2)\right|^{2+\delta}dz_1dz_2.$$

a) First case: If $y=a_{i_0}x+b_{i_0}$ From the definition of U_n, we have

$$\lim_{n\to\infty}\frac{U_n^{\frac{2+\delta}{2}}n^{2+\delta}}{n\beta_n^2}=\lim_{n\to\infty}\frac{n^{\frac{\delta}{2}}}{\beta_n^{1-\delta/2}}\left(\frac{\phi_{i_0}(x)}{K^2(0,0)}\right)^{\frac{2+\delta}{2}}=+\infty$$

b) Second case: If $(x,y)\neq(w_{1j},w_{2j})$ and $y\neq a_ix+y_i$

From the expression of U_n, we obtain

$$\frac{U_n^{1+\frac{\delta}{2}}n^{2+\delta}}{n\beta_n^2}=(n\beta_n)^{\frac{\delta}{2}}\frac{1}{K(0,0)}f(x,y)\int K^2(t_1,t_2)dt_1dt_2$$

Since $n\beta_n\to+\infty$, then $\dfrac{U_n^{1+\frac{\delta}{2}}n^{2+\delta}}{n\beta_n^2}\to+\infty$.

c) Third case: If $(x,y)=(w_{1j_0},w_{2j_0})$

$\dfrac{U_n^{1+\frac{\delta}{2}}n^{2+\delta}}{n\beta_n^2}=(a'_{j_0})^{1+\frac{\delta}{2}}\dfrac{n^{\delta/2}}{\beta_n^2}$. Then, $\dfrac{U_n^{1+\frac{\delta}{2}}n^{2+\delta}}{n\beta_n^2}\to+\infty$. Therefore $S_1\to 0$. Using the same arguments, we show that S_2 and S_3 converge to zero. Therefore $nE\left|Z_{n1}\right|^{2+\delta}\to 0$. From Liapounov's theorem, we have $\dfrac{Z_n}{var(Z_n)}\to N(0,1)$.

5.4 *Proof of the corollary 4.1*

Beginning by the following decomposition:

$$\hat{a}_n(x,y)-a'(x,y)=(\hat{a}_n(x,y)-E(\hat{a}_n(x,y)))+(E(\hat{a}_n(x,y))-a'(x,y)).$$

The second term of right hand side of the last equality can be written as follows:

$$E(\hat{a}_n(x,y))-a'(x,y)=\frac{1}{K(0,0)}EK\left(\frac{x-x_i}{\beta_n},\frac{y-y_i}{\beta_n}\right)-a'(x,y)$$

$$=\frac{1}{K(0,0)}\int K\left(\frac{x-z_1}{\beta_n},\frac{y-z_2}{\beta_n}\right)f(z_1,z_2)dz_1dz_2$$

$$+\frac{1}{K(0,0)}\sum_{j=1}^{q}a'_jK\left(\frac{x-w_{1j}}{\beta_n},\frac{y-w_{2j}}{\beta_n}\right)-a'(x,y)$$

$$+\frac{1}{K(0,0)}\sum_{i=1}^{q'}\int K\left(\frac{x-v_1}{\beta_n},\frac{y-a_iv_1-b_i}{\beta_n}\right)\Phi_i(v_1)dv_1$$

$$\stackrel{\Delta}{=}H_1+H_2+H_2$$

Showing that $H_1=O(\beta_n^2)$, $H_2=O(\beta_n^2)$ and $H_3=O(\beta_n^2)$. Indeed, $\frac{1}{\beta_n^2}H_1=\frac{1}{\beta_n^2K(0,0)}\int K\left(\frac{x-z_1}{\beta_n},\frac{y-z_2}{\beta_n}\right)f(z_1,z_2)dz_1dz_2$. Since K is a kernel, $\lim_{n\to\infty}\frac{1}{\beta_n^2}H_1=f((x,y)$

If $(x,y)\neq(w_{j1},w_{j2})$, we have

$$\frac{1}{\beta_n^2}H_2=O\left(\frac{1}{\beta_n}K_1\left(\frac{1}{\beta_n}\right)\frac{1}{\beta_n}K_2\left(\frac{1}{\beta_n}\right)\right).\tag{14}$$

Since $\frac{1}{\beta_n}K_1\left(\frac{1}{\beta_n}\right)$ converges to zero, $H_2=O(\beta_n^2)$.

