PROBABILITY DISTRIBUTION RELATIONSHIPS

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1. INTRODUCTION

In spite of the variety of the probability distributions, many of them are related to each other by different kinds of relationship. Deriving the probability distribution from other probability distributions are useful in different situations, for example, parameter estimations, simulation, and finding the probability of a certain distribution depends on a table of another distribution. The relationships among the probability distributions could be one of the two classifications: the transformations and limiting distributions. In the transformations, there are three most popular techniques for finding a probability distribution from another one. These three techniques are:

1 - The cumulative distribution function technique

2 - The transformation technique

3 - The moment generating function technique.

The main idea of these techniques works as follows:

For given functions $g_i(X_1, X_2, ..., X_n)$, for $i = 1, 2, ..., k$ where the joint distribution of random variables (r.v.'s) $X_1, X_2, ..., X_n$ is given, we define the functions

$$Y_i = g_i(X_1, X_2, ..., X_n), \quad i = 1, 2, ..., k \tag{1}$$

The joint distribution of $Y_1, Y_2, ..., Y_n$ can be determined by one of the suitable method sated above. In particular, for $k = 1$, we seek the distribution of

$$Y = g(X) \tag{2}$$

For some function $g(X)$ and a given r.v. X.

The equation (1) may be linear or non-linear equation. In the case of linearity, it could be taken the form

$$Y = \sum_{i=1}^{n} a_i X_i \tag{3}$$

Many distributions, for this linear transformation, give the same distributions for different values for a_i such as: normal, gamma, chi-square and Cauchy for continuous distributions and Poisson, binomial, negative binomial for discrete distributions as indicated in the Figures by double rectangles. On the other hand, when $a_i = 1$, the equation (3) gives another distribution, for example, the sum of the exponential r.v.'s gives the Erlang distribution and the sum of geometric r.v.'s gives negative- binomial distribution as well as the sum of Bernoulli r.v.'s gives the binomial distribution. Moreover, the difference between two r.v.'s give another distribution, for example, the difference between the exponential r.v.'s gives Laplace distribution and the difference between Poisson r.v.'s gives Skellam distribution, see Figures 1 and 2.

In the case of non-linearity of equation (1), the derived distribution may give the same distribution, for example, the product of log-normal and the Beta distributions give the same distribution with different parameters; see, for example, (Crow and Shimizu, 1988), (Kotlarski, 1962), and (Krysicki, 1999). On the other hand, equation (1) may be give different distribution as indicated in the Figures.

The other classification is the asymptotic or approximating distributions. The asymptotic theory or limiting distribution provides in some cases exact but in most cases approximate distributions. These approximations of one distribution by another one exist. For example, for large n and small p the binomial distribution can be approximated by the Poisson distribution. Other approximations can be given by the central limit theorem. For example, for large n and constant p, the central limit theorem gives a normal approximation of the binomial distribution. In the first case, the binomial distribution is discrete and the approximating Poisson distribution is also discrete. While, in the second case, the binomial distribution is discrete and the approximating normal distribution is continuous. In most cases, the normal or standard normal plays a very predominant role in other distributions.

The most important use of the relationships between the probability distributions is the simulation technique. Many of the methods in computational statistics require the ability to generate random variables from known probability distributions. The most popular method is the inverse transformation technique which deals with the cumulative distribution function, $F(x)$, of the distribution to be simulated. By setting

$$F(x) = U$$

Where $F(x)$ and U are defined over the interval $(0,1)$ and U is a r.v. follows the uniform distribution. Then, x is uniquely determine by the relation

$$x = F^{-1}(U) \tag{4}$$

Unfortunately, the inverse transformation technique can not be used for many distributions because a simple closed form solution of (4) is not possible or it is so complicated as to be impractical. When this is the case, another distribution with a simple closed form can be used and derived from another or other distributions. For example, to generate an Erlang deviate we only need the sum m exponential deviates each with expected value $1/m$. Therefore, the Erlang variate x is expressed as

$$x = \sum_{i=1}^{m} y_i = -\frac{1}{\theta} \sum_{i=1}^{m} \ln U_i$$

Where y_i is an exponential deviate with parameter θ, generated by the inverse transform technique and U_i is a random number from the uniform distribution. Therefore, a complicated situation as in simulation models can be replaced by a comparatively simple closed form distribution or asymptotic model if the basic conditions of the actual situation are compatible with the assumptions of the model.

