RATIO-CUM-PRODUCT TYPE EXPONENTIAL ESTIMATOR

H.P. Singh, L.N. Upadhyay, R. Tailor

1. INTRODUCTION

Consider a finite population $U=(U_1,U_2,...,U_N)$ of N units. Suppose two auxiliary variables X_1 and X_2 are observed on U_i (i=1,2,3,...,N), where X_1 is positively and X_2 is negatively correlated with the study variable Y. A simple random sample of size n is drawn without replacement from population U to estimate the population mean $\bar{Y}$ of the study variable Y assuming the knowledge of the population means $\bar{X}_1$ and $\bar{X}_2$ of the auxiliary variables X_1 and X_2 respectively. Singh (1967) proposed a ratio-cum-product estimator for population mean $\bar{Y}$ as

$$t_{RP}=\bar{y}(\bar{X}_1/\bar{x}_1)(\bar{x}_2/\bar{X}_2), \tag{1.1}$$

where

$$\bar{y}=\sum_{i=1}^{n} y_i/n,\ \bar{x}_1=\sum_{i=1}^{n} x_{1i}/n \text{ and } \bar{x}_2=\sum_{i=1}^{n} x_{2i}/n .$$

Bahl and Tuteja (1991) envisaged ratio and product type exponential estimators for $\bar{Y}$ respectively as

$$t_{\mathrm{Re}}=\bar{y}\ \exp\left[\frac{\bar{X}_1-\bar{x}_1}{\bar{X}_1+\bar{x}_1}\right], \tag{1.2}$$

$$t_{Pe}=\bar{y}\ \exp\left[\frac{\bar{x}_2-\bar{X}_2}{\bar{x}_2+\bar{X}_2}\right]. \tag{1.3}$$

It is well known under simple random sampling without replacement (SRSWOR) that the variance/mean squared error (MSE) of usual unbiased estimator $\bar{y}$ is

$$V(\bar{y})=\theta S_0^{\ 2}=MSE(\bar{y}), \tag{1.4}$$

where

$$\theta=(1-f)/n\ ,\ f=n/N\ \text{ and } S_0^2=\sum_{j=1}^{N}(y_j-\overline{Y})^2/(N-1)\,.$$

To the first degree of approximation, mean squared errors of t_{RP}, t_{Re} and t_{Pe} are respectively given by

$$MSE(t_{RP})=\theta\overline{Y}^2[C_0^2+C_1^2(1-2K_{01})+C_2^2\{1+2(K_{02}-K_{12})\}] \tag{1.5}$$

$$MSE(t_{\mathrm{Re}})=\theta\overline{Y}^2[C_0^2+\{C_1^2(1-4K_{01})/4\}] \tag{1.6}$$

$$MSE(t_{Pe})=\theta\overline{Y}^2[C_0^2+\{C_2^2(1-4K_{02})/4\}]\,, \tag{1.7}$$

where $C_o=\dfrac{S_o}{\overline{Y}}$, $C_i=\dfrac{S_i}{\overline{X}_i}$, $K_{oi}=\rho_{oi}\dfrac{C_o}{C_i}$ (i=1,2), $K_{12}=\rho_{12}\dfrac{C_1}{C_2}$,

$S_i^2=\sum_{j=1}^{N}(x_{ij}-\overline{X}_i)^2/(N-1)$, $S_{oi}^2=\sum_{j=1}^{N}(y_i-\overline{Y})(x_{ij}-\overline{X}_i)/(N-1)$, $(i=1,2)$;

$\rho_{0i}=(S_{0i}/S_0S_i)$: is the correlation coefficient between Y and X_i $(i=1,2)$,

$\rho_{12}=S_{12}/(S_1S_2)$: is the correlation coefficient between X_1 and X_2,

and $S_{12}=\sum_{j=1}^{N}(x_{1j}-\overline{X}_1)(x_{2j}-\overline{X}_2)/(N-1)$.