If $(x,y)=(w_{1j_0},w_{2j_0})$, we have $H_2=\frac{1}{K(0,0)}\sum_{j\neq j_0}^{q}a'_jK\left(\frac{x-w_{1j}}{\beta_n},\frac{y-w_{2j}}{\beta_n}\right)$.

Therefore $H_2=O(\beta_n^2)$.

Let us now show that $H_3=O(\beta_n^2)$. Indeed, as $y\neq a_ix+b_i$, we assume that

$x < \frac{y-b_i}{a_i}$ (same arguments in the case where $x > \frac{y-b_i}{a_i}$) and we split the integral, in the expression of H_3 as follows:

$$\frac{1}{\beta_n^2}H_3 = \frac{1}{\beta_n^2}\sum_{i=1}^{q'}\int_{-\infty}^{x-\varepsilon}K_1\left(\frac{x-v_1}{\beta_n}\right)K_2\left(\frac{y-a_iv_1-b_i}{\beta_n}\right)\phi_i(v_1)dv_1$$

$$+\frac{1}{\beta_n^2}\sum_{i=1}^{q'}\int_{x-\varepsilon}^{x+\varepsilon}K_1\left(\frac{x-v_1}{\beta_n}\right)K_2\left(\frac{y-a_iv_1-b_i}{\beta_n}\right)\phi_i(v_1)dv_1$$

$$+\frac{1}{\beta_n^2}\sum_{i=1}^{q'}\int_{x+\varepsilon}^{\frac{y-b_i}{a_i}-\varepsilon}K_1\left(\frac{x-v_1}{\beta_n}\right)K_2\left(\frac{y-a_iv_1-b_i}{\beta_n}\right)\phi_i(v_1)dv_1$$

$$+\frac{1}{\beta_n^2}\sum_{i=1}^{q'}\int_{\frac{y-b_i}{a_i}-\varepsilon}^{\frac{y-b_i}{a_i}+\varepsilon}K_1\left(\frac{x-v_1}{\beta_n}\right)K_2\left(\frac{y-a_iv_1-b_i}{\beta_n}\right)\phi_i(v_1)dv_1$$

$$+\frac{1}{\beta_n^2}\sum_{i=1}^{q'}\int_{\frac{y-b_i}{a_i}+\varepsilon}^{+\infty}K_1\left(\frac{x-v_1}{\beta_n}\right)K_2\left(\frac{y-a_iv_1-b_i}{\beta_n}\right)\phi_i(v_1)dv_1,$$

where ε is a nonnegative real sufficiently small for having $x+\varepsilon < \frac{y-b_i}{a_i}-\varepsilon$. We denote the five terms of the last equality: $I_{1,}I_{2,}I_{3,}I_4$ and I_5. Since the functions K_1 and K_2 are decreasing and even, we can write $I_1 \le \frac{1}{\beta_n^2}\sup_{v_1\in]-\infty,x-\varepsilon[}K_1\left(\frac{x-v_1}{\beta_n}\right)\sup_{v_1\in]-\infty,x-\varepsilon[}K_2\left(\frac{y-a_iv_1-b_i}{\beta_n}\right)\int_{-\infty}^{+\infty}\phi_i(v_1)dv_1$. The two "sup" reach values respectively different from x and from $\frac{y-b_i}{a_i}$, hence

$$I_1 = O\left(\frac{1}{\beta_n^2}K_1\left(\frac{1}{\beta_n}\right)K_2\left(\frac{1}{\beta_n}\right)\right). \tag{15}$$

as above, it is shown that

$$I_3 = O\left(\frac{1}{\beta_n^2}K_1\left(\frac{1}{\beta_n}\right)K_2\left(\frac{1}{\beta_n}\right)\right) \quad \text{and} \quad I_5 = O\left(\frac{1}{\beta_n^2}K_1\left(\frac{1}{\beta_n}\right)K_2\left(\frac{1}{\beta_n}\right)\right).$$