The relationships among the probability distributions have been represented by (Leemis, 1986), (Taha, 2003) and (Rider, 2004) in limited attempts. The first and second authors have presented a diagram to show the relationships among probability distributions. The diagrams have twenty eight and nineteen distributions including: continuous, discrete and limiting distributions, respectively. The Rider's diagram divided into four categories: discrete, continuous, semi-bounded, and unbounded distributions. The diagram includes only twenty distributions. This paper presents four diagrams. The first one shows the relationships among the continuous distributions. The second diagram presents the discrete distributions as well as the analogue continuous distributions. The third diagram is concerned to the limiting distributions in both cases: continuous and discrete. The Balakrishnan skew-normal density and its relationships with other distributions are shown in the fourth diagram. It should be mentioned that the first diagram and the fourth one are connected. Because the fourth diagram depends on some continuous distributions such as: standard normal, chi-square, the standard Cauchy, and the student's t-distribution.

Throughout the paper, the words "diagram" and "figures" shall be used synonymously.

2. THE MAIN FEATURES OF THE FIGURES

Many distributions have their genesis in a prime distribution, for example, Bernoulli and uniform distributions form the bases to all distributions in discrete and continuous case, respectively. The main features of Figure 1 explain the continuous distribution relationships using the transformation techniques. These transformations may be linear or non-linear. The uniform distribution forms the base to all other distributions.

The Appendix contains the well known distributions which are used in this paper and are obtained from the following web site: http://www.mathworld.wolfram.com. It is written in the Appendix in concise and compact way. We do not present proofs in the present collection of results. For surveys of this materials and additional results we refer to (Johnson *et al.*, 1994 and 1995).

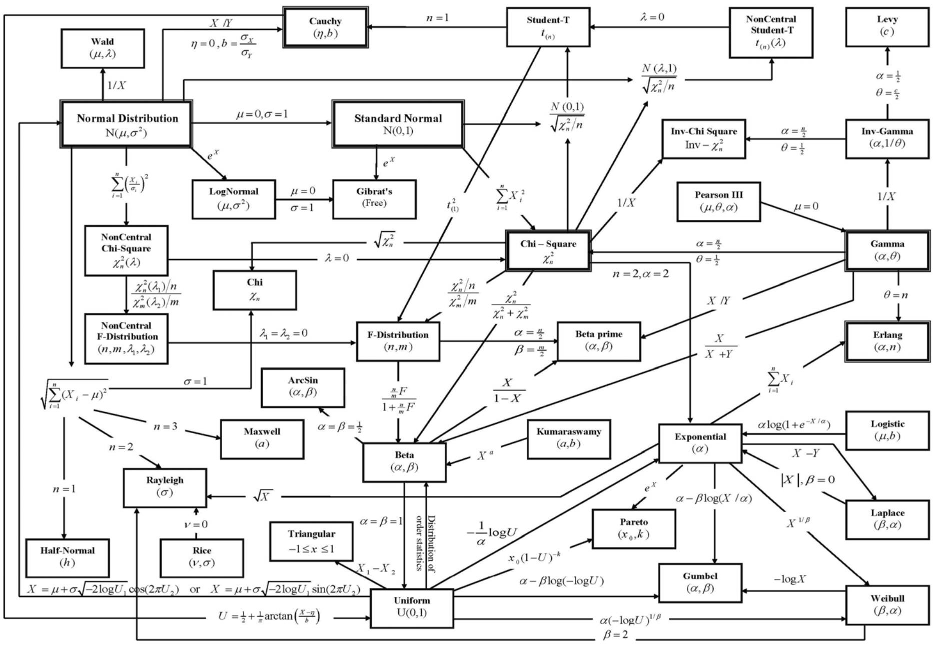

Figure 1 – Continuous distribution relationships.