In this paper we have suggested a ratio-cum-product type exponential estimator for the population mean $\overline{Y}$ of the study variate Y using auxiliary information on X_1 and X_2. The mean squared error of the suggested estimator has been derived under large sample approximation. Numerical illustration is given in support of the present study.

2. PROPOSED RATIO-CUM-PRODUCT TYPE EXPONENTIAL ESTIMATOR

Motivated by Singh (1967) we propose a ratio-cum-product type exponential estimator for population mean $\overline{Y}$ as

$$t_{RPe}=\overline{y}\exp\left[\frac{\overline{X}_1-\overline{x}_1}{\overline{X}_1+\overline{x}_1}\right].\ \exp\left[\frac{\overline{x}_2-\overline{X}_2}{\overline{x}_2+\overline{X}_2}\right]$$

$$=\overline{y}\exp\left[\frac{2(\overline{X}_1\overline{x}_2-\overline{x}_1\overline{X}_2)}{(\overline{X}_1+\overline{x}_1)(\overline{x}_2+\overline{X}_2)}\right] \tag{2.1}$$

It is to be mentioned that when no auxiliary information is used, the estimator t_{RPe} reduces to the usual unbiased estimator $\overline{y}$. If the information on auxiliary variate X_1 is used, then the estimator t_{RPe} tends to the ratio-type exponential estimator t_{Re} given by (1.2). On the other hand, if the information on only auxiliary variate X_2 is available, the estimator t_{RPe} reduces to the product-type exponential estimator t_{Pe} given by (1.3).

To the first degree of approximation, the mean squared error of t_{RPe} is given by

$$MSE(t_{RPe}) = \theta\overline{Y}^2[C_0^2 + \{C_1^2(1-4K_{01})/4\} + \{C_2^2(1+4K_{02}-2K_{12})/4\}] \tag{2.2}$$

3. EFFICIENCY COMPARISONS

In this section we have made comparison of the suggested estimator t_{RPe} with the existing estimators such as usual unbiased estimator $\overline{y}$, ratio and product estimators t_R and t_P, Singh's (1967) estimator t_{RP} and Bahl and Tuteja's (1991) estimators t_{Re} and t_{Pe}.

It follows from (1.4), (1.5), (1.6), (1.7) and (2.2) that the proposed ratio-cum-product exponential estimator t_{RPe} is more efficient than:

(i) the usual unbiased estimator $\overline{y}$ if

$$K_{01} > (1/4) \quad \text{and} \quad K_{02} < \{(2K_{12}-1)/4\}, \tag{3.1}$$

(ii) the Singh's (1967) estimator t_{RP} if

$$K_{01} < (3/4) \quad \text{and} \quad K_{02} > \{3(2K_{12}-1)/4\}, \tag{3.2}$$

(iii) the ratio-type exponential estimator t_{Re} if

$$K_{02} < \{(2K_{12}-1)/4\}, \tag{3.3}$$

(iv) the product-type exponential estimator t_{Pe} if

$$K_{01} > \{(1-2K_{21})/4\}. \tag{3.4}$$

Further to compare the ratio-cum-product type exponential estimator t_{RPe} with usual ratio estimator $t_R = \overline{y}(\overline{X}_1/\overline{x}_1)$ and product estimator $t_P = \overline{y}(\overline{x}_2/\overline{X}_2)$ we write the mean squared errors of t_R and t_P to the first degree of approximation, respectively as

$$MSE(t_R) = \theta \bar{Y}^2 [C_0^2 + C_1^2(1-2K_{01})] \tag{3.5}$$

$$MSE(t_P) = \theta \bar{Y}^2 [C_0^2 + C_2^2(1+2K_{02})] \tag{3.6}$$

From (2.2), (3.5) and (3.6) we note that the proposed estimator t_{RPe} is more efficient than:

(i) the usual ratio estimator t_R if

$$K_{01} < (3/4) \quad \text{and} \quad K_{02} < \{(2K_{12}-1)/4\}, \tag{3.7}$$

(ii) the usual product estimator t_P if

$$K_{01} > (1/4) \quad \text{and} \quad K_{02} > -(2K_{12}+3)/4 . \tag{3.8}$$

4. CLASS OF ALMOST UNBIASED ESTIMATORS OF POPULATION MEAN $\bar{Y}$

It is observed that the suggested ratio-cum-product type exponential estimator is biased. In some applications biasedness is disadvantageous. This led authors to investigate unbiased estimators of the population mean $\bar{Y}$.

