1. INTRODUCTION

Consider a finite population $U = (U_1, U_2, ..., U_N)$ of $N$ units. Suppose two auxiliary variables $X_1$ and $X_2$ are observed on $U_i$ ($i=1,2,3,...,N$), where $X_1$ is positively and $X_2$ is negatively correlated with the study variable $Y$. A simple random sample of size $n$ is drawn without replacement from population $U$ to estimate the population mean $\bar{Y}$ of the study variable $Y$ assuming the knowledge of the population means $\bar{X}_1$ and $\bar{X}_2$ of the auxiliary variables $X_1$ and $X_2$ respectively. Singh (1967) proposed a ratio-cum-product estimator for population mean $\bar{Y}$ as

$$t_{RP} = \bar{y}(\bar{X}_1/\bar{X}_1)(\bar{X}_2/\bar{X}_2),$$

where

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i, \quad \bar{X}_1 = \frac{1}{n} \sum_{i=1}^{n} x_{1i}, \quad \text{and} \quad \bar{X}_2 = \frac{1}{n} \sum_{i=1}^{n} x_{2i}.$$  

Bahl and Tuteja (1991) envisaged ratio and product type exponential estimators for $\bar{Y}$ respectively as

$$t_{Re} = \bar{y} \exp \left( \frac{\bar{X}_1 - \bar{X}_1}{\bar{X}_1 + \bar{X}_1} \right),$$

$$t_{Pv} = \bar{y} \exp \left( \frac{\bar{X}_2 - \bar{X}_2}{\bar{X}_2 + \bar{X}_2} \right).$$

It is well known under simple random sampling without replacement (SRSWOR) that the variance/mean squared error (MSE) of usual unbiased estimator $\bar{Y}$ is

$$V(\bar{Y}) = \theta S_0^2 = MSE(\bar{Y}),$$
where
\[ \theta = (1 - f)/n, \quad f = n/N \] and \[ S_0^2 = \sum_{j=1}^{N} (y_j - \bar{Y})^2 / (N - 1). \]

To the first degree of approximation, mean squared errors of \( t_{RP} \), \( t_{Re} \) and \( t_{Ph} \) are respectively given by
\[ \text{MSE}(t_{RP}) = \theta \bar{Y}^2 \left[ C_o^2 + C_i^2 (1 - 2K_{01}) + C_2^2 \{1 + 2(K_{02} - K_{12})}\right] \tag{1.5} \]
\[ \text{MSE}(t_{Re}) = \theta \bar{Y}^2 \left[ C_0^2 + \{C_1^2 (1 - 4K_{01})/4\}\right] \tag{1.6} \]
\[ \text{MSE}(t_{Ph}) = \theta \bar{Y}^2 \left[ C_0^2 + \{C_2^2 (1 - 4K_{02})/4\}\right], \tag{1.7} \]
where \( C_o = \frac{S_o}{\bar{Y}}, \quad C_i = \frac{S_i}{\bar{X}_i}, \quad K_{wl} = \rho_{ai} \frac{C_o}{C_i} \) \( (i=1,2), \quad K_{12} = \rho_{12} \frac{C_1}{C_2}, \)
\[ S_i^2 = \sum_{j=1}^{N} (x_{ij} - \bar{X}_i)^2 / (N - 1), \quad S^2_w = \sum_{j=1}^{N} (y_j - \bar{Y})(x_{ij} - \bar{X}_i) / (N - 1), \quad (i = 1,2); \]
\[ \rho_{0i} = (S_0 / S_0 S_i) : \text{is the correlation coefficient between } Y \text{ and } X_i \quad (i = 1,2), \]
\[ \rho_{12} = S_{12} / (S_1 S_2): \text{is the correlation coefficient between } X_1 \text{ and } X_2, \]
and \[ S_{12} = \sum_{j=1}^{N} (x_{1j} - \bar{X}_1)(x_{2j} - \bar{X}_2) / (N - 1). \]

In this paper we have suggested a ratio-cum-product type exponential estimator for the population mean \( \bar{Y} \) of the study variate \( Y \) using auxiliary information on \( X_1 \) and \( X_2 \). The mean squared error of the suggested estimator has been derived under large sample approximation. Numerical illustration is given in support of the present study.