The main features of Figure 2 can be expressed as follows. If $X_1, X_2, \ldots$ is a sequence of independent Bernoulli r.v.'s, the number of successes in the first n trials has a binomial distribution and the number of failures before the first success has a geometric distribution. The number of failure before the kth success (the sum of k independent geometric r.v.'s) has a Pascal or negative binomial distribution. The sampling without replacement, the number of successes in the first n trials has a hyper-geometric distribution. Moreover, the exponential distribution is limit of geometric distribution, and the Erlang distribution is limit of negative binomial distribution. The other relationships of discrete distribution as well as the analogue continuous distribution can be seen clearly in Figure 2.

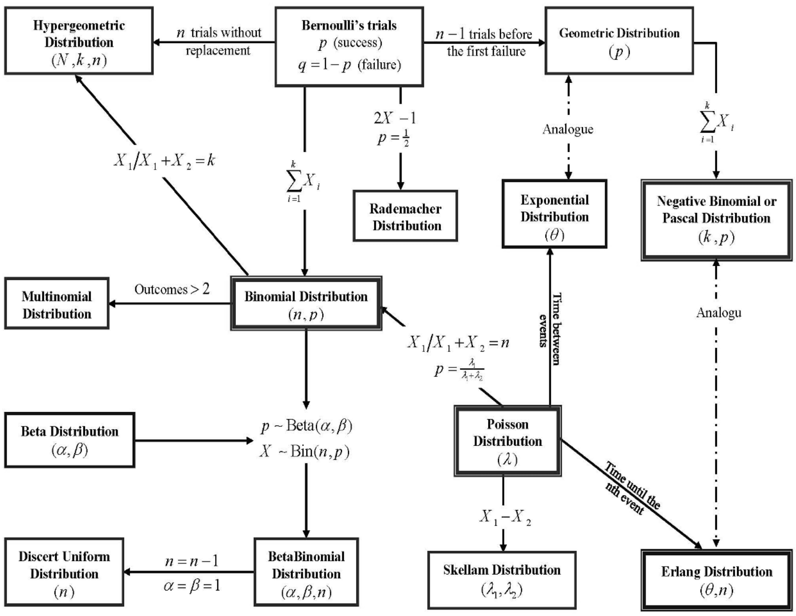

Figure 2 – Bernoulli's trials and its related distributions.

Figure 3 is depicted the asymptotic distributions together with the conditions of limiting. Limiting distributions, in Figure 3 and 4, are indicated with a dashed arrow. The standard normal and the binomial distribution play a very predominant role in other distributions.

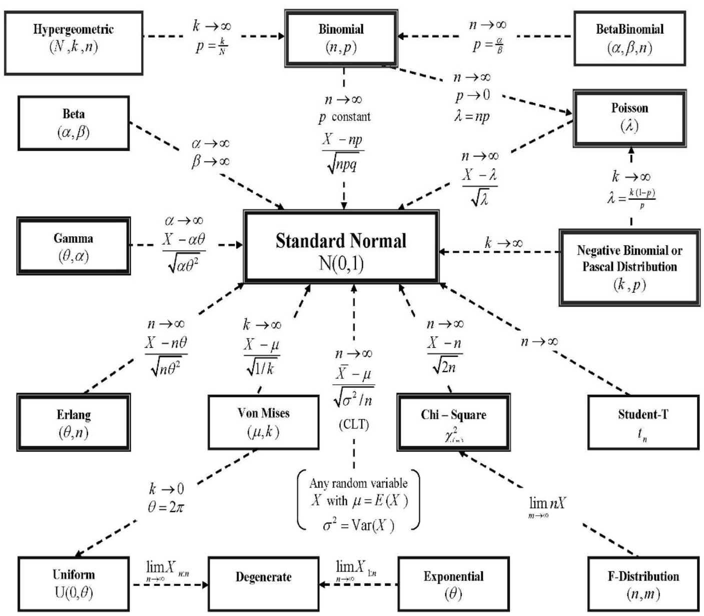

Figure 3 – Limiting distributions.