4.1 *Bias Subtraction Method*

The bias of t_{RPe} to the first degree of approximation is given by

$$B(t_{RPe}) = (\theta/2)\bar{Y}\left[\frac{3}{4}\frac{S_1^2}{\bar{X}_1^2} - \frac{S_{01}}{\bar{Y}\bar{X}_1} + \frac{S_{02}}{\bar{Y}\bar{X}_2} - \frac{S_{12}}{\bar{X}_1\bar{X}_2} - \frac{S_2^2}{4\bar{X}_2^2}\right] \tag{4.1}$$

Replacing $\bar{Y}, S_1^2, S_2^2, S_{01}$ and S_{12} by their unbiased estimators $\bar{y}$,

$s_1^2 = \sum_{j=1}^{n}(x_{1j}-\bar{x}_1)^2/(n-1)$, $s_2^2 = \sum_{j=1}^{n}(x_{2j}-\bar{x}_2)^2/(n-1)$,

$s_{01} = \sum_{j=1}^{n}(y_j-\bar{y})(x_{1j}-\bar{x}_1)/(n-1)$, $s_{02} = \sum_{j=1}^{N}(y_j-\bar{y})(x_{2j}-\bar{x}_2)/(n-1)$ and $s_{12} = \sum_{j=1}^{N}(x_{1j}-\bar{x}_1)(x_{2j}-\bar{x}_2)/(n-1)$ respectively in (4.1) we get a consistent estimate of the bias B(t_{RPe}) as

$$\hat{B}(t_{RPe}) = (\theta/2)\bar{y}\left[\frac{3}{4}\frac{s_1^2}{\bar{X}_1^2} - \frac{s_{01}}{\bar{y}\,\bar{X}_1} + \frac{s_{02}}{\bar{y}\bar{X}_2} - \frac{s_{12}}{\bar{X}_1\bar{X}_2} - \frac{s_2^2}{4\bar{X}_2^2}\right] \quad (4.2)$$

Thus an almost unbiased estimator of the population mean $\bar{Y}$ is given by

$$t_{RPe}^{(u)} = \bar{y}\left[\exp\left\{\frac{2(\bar{X}_1\bar{x}_2 - \bar{x}_1\bar{X}_2)}{(\bar{X}_1+\bar{x}_1)(\bar{x}_2+\bar{X}_2)}\right\} - \frac{(1-f)}{2n}\left\{\frac{3}{4}\frac{s_1^2}{\bar{X}_1^2} - \frac{s_{01}}{\bar{y}\bar{X}_1} + \frac{s_{02}}{\bar{y}\bar{X}_2} - \frac{s_{12}}{\bar{X}_1\bar{X}_2} - \frac{s_2^2}{4\bar{X}_2^2}\right\}\right] \quad (4.3)$$

It can be easily shown to the first degree of approximation that the variance of $t_{RPe}^{(u)}$ is

$$Var(t_{RPe}^{(u)}) = MSE(t_{RPe}) \quad (4.4)$$

Thus if the bias is of considerable importance then the estimator $t_{RPe}^{(u)}$ is to be preferred over t_{RPe}.

