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SOME PROPERTIES OF KEMP FAMILY OF DISTRIBUTIONS

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1. INTRODUCTION

Kemp (1968) considered a wide class of univariate discrete distributions called the generalized hypergeometric probability distributions (GHPD), which has the following probability generating function (p.g.f.).

$$G(t) = \frac{{}_pF_q(\underline{a};\underline{b};\theta t)}{{}_pF_q(\underline{a};\underline{b};\theta)} \qquad (1)$$

where ${}_pF_q(\underline{a};\underline{b};\theta)$ is the generalized hypergeometric series (cf. Slater, 1966 and Mathai and Saxena, 1973) , in which a' s, b's and θ are assumed to be appropriate reals and the domain of G(.) is an open interval containing the region of convergence of ${}_pF_q(\underline{a};\underline{b};\theta)$. (Dacey, 1972) has listed more than fifty p.g.f.'s involving the generalized hypergeometric series, in which more than twenty are special cases of (1). These special cases include several well-known discrete distributions such as binomial, Poisson, negative binomial, hypergeometric, inverse hypergeometric, negative hypergeometric, Polya, inverse Polya, Waring, Yule etc. (Kemp, 1968) has studied the properties of the GHPD by considering certain differential equations satisfied by various generating functions of the GHPD. For a detailed account of GHPD see (Johnson et al., 2005). (Moothathu and Kumar, 1997) introduced and studied a bivariate version of GHPD and (Kumar, 2002, 2009) introduced extended versions of GHPD.

Here in section 2 we show that all the moments of the GHPD exists finitely and obtain an expression for raw moments of the GHPD. In section 3 we derive certain useful and simple recurrence relations for probabilities, raw moments and factorial moments of the GHPD.

Throughout the paper let us adopt the following simplifying notations, for i = 0, 1, ...

$$H_i = {}_pF_q(\underline{a} + \underline{i}_p; \underline{b} + \underline{i}_q; \theta),$$

$$D_i = \frac{\prod_{j=1}^{p}(a_j + i)_n}{\prod_{k=1}^{q}(b_k + i)_n}$$

and $(a)_0 = 1$, $(a)_n = a(a+1)...(a+n-1)$, $n \geq 1$. Further we need the following series representation in the sequel.

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} B(s,r) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} B(s, r-s) \tag{2}$$

2. MOMENTS OF GHPD

Let X be a random variable having GHPD with the following p.g.f., in which for $r \geq 0$, $P_r(\underline{a};\underline{b}) = P(X = r)$ denote its probability mass function.

$$\begin{aligned} G(t) &= \sum_{r=0}^{\infty} P_r(\underline{a};\underline{b}) t^r \\ &= H_0^{-1}\ {}_pF_q(\underline{a};\underline{b};\theta t) \end{aligned} \tag{3}$$

Now we have the following result.

Result 2.1 For any positive integer r, the r-th raw moment μ_r of GHPD exists finitely.

Proof. In (3), $|\theta| < 1$ when $p = q+1$. Then for $\theta \neq 0$, for any positive integer r and for any t in ($-|\theta|^{-1}$, $|\theta|^{-1}$), $G^{(r)}(t) = \frac{d^r G(t)}{dt^r}$ exists and is continuous. When $p \leq q$ and/or when $\theta = 0$, obviously $G^{(r)}(t)$ exists and is continuous at every real number t. Thus $G^{(r)}(t)$ exists and is continuous for every t in an open interval containing unity. Hence the r-th factorial moment of GHPD is

$$\begin{aligned} \mu_{[r]} &= E(X^{[r]}) \\ &= [G^{(r)}(t)]_{t=1} \\ &= \frac{(a_1)_r \cdots (a_p)_r}{(b_1)_r \cdots (b_q)_r} \frac{\theta^r H_r}{H_0}\ , \end{aligned}$$

which is finite, since ${}_pF_q(\underline{a};\underline{b};\theta)$ in (3) is assumed to be convergent. From (Johnson *et al.*, 2005) we have

$$\mu_r = \sum_{j=0}^{r} S(r, j)\mu_{[j]} \tag{4}$$

where $S(r, j)$ are the Stirling numbers of the second kind. Hence $\mu_r = E(X^r)$ exists finitely.

Here onwards we shall denote $\mu_r = \mu_r(\underline{a};\underline{b})$. Therefore the characteristic function

$$\varphi(t) = G(e^{it}) = H_0^{-1}\ {}_pF_q(\underline{a};\underline{b};\theta e^{it}) \tag{5}$$

$$= \sum_{r=0}^{\infty} \mu_r(\underline{a};\underline{b})\frac{(it)^r}{r!} \tag{6}$$

On expanding ${}_pF_q(.)$ and the exponential function e^{it} in (5) we have the following.

$$\varphi(t) = H_0^{-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{k=1}^{q}(b_k)_n}\frac{\theta^n}{n!}\sum_{r=0}^{\infty}\frac{(nit)^r}{r!} \tag{7}$$