(Sharafi and Behboodian, 2008) have introduced the Balakrishnan skew-normal density ($SNB_n(\lambda)$) and studied its properties. They defined the $SNB_n(\lambda)$ with integer $n \geq 1$ by

$$f_n(x;\lambda) = c_n(\lambda)\,\phi(x)\,\Phi^n(\lambda x), \quad x, \lambda \in \mathbb{R}, n \in \mathbb{Z}^+ \tag{5}$$

where the coefficient $c_n(\lambda)$ is given by

$$c_n(\lambda) = \frac{1}{\int_{-\infty}^{\infty} \phi(x)\,\Phi^n(\lambda x)\,dx} = \frac{1}{E(\Phi^n(\lambda U))},$$

where $U \sim N(0,1)$.

An special case arises when $n = 1$ and $c_n(\lambda) = 2$ which gives the skew-normal density $(SN(\lambda))$, see (Azzalini, 1985).

The probability distribution of the Balakrishnan skew-normal density and its relationships of the other distributions are also discussed. The most important properties of $SNB_n(\lambda)$ are:

(i) $SNB_n(\lambda)$ is strongly unimodal,

(ii) $c_1(\lambda) = 2$,

$$c_2(\lambda) = \left[\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho\right]^{-1},$$

$$c_3(\lambda) = \left[\frac{1}{8} + \frac{3}{4\pi}\sin^{-1}\rho\right]^{-1},$$

Where ρ is denoted the correlation coefficient. For $n \geq 4$, there is no closed form for $c_n(\lambda)$. But some approximate values can be found in (Steck, 1962). The bivariate case of $SNB_n(\lambda)$ and the location and scale parameters are also presented.

Figure 4 explains the main features of their results. The dotted arrows indicate asymptotic distribution.

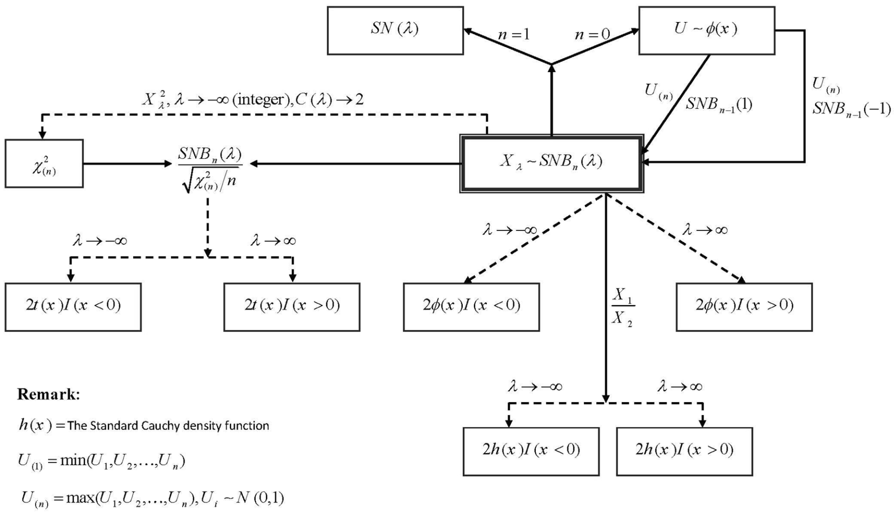

Figure 4 – The balakrishnan skew-normal density $SNB_n(\lambda)$.

3. CONCLUDING REMARKS

It is a reasonable assertion that all probability distributions are someway related to one another. In this paper, four diagrams summarize the most popular relationships among the probability distributions. The relationships among the probability distributions are one of the two classifications: transformation and limiting. Each diagram explains itself. The advantages of using these diagrams are: the student at the senior undergraduate level or beginning graduate level in statistics or engineering can use the diagrams to supplement course material. Besides, the researchers can use the diagrams for fasting search for the relationships among the distributions. These diagrams are just start points. Similar diagrams can be constructed to summarize many statistical theorems such as: the characterizations of distributions based on; order statistics, Records and Moments, see (Gather *et al.*, 1998).