4.2 *Jack knife Method*

Let a simple random sample of size n=g m drawn without replacement and split at random into g sub samples, each of size m. Then we consider the jack-knife ratio-cum-product type exponential estimator of the population mean $\bar{Y}$ as

$$t_{RPeJ} = \frac{1}{g}\sum_{j=1}^{g}\bar{y}'_j \exp\left\{\frac{2(\bar{X}_1\bar{x}'_{2j} - \bar{x}'_{1j}\bar{X}_2)}{(\bar{X}_1+\bar{x}'_{1j})(\bar{x}'_{2j}+\bar{X}_2)}\right\} \quad (4.5)$$

where $\bar{y}'_j = (n\bar{y} - m\bar{y}_j)/(n-m)$ and $\bar{x}'_{ij} = (n\bar{x}_i - m\bar{x}_{ij})/(n-m)$, i=1,2; are the sample means based on a sample of (n-m) units obtained by omitting the j[th] group and $\bar{y}$ and $\bar{x}_{ij}$ (i=1,2; j=1,2,...,g) are the sample means based on the j[th] sub samples of size m=n/g.

To the first degree of approximation, the bias of t_{RPeJ} is given by

$$B(t_{RPeJ}) = \frac{(N-n+m)}{2N(n-m)}\bar{Y}\left[C_1^2\left(\frac{3}{4} - K_{01}\right) + C_2^2\left(K_{02}\frac{K_{12}}{2} - \frac{1}{4}\right)\right] \quad (4.6)$$

From (4.1) and (4.6) we have

$$\frac{B(t_{RPe})}{B(t_{RPeJ})} = \frac{(N-n)(n-m)}{n(N-n+m)} = \delta\,(Say)$$

or $B(t_{RPe}) - \delta\, B(t_{RPeJ}) = 0$ or $\lambda B(t_{RPe}) - \lambda\delta\, B(t_{RPeJ}) = 0$,

where λ is a scalar. For any scalar λ, we have

$$\lambda E(t_{RPe} - \bar{Y}) - \lambda\delta\, E(t_{RPeJ} - \bar{Y}) = 0 \text{ or } \lambda E(t_{RPe} - \bar{y}) - \lambda\delta\, E(t_{RPeJ} - \bar{y}) = 0 \text{ or}$$

$$E[\lambda t_{RPe} - \lambda\delta\, t_{RPeJ} - \bar{y}\{\lambda(1-\delta) - 1\}] = \bar{Y}.$$

Hence the general class of almost unbiased ratio-cum-product type exponential estimator of $\bar{Y}$ as

$$t_{RPe(u)} = \left[\bar{y}\{1-\lambda(1-\delta)\} + \lambda\bar{y}\exp\left\{\frac{2(\bar{X}_1\bar{x}_2 - \bar{x}_1\bar{X}_2}{(\bar{X}_1+\bar{x}_1)(\bar{X}_2+\bar{x}_2)}\right\} - \frac{\lambda\delta}{g}\sum_{j=1}^{g}\bar{y}'_j\exp\left\{\frac{2(\bar{X}_1\bar{x}'_{2j} - \bar{x}'_{1j}\bar{X}_2}{(\bar{X}_1+\bar{x}'_{1j})(\bar{x}'_{2j}+\bar{X}_2)}\right\}\right] \tag{4.7}$$

See Singh (1987 a) and Singh and Tailor (2005).

Remark 4.1. For λ =0, the estimator $t_{RPe(u)}$ reduces to the conventional unbiased estimator $\bar{y}$ while for $\lambda = (1-\delta)^{-1}$, $t_{RPe(u)}$ yields an almost unbiased estimator for $\bar{Y}$ as

$$t^1_{RPe(u)} = \left[\frac{(N-n+m)}{N}g\,\bar{y}\exp\left\{\frac{2(\bar{X}_1\bar{x}_2 - \bar{x}_1\bar{X}_2}{(\bar{X}_1+\bar{x}_1)(\bar{X}_2+\bar{x}_2)}\right\} - \frac{(N-n)(g-1)}{Ng}\sum_{j=1}^{g}\bar{y}_j\exp\left\{\frac{2(\bar{X}_1\bar{x}_{2j} - \bar{x}_{1j}\bar{X}_2}{(\bar{X}_1+\bar{x}_{1j})(\bar{x}_{2j}+\bar{X}_2)}\right\}\right] \tag{4.8}$$

which is *jackknifed* version of the suggested estimator t_{RPe}.