2. PROPOSED RATIO-CUM-PRODUCT TYPE EXPONENTIAL ESTIMATOR

Motivated by Singh (1967) we propose a ratio-cum-product type exponential estimator for population mean \( \bar{Y} \) as
\[ t_{RP} = \bar{Y} \exp \left[ \frac{\bar{X}_1 - \bar{X}_1}{\bar{X}_1 + \bar{X}_1} \right] \exp \left[ \frac{\bar{X}_2 - \bar{X}_2}{\bar{X}_2 + \bar{X}_2} \right] \]
\[ = \bar{Y} \exp \left[ \frac{2(\bar{X}_1 \bar{X}_2 - \bar{X}_1 \bar{X}_2)}{(\bar{X}_1 + \bar{X}_1)(\bar{X}_2 + \bar{X}_2)} \right] \tag{2.1} \]
It is to be mentioned that when no auxiliary information is used, the estimator $t_{RP}$ reduces to the usual unbiased estimator $\bar{Y}$. If the information on auxiliary variate $X_1$ is used, then the estimator $t_{RP}$ tends to the ratio-type exponential estimator $t_{Re}$ given by (1.2). On the other hand, if the information on only auxiliary variate $X_2$ is available, the estimator $t_{RP}$ reduces to the product-type exponential estimator $t_{Pe}$ given by (1.3).

To the first degree of approximation, the mean squared error of $t_{RP}$ is given by

$$MSE(t_{RP}) = \sigma Y^2 \left[ C_0^2 + \left\{ C_1^2 \left( 1 - 4K_{01} \right)/4 \right\} + \left\{ C_2^2 \left( 1 + 4K_{02} - 2K_{12} \right)/4 \right\} \right]$$

(2.2)

3. EFFICIENCY COMPARISONS

In this section we have made comparison of the suggested estimator $t_{RP}$ with the existing estimators such as usual unbiased estimator $\bar{Y}$, ratio and product estimators $t_R$ and $t_p$, Singh’s (1967) estimator $t_{RP}$ and Bahl and Tuteja’s (1991) estimators $t_{Re}$ and $t_{Pe}$.

It follows from (1.4), (1.5), (1.6), (1.7) and (2.2) that the proposed ratio-cum-product type exponential estimator $t_{RP}$ is more efficient than:

(i) the usual unbiased estimator $\bar{Y}$ if

$$K_{01} > (1/4) \quad \text{and} \quad K_{02} < \{(2K_{12} - 1)/4\},$$

(3.1)

(ii) the Singh’s (1967) estimator $t_{RP}$ if

$$K_{01} < (3/4) \quad \text{and} \quad K_{02} > \{3(2K_{12} - 1)/4\},$$

(3.2)

(iii) the ratio-type exponential estimator $t_{Re}$ if

$$K_{02} < \{(2K_{12} - 1)/4\},$$

(3.3)

(iv) the product-type exponential estimator $t_{Pe}$ if

$$K_{01} > \{(1 - 2K_{21})/4\}.$$

(3.4)

Further to compare the ratio-cum-product type exponential estimator $t_{RP}$ with usual ratio estimator $t_R = \bar{Y}(\bar{X}_1 / \bar{X}_1)$ and product estimator $t_p = \bar{Y}(\bar{X}_2 / \bar{X}_2)$ we write the mean squared errors of $t_R$ and $t_p$ to the first degree of approximation, respectively as
MSE\( (t_R) = \theta \bar{Y}^2 \left[ C_0^2 + C_1^2 (1 - 2K_{01}) \right] \)

(3.5)

MSE\( (t_P) = \theta \bar{Y}^2 \left[ C_0^2 + C_2^2 (1 + 2K_{02}) \right] \)