On the other hand for all v belonging to $[x-\varepsilon, x+\varepsilon]$ we have $y \neq a_i v - b_i$. Therefore we have

$$I_2 \leq \frac{1}{\beta_n^2} \sup_{v\in[x-\varepsilon,x+\varepsilon]} K_2\left(\frac{y-a_i v-b_i}{\beta_n}\right) \times \int_{-\infty}^{+\infty} K_1\left(\frac{x-v_1}{\beta_n}\right)\varphi_i(v_1)dv_1.$$

Since $x \to \frac{1}{\beta_n} K_1\left(\frac{x}{\beta_n}\right)$ is a kernel, we conclude that $I_2 = O\left(\frac{1}{\beta_n} K_2\left(\frac{1}{\beta_n}\right)\right)$.

In the same manner we increase the expression of I_4. Thus we obtain $H_3 = O(\beta_n^2)$.

Then $E(\hat{a}_n(x,y)) - a'(x,y) = O(\beta_n^2)$. From theorem 4.1, we deduce the result of this corollary.

6. NUMERICAL APPLICATION

In this section, we study a concrete example which validates theoretical results. It is to study the structural fissure of the agricultural soil. We observe two dependent variables X and Y. The variable X presents the resistance of soil measured by using "penetration" method at several locations at same depth of 30cm. The variable Y is the humidity of soil measured in the laboratory on samples taken at same locations. Observing (X,Y) at 1000 locations, we have 1000 observations: $((x_1,y_1),(x_2,y_2),...(x_{1000},y_{1000})$, of the pair (X,Y). Knowledge of the conjoint density $f(x,y)$ of the pair (X,Y) permits, for example, to calculate the probability that the resistance and the humidity be between respectively two critical values (r_1, r_2) and (h_1, h_2). These critical values determine whether to drain the ground or leave without drainage. So it is interesting to estimate the conjoint density f. Then, we calculate the kernel density:

$$f_n(x,y) = \sum_{i=1}^{n} \frac{1}{nh_n^2} K\left(\frac{x-x_i}{h_n}, \frac{y-y_i}{h_n}\right)$$

where the kernels are chosen $K(x,y) = K_1(x)K_2(y)$ with $K_1(x) = K_2(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$, $n = 1000$ and $h_n = n^{-3/5}$. The kernel estimate function is presented by figure 1 in the annexe.

From this figure, we note that there are a jump point localized in the block $[1,2]\times[3.5,4.5]$ and a jump line passing through points $(0,7)$ and $(1,6)$. Then we propose for its joint probability measure the following model:

$$d\mu = f(x,y)dxdy + a_1'\delta_{(\omega_{11},\omega_{21})} + \varphi(u)\delta_{(u,au+b)}.$$

The jump point: (w_{11}, w_{21}) is localized in the block $[1,2]\times[3.5,4.5]$. From the fact that $(0,7)$ and $(1,6)$ are belonging to the jump line, it is easy to see that the equation of the jump line is: $y = -x + 7$. In order to calculate the estimators $\hat{f}(x,y)$, $\hat{a}_n'(x,y)$ and $\hat{\phi}(x,y)$, defined in the section 2, we must choose the spectral windows.This amounts to choose $W^{(1)}$, $W^{(2)}$, $W^{(3)}$ and $W^{(4)}$ satisfying:

$$W^{(1)}(M_n^{(1)}x) = W^{(2)}(M_n^{(2)}x) \qquad \forall x \in \left]-\frac{1}{M_n^{(1)}}, \frac{1}{M_n^{(1)}}\right[$$

$$W^{(3)}(L_n^{(1)}x) = W^{(4)}(L_n^{(2)}x) \qquad \forall x \in \left]-\frac{1}{L_n^{(1)}}, \frac{1}{L_n^{(1)}}\right[$$

To simplify, we take $W^{(1)} = W^{(3)}$ and $W^{(2)} = W^{(4)}$ with $M_n^{(1)} = L_n^{(1)}$ and $M_n^{(2)} = L_n^{(2)}$. Choosing $M_n^{(1)} = n^p$ and $M_n^{(2)} = n^q$ with $0 < q < p < 3/5$, $\beta_n = n^{-3/5}$. These parameters satisfy the hypothesis given in § 2.1 and § 2.2.