A large number of almost unbiased ratio-cum-product type exponential estimators from (4.8) can be generated by substituting the suitable values of the scalar λ.

5. SEARCH OF ASYMPTOTICALLY OPTIMUM ALMOST UNBIASED RATIO-CUM-PRODUCT TYPE EXPONENTIAL ESTIMATOR IN THE CLASS OF ESTIMATORS $t_{RPe(u)}$ AT (4.7)

We write the estimator $t_{RPe(u)}$ at (4.8) as

$$t_{RPe(u)} = [\bar{y}\{1-\lambda(1-\delta)\} + \lambda\, t_{RPe} - \lambda\delta\, t_{RPe\,J}] \tag{5.1}$$

The variance of $t_{RPe(u)}$ is given by

$$\begin{aligned} V(t_{RPe(u)}) = \Big[& \lambda^2\{(1-\delta)^2 Var(\bar{y}) + Var(t_{RPe}) + \delta^2 Var(t_{RPeJ}) \\ & - 2\delta Cov(t_{RPe}, t_{RPeJ}) - 2(1-\delta)Cov(\bar{y}, t_{RPe}) + 2\delta(1-\delta)Cov(\bar{y}, t_{RPej})\} \\ & -2\lambda\{(1-\delta)Var(\bar{y}) - Cov(\bar{y}, t_{RPe}) + \delta\, Cov(\bar{y}, t_{RPeJ})\} + Var(\bar{y})\Big] \end{aligned} \tag{5.2}$$

To the first degree of approximation, it can be easily shown that

$$Var(t_{RPeJ}) = Var(t_{RPe}) = Cov(t_{RPe}, t_{RPeJ}) = MSE(t_{RPe}) \tag{5.3}$$

and

$$\mathrm{Cov}(\bar{y}, t_{RPe}) = \mathrm{Cov}(\bar{y}, t_{RPeJ}) = (\theta/2)\ [2C_0^2 + K_{02}C_2^2 - K_{01}C_1^2)], \tag{5.4}$$

where MSE(t_{RPe}) is given by (2.2).

Substitution of (1.4), (5.3) and (5.4) in (5.2), we get the variance of $t_{RPe}^{(u)}$ to the first degree of approximation as

$$\begin{aligned} Var(t_{RPe(u)}) = (\theta^2)\bar{Y}^2[C_0^2 + (\lambda^2/4)(1-\delta)^2(C_1^2 + C_2^2 - 2K_{12}C_2^2) \\ + \lambda(1-\delta)(K_{02}C_2^2 - K_{01}C_1^2)] \end{aligned} \tag{5.5}$$

which is minimized for

$$\lambda = \frac{2(K_{01}C_1^2 - K_{02}C_2^2}{(1-\delta)(C_1^2 + C_2^2 - 2K_{12}C_2^2)} = \lambda_0 \ \text{(say)} \tag{5.6}$$

Thus the resulting minimum variance of $t_{RPe(u)}$ is given by

$$Var\min(t_{RPe(u)}) = \theta\bar{Y}^2\left[C_0^2 - \frac{(K_{01}C_1^2 - K_{02}C_2^2)^2}{(C_1^2 + C_2^2 - 2K_{12}C_2^2)}\right] \tag{5.7}$$

From (1.4), (2.2) and (5.7) we have

$$Var(\bar{y}) - Var\min(t_{RPe(u)}) = \theta\bar{Y}^2\left[\frac{(K_{01}C_1^2 - K_{02}C_2^2)^2}{(C_1^2 + C_2^2 - 2K_{12}C_2^2)}\right] \geq 0 \tag{5.8}$$

$$Var(t_{RPe}) - Var\min(t_{RPe(u)}) = \left[\frac{\theta\overline{Y}^2\{C_1^2(1-2K_{01}) + C_2^2(1+2K_{02}-2K_{12})\}^2}{4(C_1^2 + C_2^2 - 2K_{12}C_2^2)}\right] \geq 0 \quad (5.9)$$