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APPENDIX

TABLE 1

Continues distribution

	Domain	*Parameters*	*PDF*
Beta Distribution	$[0,1]$	$\alpha>0\,,\beta>0$	$f(x)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$
Beta Prime Distribution	$[0,\infty)$	$\alpha>0\,,\beta>0$	$f(x)=\frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$
Cauchy Distribution	$(-\infty,\infty)$	$\eta\in\mathbb{R}\,,b>0$	$f(x)=\frac{1}{\pi}\frac{b}{(x-\eta)^2+b^2}$
Chi Distribution	$[0,\infty)$	$n>0$	$f(x)=\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})}x^{n-1}e^{-\frac{1}{2}x^2}$
Chi-Squared Distribution	$[0,\infty)$	$n>0$	$f(x)=\frac{x^{\frac{n}{2}-1}e^{-\frac{1}{2}x}}{\Gamma(\frac{n}{2})2^{\frac{n}{2}}}$
Degenerate Distribution	$\{x_0\}$	$x_0\in\mathbb{R}$	$f(x)=\delta(x-x_0)=\begin{cases}1 & , \quad x=x_0\\ 0 & , \quad x\neq x_0\end{cases}$
Erlang Distribution	$[0,\infty)$	$n=1,2,3,\cdots,\alpha>0$	$f(x)=\frac{x^{n-1}e^{-\frac{1}{\alpha}x}}{(n-1)!\alpha^n}$
Exponential Distribution	$[0,\infty)$	$\theta>0$	$f(x)=\frac{1}{\alpha}e^{-\frac{x}{\alpha}}$
F- Distribution	$[0,\infty)$	$n_1>0,n_2>0$	$f(x)=\frac{n_1^{\frac{n_1}{2}}n_2^{\frac{n_2}{2}}x^{\frac{n_2}{2}-1}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)(n_1+n_2x)^{\frac{n_1+n_2}{2}}}$
Gamma Distribution	$[0,\infty)$	$\alpha>0\,,\theta>0$	$f(x)=\frac{x^{\theta-1}e^{-\frac{1}{\alpha}x}}{\Gamma(\theta)\alpha^{\theta}}$
Gibrat's Distribution	$(0,\infty)$	None	$f(x)=\frac{1}{x\sqrt{2\pi}}e^{-(\ln x)^2/2}$
Gumbel Distribution	$(-\infty,\infty)$	$\beta>0\,,\alpha$	$f(x)=\frac{1}{\beta}e^{\left[\left(\frac{\alpha-x}{\beta}\right)-e^{\left(\frac{\alpha-x}{\beta}\right)}\right]}$
Half-Normal Distribution	$[0,\infty)$	$h>0$	$f(x)=\frac{2h}{\pi}e^{-h^2x^2/\pi}$
Inverse Chi-Square Distribution	$(0,\infty)$	$\nu>0$	$f(x)=\frac{2^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}x^{-\frac{\nu}{2}-1}e^{-\frac{1}{2x}}$
Inverse-Gamma Distribution	$(0,\infty)$	$\alpha>0\,,\theta>0$	$f(x)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}x^{-\alpha-1}e^{-\frac{\theta}{x}}$
Kumaraswamy Distribution	$[0,1]$	$a>0\,,b>0$	$f(x)=abx^{a-1}(1-x^a)^{b-1}$
Laplace Distribution	$(-\infty,\infty)$	$b>0,\mu$	$f(x)=\frac{1}{2b}e^{-\lvert x-\mu\rvert/b}$
Levy Distribution	$[0,\infty)$	$c>0$	$f(x)=\sqrt{\frac{c}{2\pi}}\frac{e^{-c/2x}}{x^{3/2}}$