(3.6)

From (2.2), (3.5) and (3.6) we note that the proposed estimator \( t_{RP_e} \) is more efficient than:

(i) the usual ratio estimator \( t_R \) if

\[ K_{01} < (3/4) \quad \text{and} \quad K_{02} < \{(2K_{12} - 1)/4\}, \]

(3.7)

(ii) the usual product estimator \( t_P \) if

\[ K_{01} > (1/4) \quad \text{and} \quad K_{02} > -(2K_{12} + 3)/4. \]

(3.8)

4. CLASS OF ALMOST UNBIASED ESTIMATORS OF POPULATION MEAN \( \bar{Y} \)

It is observed that the suggested ratio-cum-product type exponential estimator is biased. In some applications biasedness is disadvantageous. This led authors to investigate unbiased estimators of the population mean \( \bar{Y} \).

4.1 Bias Subtraction Method

The bias of \( t_{RP_e} \) to the first degree of approximation is given by

\[ B(t_{RP_e}) = (\theta / 2)\bar{Y} \left[ \frac{3}{4} \frac{S^2_{11}}{X^2_{1}} - \frac{S_{01}}{Y X_{1}} + \frac{S_{02}}{Y X_{2}} - \frac{S_{12}}{X_{1} X_{2}} - \frac{S^2_{22}}{4 X^2_{2}} \right] \]

(4.1)

Replacing \( \bar{Y} \), \( S^2_{11}, S^2_{22}, S_{01} \) and \( S_{12} \) by their unbiased estimators \( \bar{y}, \)

\[ x^2_{1} = \sum_{j=1}^{n} (x_{1j} - \bar{x}_{1})^2 / (n-1), \quad x^2_{2} = \sum_{j=1}^{n} (x_{2j} - \bar{x}_{2})^2 / (n-1), \]

\[ s_{01} = \sum_{j=1}^{n} (y_{j} - \bar{y})(x_{1j} - \bar{x}_{1}) / (n-1), \quad s_{02} = \sum_{j=1}^{n} (y_{j} - \bar{y})(x_{2j} - \bar{x}_{2}) / (n-1) \quad \text{and} \]

\[ s_{12} = \sum_{j=1}^{n} (x_{1j} - \bar{x}_{1})(x_{2j} - \bar{x}_{2}) / (n-1) \]

respectively in (4.1) we get a consistent estimate of the bias \( B(t_{RP_e}) \) as
\[
\hat{B}(t_{RPy}) = (\theta / 2) \sqrt{\left[ \frac{3}{4} s_1^2 - \frac{s_{01}}{\bar{y} \bar{X}_1} + \frac{s_{02}}{\bar{y} \bar{X}_2} - \frac{s_{12}}{\bar{X}_1 \bar{X}_2} - \frac{s_2^2}{4 \bar{X}_2^2} \right]} \tag{4.2}
\]

Thus an almost unbiased estimator of the population mean \( \bar{Y} \) is given by

\[
t_{RPy}^{(a)} = \bar{Y} \left[ \exp \left\{ \frac{2(\bar{X}_1 \bar{x}_2 - \bar{x}_1 \bar{X}_2)}{(\bar{X}_1 + \bar{x}_1)(\bar{X}_2 + \bar{x}_2)} \right\} \frac{(1-f)}{2n} \left[ \frac{3}{4} \bar{x}_1^2 - \frac{s_{01}}{\bar{x}_1 \bar{X}_1} + \frac{s_{02}}{\bar{x}_1 \bar{X}_2} - \frac{s_{12}}{\bar{X}_1 \bar{X}_2} - \frac{s_2^2}{4 \bar{X}_2^2} \right]\right]
\tag{4.3}
\]

It can be easily shown to the first degree of approximation that the variance of \( t_{RPy}^{(a)} \) is

\[
Var(t_{RPy}^{(a)}) = MSE(t_{RPy}) \tag{4.4}
\]

Thus if the bias is of considerable importance then the estimator \( t_{RPy}^{(a)} \) is to be preferred over \( t_{RPy} \).