It follows from (5.8) and (5.9) that the proposed class of estimators $t_{RPe(u)}$ is more efficient than usual unbiased estimator $\overline{y}$ and the ratio-cum-product type exponential estimator t_{RPe} at its optimum condition (i. e. when λ coincides exactly with its optimum value λ_0 given by (5.6)). The optimum value λ_0 of λ can be obtained quite accurately either through past data or experience gathered in due course of time.

6. A GENERALIZED VERSION OF THE SUGGESTED RATIO-CUM-PRODUCT TYPE EXPONENTIAL ESTIMATOR t_{RPe}

We define the following class of ratio-cum-product type exponential estimator for the population mean $\overline{Y}$ as

$$t_{RPe}^{(a_1,a_2)} = \overline{y}\exp\left\{\frac{a_1(\overline{X}_1 - \overline{x}_1)}{(\overline{X}_1 - \overline{x}_1)}\right\}\exp\left\{\frac{a_2(\overline{x}_2 - \overline{X}_2)}{(\overline{x}_2 - \overline{X}_2)}\right\} = \overline{y}\exp\left[\frac{a_1(\overline{X}_1 - \overline{x}_1)}{(\overline{X}_1 + \overline{x}_1)} + \frac{a_2(\overline{x}_2 - \overline{X}_2)}{(\overline{x}_2 + \overline{X}_2)}\right], \quad (6.1)$$

where a_1 and a_2 are suitably chosen constants. For (a_1, a_2)=(0,0), (1,1), (1,0), and (0,1), $t_{RPe}^{(a_1,a_2)}$ respectively reduce to $\overline{y}$, t_{RPe}, t_{Re} and t_{Pe}. To the first degree of approximation the bias and mean squared error of $t_{RPe}^{(a_1,a_2)}$ are respectively given by

$$B(t_{RPe}^{(a_1,a_2)}) = \theta\overline{Y}\left[\frac{a_2C_2^2}{2}\left(K_{02} + \frac{(2-a_2)}{4}\right) - \frac{a_1C_1^2}{2}\left(K_{01} - \frac{(2+a_1)}{4}\right) - \frac{a_1a_2\rho_{12}C_1C_2}{4}\right], \quad (6.2)$$

$$MSE(t_{RPe}^{(a_1,a_2)}) = \theta\overline{Y}^2\left[C_0^2 + \frac{a_1C_1^2}{4}(a_1 - 4K_{01}) + \frac{a_2C_2^2}{4}(a_2 + 4K_{02}) - \frac{a_1a_2\rho_{12}C_1C_2}{2}\right]. \quad (6.3)$$

We mention that to the first degree of approximation the biases and MSEs of the estimators $\overline{y}$, t_{RPe}, t_{Re} and t_{Pe} can be easily obtained from (6.2) and (6.3) just be putting (a_1, a_2)=(0,0), (1,1), (1,0), and (0,1) respectively.

The MSE $(t_{RPe}^{(a_1,a_2)})$ at (6.3) is minimized for

$$a_1 = \frac{2(K_{01} - K_{02}K_{21})}{(1 - K_{12}K_{21})} = a_{10}(say),\ a_2 = -\frac{2(K_{02} - K_{01}K_{12})}{(1 - K_{12}K_{21})} = a_{20}(say). \tag{6.4}$$

Substitution of (6.4) in (6.1) yields asymptotically optimum estimator (AOE) in the class of estimators $t_{RPe}^{(a_1,a_2)}$ as

$$t_{RPe}^{(a_{10},a_{20})} = \bar{y}\exp\left[\left\{\frac{a_{10}(\bar{X}_1 - \bar{x}_1)}{(\bar{X}_1 + \bar{x}_1)}\right\} + \left\{\frac{a_{20}(\bar{x}_2 - \bar{X}_2)}{(\bar{x}_2 + \bar{X}_2)}\right\}\right]. \tag{6.5}$$