First, choosing $W^{(1)}$ as a nonnegative, even and integrable function. We propose:

$$W^{(1)}(t) = \begin{cases} \frac{64}{63}t + \frac{64}{63} & \text{if } t \in [-1, -1/8[\\ 8/9 & \text{if } t \in [-1/8, 1/8] \\ -\frac{64}{63}t + \frac{64}{63} & \text{if } t \in]1/8, 1] \\ 0 & \text{otherwise} \end{cases}$$

It is easy to show that $\int W^{(1)}(t)dt = 1$.

Choosing now a nonnegative, even and integrable function $W^{(2)}$ such that $W^{(1)}$ and $W^{(2)}$ satisfying (3). We propose:

$$W^{(2)}(t) = \begin{cases} W^{(1)}\left(\frac{M_n^{(1)}}{M_n^{(2)}}t\right) & \text{if } t \in [-1/8, 1/8] \\ 4/7\left(1 - \frac{M_n^{(2)}}{M^{(1)}}\right) & \text{if } t \in [-1, -1/8[\cup]1/8, 1] \\ 0 & \text{otherwise} \end{cases}$$

$\int W^{(2)}(t)dt = \int_{-1}^{-1/8} W^{(2)}(t)dt + \int_{-1/8}^{1/8} W^{(2)}(t)dt + \int_{1/8}^{1} W^{(2)}(t).dt$

$\int W^{(2)}(t)dt = \left(1 - \frac{M_n^{(2)}}{M^{(1)}}\right) + \int_{-1/8}^{1/8} W^{(2)}(t)dt$. From the definition of $W^{(2)}$, we have $\int_{-1/8}^{1/8} W^{(2)}(t)dt = 2\int_0^{1/8} W^{(1)}\left(\frac{M_n^{(1)}}{M_n^{(2)}} t\right)dt$. Putting $u = \frac{M_n^{(1)}}{M_n^{(2)}} t$, we have

$\int_{-1/8}^{1/8} W^{(2)}(t)dt = 2\left(\frac{M_n^{(2)}}{M_n^{(1)}}\right)\int_0^{1/8\left(\frac{M_n^{(1)}}{M_n^{(2)}}\right)} W^{(1)}(u)du.$

Since $\frac{M_n^{(1)}}{M_n^{(2)}} \to \infty$, for n large enough we have $\frac{M_n^{(1)}}{M_n^{(2)}} > 1$. Thus, we obtain

$\int_{-1/8}^{1/8} W^{(2)}(t)dt = 2\left(\frac{M_n^{(2)}}{M_n^{(1)}}\right)\int_0^1 W^{(1)}(u)du.$ $W^{(1)}$ being even and $\int W^{(1)}(t)dt = 1$, we deduce $\int_{-1/8}^{1/8} W^{(2)}(t)dt = \left(\frac{M_n^{(2)}}{M_n^{(1)}}\right)$. Thus, $\int W^{(2)}(t)dt = 1$.

Let us show that (3) is satisfied. Indeed, let t a real number belonging to $\left]-\frac{1}{M_n^{(1)}}, \frac{1}{M_n^{(1)}}\right[$, since $\frac{M_n^{(2)}}{M_n^{(1)}}$ converges to zero, for n large enough, we have $-1/8 < -\frac{M_n^{(2)}}{M_n^{(1)}} \le M_n^{(2)} t \le \frac{M_n^{(2)}}{M_n^{(1)}} < 1/8$. Therefore, from (16), we have: $W^{(2)}(M_n^{(2)} t) = W^{(1)}\left(\frac{M_n^{(1)}}{M_n^{(2)}} M_n^{(2)} t\right) = W^{(1)}(M_n^{(1)} t).$

The graphic (fig 2) of the estimate $\hat{f}(x, y)$ defined in the section 2 is given in the annexe.