Logistic Distribution	$(-\infty,\infty)$	$b>0,\mu$	$f(x)=\dfrac{e^{-\left(\frac{x-\mu}{b}\right)}}{b\left[1+e^{-\left(\frac{x-\mu}{b}\right)}\right]^2}$
Lognormal Distribution	$[0,\infty)$	$\sigma>0,\mu$	$f(x)=\dfrac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\dfrac{[\ln x-\mu]^2}{2\sigma^2}\right)$
Maxwell Distribution	$[0,\infty)$	$a>0$	$f(x)=\sqrt{\dfrac{2}{\pi}}\dfrac{x^2e^{-x^2/2a^2}}{a^3}$
Noncentral Chi-Squared Distribution	$[0,\infty)$	$n>0,\lambda>0$	$f(x)=\dfrac{x^{\frac{n}{2}-1}e^{-\left(\frac{x+\lambda}{2}\right)}}{2^{\frac{n}{2}}}\sum_{k=0}^{\infty}\dfrac{(\lambda x)^k}{2^{2k}k!\Gamma(k+\frac{n}{2})}$
Noncentral F-Distribution	$[0,\infty)$	$n_1,n_2,\lambda_1,\lambda_2>0$	$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\dfrac{n_1^{\frac{k+n_1}{2}}n_2^{\frac{l+n_2}{2}}x^{\frac{k+n_1}{2}-1}\lambda_1^k\lambda_2^l}{2^{k+l}e^{\frac{\lambda_1+\lambda_2}{2}}B\left(k+\frac{n_1}{2},l+\frac{n_2}{2}\right)}\times$ $(n_2+n_1x)^{(k+l)-\left(\frac{n_1+n_2}{2}\right)}$
Noncentral Student's t-Distribution	$(-\infty,\infty)$	$n>0,\lambda>0$	$f(x)=\dfrac{n^{\frac{n}{2}}n!}{2^ne^{\lambda^2/2}(n+x^2)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\times$ $\left\{\dfrac{\sqrt{2}\lambda x\,{}_1F_1\left(\frac{n}{2}+1;\frac{3}{2};\frac{\lambda^2x^2}{2(n+x^2)}\right)}{(n+x^2)\Gamma\left(\frac{n+1}{2}\right)}\dfrac{{}_1F_1\left(\frac{n}{2}+1;\frac{1}{2};\frac{\lambda^2x^2}{2(n+x^2)}\right)}{\sqrt{n+x^2}\Gamma\left(\frac{n}{2}+1\right)}\right\}$
Normal Distribution	$(-\infty,\infty)$	$\sigma^2>0,\mu$	$f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Pareto Distribution	$[x_0,\infty)$	$x_0>0,k>0$	$f(x)=\dfrac{kx_0^k}{x^{k+1}}$
Pearson Type III Distribution	$[0,\infty)$	$\theta,\alpha>0,\mu$	$f(x)=\dfrac{1}{\Gamma(\alpha)\theta}\left(\dfrac{x-\mu}{\theta}\right)^{\alpha-1}e^{-\left(\frac{x-\mu}{\theta}\right)}$
Rayleigh Distribution	$[0,\infty)$	$\sigma>0$	$f(x)=\dfrac{xe^{-\frac{x^2}{2\sigma^2}}}{\sigma^2}$
Rice Distribution	$[0,\infty)$	$\sigma>0,\nu>0$	$f(x)=\dfrac{x}{\sigma^2}e^{-\frac{x^2+\nu^2}{2\sigma^2}}I_o\left(\dfrac{x\nu}{\sigma^2}\right)$
Standard Normal Distribution	$(-\infty,\infty)$	None	$f(x)=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$
Student's t-Distribution	$(-\infty,\infty)$	$n>0$	$f(x)=\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)(1+x^2/n)^{\frac{n+1}{2}}}$
Triangular Distribution	$[a,b]$	$a<c<b$	$f(x)=\begin{cases}\dfrac{2(x-a)}{(b-a)(c-a)} & a\le x\le c\\ \dfrac{2(b-x)}{(b-a)(b-c)} & c\le x\le b\end{cases}$
Uniform Distribution	$[a,b]$	a,b	$f(x)=\dfrac{1}{b-a}$
Wald Distribution	$(0,\infty)$	$\lambda>0,\mu>0$	$f(x)=\left[\dfrac{\lambda}{2\pi x^3}\right]^{1/2}\exp\left[\dfrac{-\lambda(x-\mu)^2}{2\mu^2x}\right]$
Weibull Distribution	$[0,\infty)$	$\alpha,\beta>0$	$f(x)=\dfrac{\beta}{\alpha^\beta}x^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^\beta}$