The MSE of AOE $t_{RPe}^{(a_1,a_2)}$ is given by

$$MSE(t_{RPe}^{(a_{10},a_{20})}) = (\theta/2)S_0^2[1 - (\rho_{01}^2 + \rho_{02}^2 - 2\rho_{01}\rho_{02}\rho_{12})/(1 - \rho_{12}^2)]. \tag{6.6}$$

It is to be noted that the AOE $t_{RPe}^{(a_{10},a_{20})}$ can be used in practice only when the optimum values a_{10} and a_{20} of the scalars a_1 and a_2 respectively are known. However it may happen in some practical situations that the optimum values a_{10} and a_{20} are not known. In such situation it is worth advisable to estimate them from the sample data at hand, let

$$\hat{a}_{10} = \frac{2(\hat{K}_{01} - \hat{K}_{02}\hat{K}_{21})}{(1 - \hat{K}_{12}\hat{K}_{21})},\ \hat{a}_{20} = -\frac{2(\hat{K}_{02} - \hat{K}_{01}\hat{K}_{12})}{(1 - \hat{K}_{12}\hat{K}_{21})}; \tag{6.7}$$

be consistent estimators of a_{10} and a_{20} respectively, where

$\hat{K}_{01} = \hat{\rho}_{01}\dfrac{\hat{C}_0}{\hat{C}_1}$; $\hat{K}_{21} = \hat{\rho}_{12}\dfrac{\hat{C}_2}{\hat{C}_1}$; $\hat{K}_{02} = \hat{\rho}_{02}\dfrac{\hat{C}_0}{\hat{C}_2}$; $\hat{K}_{12} = \hat{\rho}_{12}\dfrac{\hat{C}_1}{\hat{C}_2}$; $\hat{\rho}_{0i} = \dfrac{s_{0i}}{s_0 s_i}$ and $\hat{\rho}_{12} = \dfrac{s_{12}}{s_1 s_2}$.

Replacing a_{10} and a_{20} by their consistent estimators $\hat{a}_{10}$ and $\hat{a}_{20}$ respectively in (6.5) we get a ratio-cum-product type exponential estimator $t_{RPe}^{(\hat{a}_{10},\hat{a}_{20})}$ (based estimated optimum values) of the population mean $\bar{Y}$ as

$$t_{RPe}^{(\hat{a}_{10},\hat{a}_{20})} = \bar{y}\exp\left\{\frac{\hat{a}_{10}(\bar{X}_1 - \bar{x}_1)}{(\bar{X}_1 + \bar{x}_1)}\right\}\exp\left\{\frac{\hat{a}_{20}(\bar{x}_2 - \bar{X}_2)}{(\bar{x}_2 + \bar{X}_2)}\right\} \tag{6.8}$$

The MSE of AOE $t_{RPe}^{(a_1,a_2)}$ is given by

$$MSE(t_{RPe}^{(\hat{a}_{10},\hat{a}_{20})}) = (\theta/2)S_0^2[(\rho_{01}^2 + \rho_{02}^2 - 2\rho_{01}\rho_{02}\rho_{12})/(1 - \rho_{12}^2)] = MSE(t_{RPe}^{(a_{10},a_{20})}) \tag{6.9}$$

Thus if the optimum values a_{10} and a_{20} of a_1 and a_2 respectively are not known, then it is worth advisable to prefer the estimator $t_{RPe}^{(\hat{a}_{10},\hat{a}_{20})}$ (based on estimated optimum value) over the AOE $t_{RPe}^{(a_{10},a_{20})}$ in practical research.

7. EMPIRICAL STUDY

To have tangible idea about the performance of the various estimators of $\bar{Y}$, we consider the two natural population data.