6.1 *Statistical Tests*

After several attempts testing points in the block $[1,2]\times[3.5,4.5]$, we found a jump at the point: $(x, y) = (1.5, 4)$. Indeed, we calculate L defined in § 4.1, we obtain:

$$L = \left(\frac{n}{\beta_n^2}\right)^{1/2} \frac{\hat{a}'(x, y)}{(K^{-2}(0,0)\hat{f}(x, y)\int K^2(t_1, t_2)dt_1 dt_2)^{1/2}} = 3.954$$

From the table of the standard gaussian with à level of signification $\alpha = 0.05$, we read the value: $Z_{\alpha/2} = 1.96$. Since $L \notin [-Z_{\alpha/2}, Z_{\alpha/2}]$, we conclude that (1.5,4) is a jump point.

For any other point the test concludes that it is not significantly a jump point. To illustrate this, taking, for example, $(x, y) = (2,4)$. Calculation of L at $(x, y) = (2,4)$ gives $L = 0.012$. Since $L \in [-Z_{\alpha/2}, Z_{\alpha/2}]$, we conclude that (2,4) is not a jump point.

7. CONCLUSIONS

We have presented in this paper some results about limits theorems of density estimate when the measure has certain mixture. A statistical test for detecting the jump point is given and applied to study the humidity and resistance of agricultural soil. this work could be applied to other cases when the distribution contains points of discontinuity that risks being badly treated by sharing interval distribution or by using Monte Carlo method. The proposed methods can be extended to other applications in several sectors. Indeed, the control of the quality for a product manufactured in the auto industry use the measure of two variables: the consumption of diesel and the pollution. Their joint distribution can follow a continuous law except some observations which are taken when there is fog and reached the constant value (point of the jump). One example in economics, it is the observation of the variables: taxes on income and purchasing power can have a joint distribution contains some point of jumps due to exemption (disabled, former soldier, ...). In oceanography when we observe, by using a camera placed at a certain depth in water, two variables: the length of the fishes and their movement speed. The joint distribution may represent some jumps due to the acceleration of movement during the passage of a predator. In Astronomy the repeated passage of an object preventing the vision of stars (cloud, bird, ...) can create a jump of data. This work could be supplemented by the study of optimal smoothing parameters using cross validation techniques that have proven in this field.

Annexe

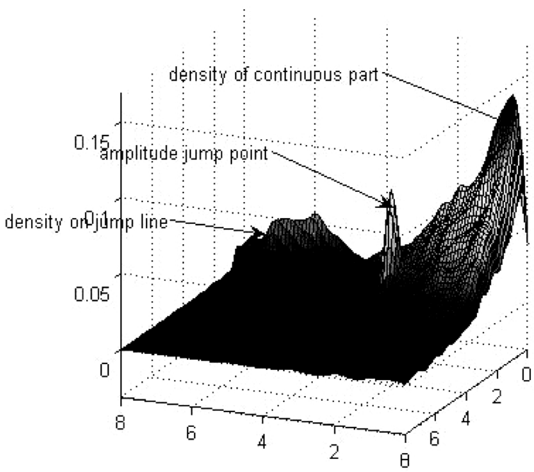

Figure 1 – The kernel density of bivariate random variable (X,Y).

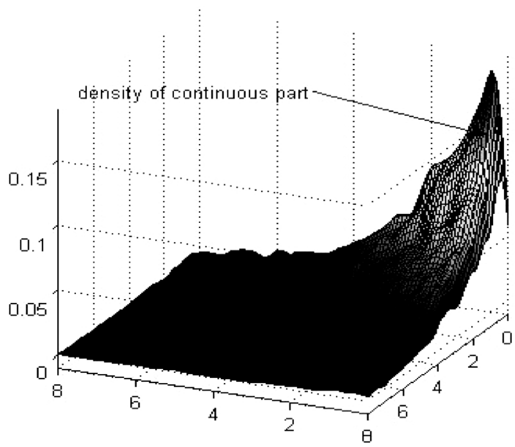

Figure 2 – The density estimate $\hat{f}(x, y)$.

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SUMMARY

Test and asymptotic normality for mixed bivariate measure

Consider a pair of random variables whose joint probability measure is the sum of an absolutely continuous measure, a discrete measure and a finite number of absolutely continuous measures on some lines called jum lines. The central limit theorem of the densities estimates is studied and its rate of convergence is given. A statistical test is developed to locate the jump points. An application on real data was conducted.