TABLE 2

Discrete distribution

	Domain	*Parameters*	*P.M.F.*
Bernoulli Distribution	$\{0,1\}$	$0 \le p \le 1,$ $q = 1-p,$	$P(x) = \begin{cases} q & x=0 \\ p & x=1 \end{cases}$
Beta Binomial Distribution	$\{0,1,...,n\}$	$\alpha, \beta > 0,$ $n = 1,2,...$	$P(x) = \binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha,\beta)}$
Binomial Distribution	$\{0,1,...,n\}$	$0 \le p \le 1,$ $q = 1-p,$ $n = 1,2,...$	$P(x) = \binom{n}{x} p^x q^{n-x}$
Discrete Uniform Distribution	$\{0,1,...,n\}$	$n = 1,2,...$	$P(x) = \frac{1}{n}$
Geometric Distribution	$\{0,1,2,...\}$	$0 \le p \le 1,$ $q = 1-p,$	$P(x) = pq^x$
Hypergeometric Distribution	$\{0,1,...,n\}$	$N = 0,1,2,...,$ $K = 0,1,...,N,$ $n = 0,1,...,N,$ $p = \frac{k}{N},\ q = 1-p$	$P(x) = \frac{\binom{Np}{x}\binom{Nq}{n-x}}{\binom{N}{n}}$
Log-Series Distribution	$\{1,2,3,...\}$	$0 < p < 1$	$P(x) = -\frac{\theta^x}{x \ln(1-\theta)}$
Pascal Distribution	$\{0,1,2,...\}$	$0 \le p \le 1,$ $q = 1-p,$ $k = 1,2,...$	$P(x) = \binom{x+k-1}{k-1} p^k q^x$
Poisson Distribution	$\{0,1,2,...\}$	$\lambda > 0$	$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$
Rademacher Distribution	$\{-1,1\}$	None	$P(x) = \begin{cases} \frac{1}{2} & x=-1 \\ \frac{1}{2} & x=1 \end{cases}$
Skellam Distribution	$\{...,-1,0,1,...\}$	$\lambda_1 > 0, \lambda_2 > 0$	$P(x) = e^{-(\lambda_1+\lambda_2)} \left(\frac{\lambda_1}{\lambda_2}\right)^{x/2} I_x\left(2\sqrt{\lambda_1\lambda_2}\right)$

REFERENCES

A. AZZALINI (1985), *A class of distributions with includes the normal ones*, "Scandinavian journal of statistics", 12, pp. 171-178.

E. L. CROW, K. SHIMIZU (1988), *Lognormal distributions: Theory and Applications*, Marcel Dekker, New York.

U. GATHER, U., KAMPUS, and N. SCHWEITZER (1998), *Characterizations of distributions via identically distributed functions of order statistics*, In: Balakrishnan, N., and Rao, C. ed., "Handbook of Statistics", 16, Elsevier Science, pp. 257-290.

N. JOHNSON, S. KOTZ and N. BALAKRISHNAN (1994), *Continuous univariate distributions*, Vol. 1, 2*nd* ed. Wiley, New York.

N. JOHNSON, S. KOTZ and N. BALAKRISHNAN (1995), *Continuous univariate distributions*, Vol. 2, 2*nd* ed., Wiley, New York.

I. KOTLARSKI (1962), *On group of n independent random variables whose product follows the beta distribution*, "Collog. Math. IX Fasc.", 2, pp. 325-332.

W. KRYSICKI (1999), *On some new properties of the beta distribution*, "Statistics & Probability Letters", 42, pp. 131-137.

L. LEEMIS (1986), *Relationships among common univariate distributions*, "The American Statistician", 40, pp. 143-146.

M. SHARAFI, J. BEHBOODIAN (2008), *The Balakrishnan skew-normal density*, "Statistical Papers", 49, pp. 769-778.

J.W. RIDER (2004), *Probability distribution relationships*, http://www.jwrider.com.

H. A. TAHA (2003), *Operations Research: An introduction*, 3*rd* ed, Macmillan Publishing Co.

SUMMARY

Probability distribution relationships

In this paper, we are interesting to show the most famous distributions and their relations to the other distributions in collected diagrams. Four diagrams are sketched as networks. The first one is concerned to the continuous distributions and their relations. The second one presents the discrete distributions. The third diagram is depicted the famous limiting distributions. Finally, the Balakrishnan skew-normal density and its relationship with the other distributions are shown in the fourth diagram.