Population –I: [Source : Steel and Torrie (1960, p. 282)]

Y: Log of leaf burn in sec., X_1: Potassium percentage, X_2: Chlorine percentage.

$\bar{Y}=0.6860$, $\bar{X}_1=4.6537$, $\bar{X}_2=0.8077$, $C_0=0.4803$, $C_1=0.2295$, $C_2=0.7493$, ρ_{01}=0.1794, $\rho_{02}=-0.4996$ and $\rho_{12}=0.4074$

Population –II: [Source: Singh (1965, p. 325)]

Y: Females employed, X_1: Females in service, X_2: Educated females

$\bar{Y}=7.46$, $\bar{X}_1=5.31$, $\bar{X}_2=179.00$, $C_0^2=0.5046$, $C_1^2=0.5757$, $C_2^2=0.0633$, ρ_{01}=0.7737, $\rho_{02}=-0.2070$ and $\rho_{12}=-0.0033$

We have computed the percent relative efficiencies (PRE(s)) of different estimators of population mean $\bar{Y}$ with respect to usual unbiased estimator $\bar{y}$ and findings are complied in Table 7.1.

TABLE 7.1

Percent relative efficiencies (PRE's) of different estimators of population mean $\bar{Y}$ with respect to usual unbiased estimator $\bar{y}$

Estimators	PRE (•, $\bar{y}$)	
	Population –I	Population –II
$\bar{y}$	100.00	100.00
t_R	94.62	208.23
t_P	53.34	102.16
t_{RP}	75.50	216.66
t_{Re}	102.95	217.95
t_{Pe}	121.25	104.38
t_{RRe} or $t_{RPe}^{(u)}$	155.23	241.81
$t_{RRe(u)}$	157.30	276.25
$t_{RPe}^{(\hat{a}_{10},\hat{a}_{20})}$	174.04	278.10

Table 7.1 exhibits that the proposed ratio-cum-product type exponential estimator $t_{RPe}^{(\hat{a}_{10},\hat{a}_{20})}$ is more efficient than all the estimators $\bar{y}$, t_R, t_P, t_{Re}, t_{Pe}, t_{RP},

t_{RPe} or $t_{RPe}^{(u)}$ and $t_{RPe(u)}$. It is interesting to note that the ratio-cum-product type exponential estimator t_{RPe} or $t_{RPe}^{(u)}$ is more efficient than Singh's (1967) ratio-cum-product estimator t_{RP}, $\overline{y}$, t_R, t_P, t_{Re}, t_{Pe} with substantial gain in efficiency in both the populations I and II. Thus the proposed ratio-cum-product type exponential estimators t_{RPe}, $t_{RPe}^{(u)}$, $t_{RPe(u)}$ and $t_{RPe}^{(\hat{a}_{10},\hat{a}_{20})}$ are to be preferred over Singh's (1967) estimator t_{RP} in practical research.

ACKNOWLEDGEMENT

Authors are thankful to the referee for his valuable suggestions regarding improvement of this paper. The research of the second author was supported by the grant of U.G.C., New Delhi.

School of Studies in Statistics HOUSILA P. SINGH
Vikram University, Ujjain, M.P., India

Department of Applied Mathematics L.N. UPADHYAYA
Indian School of Mines, Dhanbad, Jharkhand, India

School of Studies in Statistics RAJESH TAILOR
Vikram University, Ujjain, M.P., India

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SUMMARY

Ratio-cum-product type exponential estimator

This paper addresses the problem of estimating the population mean $\overline{Y}$ of the study variate Y using information on two auxiliary variables X_1 and X_2. A ratio-cum-product

type exponential estimator has been suggested and its bias and mean squared error have been derived under large sample approximation. An almost unbiased ratio-cum-product type exponential estimator has also been derived by using Jackknife technique envisaged by Quenouille (1956). A generalized version of the ratio-cum-product exponential estimator has also been given along with its properties. Numerical illustration is given in support of the present study.