Equating coefficients of $(r!)^{-1}(it)^r$ on right hand side expressions of (6) and (7) we get

$$\mu_r(\underline{a};\underline{b}) = H_0^{-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{k=1}^{q}(b_k)_n}\frac{\theta^n}{n!}n^r$$

$$= H_0^{-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{k=1}^{q}(b_k)_n}\frac{\theta^n}{n!}\sum_{m=0}^{r}S(r,m)(n)_m\ ,$$

where $S(r,m)$ are the Stirling numbers of the second kind. Writing $n! = (n)_m(n-m)!$ and rearranging the terms, we obtain

$$\mu_r(\underline{a};\underline{b}) = H_0^{-1}\sum_{m=0}^{r}S(r,m)\theta^m\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{k=1}^{q}(b_k)_n}\frac{\theta^{n-m}}{(n-m)!}$$

$$= H_0^{-1}\sum_{m=0}^{r}S(r,m)\theta^m\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n+m}}{\prod_{k=1}^{q}(b_k)_{n+m}}\frac{\theta^n}{n!}$$

Since $(A)_{n+m} = (A)_n(A+m)_n$, the above expression can be written as,

$$\mu_r(\underline{a};\underline{b}) = H_0^{-1}\sum_{m=0}^{r} S(r,m)\theta^m \frac{\prod_{j=1}^{p}(a_j)_m}{\prod_{k=1}^{q}(b_k)_m} H_m \tag{8}$$

which is a simple expression for raw moments of GHPD.

3. RECURRENCE RELATIONS

Result 3.1. The following is a recurrence relation for probabilities of GHPD, for $n \geq 0$.

$$P_{n+1}(\underline{a};\underline{b}) = \frac{\theta D_0 H_1}{(n+1)H_0} P_n(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q) \tag{9}$$

Proof. Consider the following identity obtainable from (3) on differentiation with respect to t.

$$\frac{\partial G(t)}{\partial t} = \sum_{r=0}^{\infty}(r+1)P_{r+1}(\underline{a};\underline{b})t^r$$

$$= \frac{\theta\, D_0}{H_0}\, {}_pF_q(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q;\theta t) \tag{10}$$

In (3) on replacing $\underline{a},\underline{b}$ by $\underline{a}+\underline{1}_p, \underline{b}+\underline{1}_q$ respectively, we obtain

$${}_pF_q(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q;\theta t) = H_1 \sum_{r=0}^{\infty} P_r(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)t^r \,. \tag{11}$$

By using (11) in (10) we obtain

$$\sum_{r=0}^{\infty}(r+1)P_r(\underline{a};\underline{b})t^r = \frac{\theta D_0 H_1}{H_0}\sum_{r=0}^{\infty} P_r(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)t^r \,.$$

Equating coefficients of t^n on both sides, we get the relation (9).

Result 3.2. The following is a recurrence relation for raw moments of GHPD, for $n \geq 0$, in which $\mu_0(\underline{a};\underline{b}) = 1$.

$$\mu_{n+1}(\underline{a};\underline{b}) = \frac{\theta D_0 H_1}{H_0}\sum_{s=0}^{n}\frac{n!}{s!(n-s)!}\mu_{n-s}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q) \tag{12}$$

Proof. Consider the following identity obtainable from (5) and (6) on differentiation with respect to t.

$$\frac{\partial \varphi(t)}{\partial t} = \frac{i\,\mathrm{e}^{\mathrm{it}}\,\theta\, D_0}{H_0}\; {}_pF_q(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q;\theta e^{it})$$

$$= \sum_{r=1}^{\infty} r\, i\, \mu_r(\underline{a};\underline{b})\,\frac{(it)^r}{r!}$$

By using (5) and (6) with $\underline{a},\underline{b}$ replaced by $\underline{a}+\underline{1}_p,\underline{b}+\underline{1}_q$, one has

$$\sum_{r=0}^{\infty} \mu_{r+1}(\underline{a};\underline{b})\frac{(it)^r}{r!} = \frac{\theta D_0 H_1}{H_0} e^{it} \sum_{r=0}^{\infty} \mu_r(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\frac{(it)^r}{r!}$$

$$= \frac{\theta D_0 H_1}{H_0} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\; \mu_r(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\frac{(it)^{r+s}}{r!s!}$$

$$= \frac{\theta D_0 H_1}{H_0} \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{r!}{s!(r-s)!}\mu_{r-s}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\frac{(it)^r}{r!},$$

in the light of (2). On equating coefficients of $(n!)^{-1}(it)^n$ on both sides, we obtain (12).

Result 3.3. Let $\mu_{[n]}(\underline{a};\underline{b})$ denote the n-th factorial moment of GHPD with p.g.f. (3). Then the following is a recurrence relation for factorial moments of GHPD for $n \geq 0$, in which $\mu_{[0]}(\underline{a};\underline{b}) = 1$.

$$\mu_{[n+1]}(\underline{a};\underline{b}) = \frac{\theta D_0 H_1}{H_0}\,\mu_{[n]}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q) \tag{13}$$

Proof. The factorial moment generating function $F(t)$ of GHPD with p.g.f. $G(t)$ given in (3) is the following.

$$F(t) = G(1+t) = H_0^{-1}\; {}_pF_q[\underline{a};\underline{b};\theta(1+t)]$$

$$= \sum_{r=0}^{\infty}\; \mu_{[r]}(\underline{a};\underline{b})\frac{t^r}{r!} \tag{14}$$

The relation (13) follows on differentiating the above equation with respect to t and equating coefficients of $(n!)^{-1}t^n$ on both sides, in the light of the arguments similar to those in the proof of Result 3.1.

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SUMMARY

Some properties of Kemp family of distributions

This paper studies some important properties of the generalized hypergeometric probability distribution (GHPD) of Kemp (Sankhya-Series A, 1968) by establishing the existence of all the moments of the distribution and by deriving a formula for raw moments. Here we also obtain certain recurrence relations for probabilities, raw moments and factorial moments of the GHPD.