### 4.2 Jack knife Method

Let a simple random sample of size \( n=g \) m drawn without replacement and split at random into \( g \) sub samples, each of size \( m \). Then we consider the jack-knife ratio-cum-product type exponential estimator of the population mean \( \bar{Y} \) as

\[
t_{RPy} = \frac{1}{g} \sum_{j=1}^{g} \bar{y}_j \exp \left\{ \frac{2(\bar{X}_1 \hat{\bar{x}}_{2,j} - \hat{\bar{x}}_1 \bar{X}_2)}{(\bar{X}_1 + \bar{x}_1)(\bar{X}_2 + \bar{x}_2)} \right\} \tag{4.5}
\]

where \( \bar{y}_j = (n \bar{y} - n \bar{y}_j) / (n - m) \) and \( \hat{\bar{x}}_i = (n \bar{x}_i - m \bar{x}_i) / (n - m) \), \( i=1,2 \) are the sample means based on a sample of \( (n-m) \) units obtained by omitting the \( j^{th} \) group and \( \bar{y}_j \) and \( \hat{\bar{x}}_i \) \( (i=1,2; j=1,2,...,g) \) are the sample means based on the \( j^{th} \) sub samples of size \( m=n/g \).

To the first degree of approximation, the bias of \( t_{RPy} \) is given by

\[
B(t_{RPy}) = \frac{(N - n + m)}{2N(n-m)} \bar{Y} \left[ C_1^{-1} \left( \frac{3}{4} - K_{01} \right) + C_2^{-1} \left( K_{02} \frac{K_{12}}{2} - \frac{1}{4} \right) \right] \tag{4.6}
\]

From (4.1) and (4.6) we have

\[
\frac{B(t_{RPy})}{B(t_{RPy})} = \frac{(N - n)(n - m)}{n(N - n + m)} = \delta(S_{Ay})
\]
or $B(t_{RPV}) - \delta B(t_{RPJ}) = 0$ or $\lambda B(t_{RPV}) - \lambda \delta B(t_{RPJ}) = 0$,
where $\lambda$ is a scalar. For any scalar $\lambda$, we have
\[ \lambda E(t_{RPV} - \bar{Y}) - \lambda \delta E(t_{RPJ} - \bar{Y}) = 0 \text{ or } \lambda E(t_{RPV} - \bar{Y}) - \lambda \delta E(t_{RPJ} - \bar{Y}) = 0 \text{ or } \]
\[ E[\lambda t_{RPV} - \lambda \delta t_{RPJ} - \bar{Y}(\lambda(1-\delta) - 1)] = \bar{Y}. \]

Hence the general class of almost unbiased ratio-cum-product type exponential estimator of $\bar{Y}$ as

\[ t_{RPV(a)} = \left[ \bar{Y}[1 - \lambda(1 - \delta)] + \lambda \bar{Y} \exp \left\{ \frac{2(\bar{X}_1 \bar{X}_2 - \bar{X}_1 \bar{X}_2)}{\bar{X}_1 + \bar{X}_1}(\bar{X}_2 + \bar{X}_2) \right\} \right. \]
\[ - \frac{\lambda \delta}{g} \sum_{j=1}^{g} \bar{y}_j \exp \left\{ \frac{2(\bar{X}_1 \bar{X}_2 - \bar{X}_1 \bar{X}_2)}{\bar{X}_1 + \bar{X}_1}(\bar{X}_2 + \bar{X}_2) \right\} \] \hspace{1cm} (4.7)

See Singh (1987 a) and Singh and Tailor (2005).

Remark 4.1. For $\lambda = 0$, the estimator $t_{RPV(a)}$ reduces to the conventional unbiased estimator $\bar{Y}$ while for $\lambda = (1 - \delta)^{-1}$, $t_{RPV(a)}$ yields an almost unbiased estimator for $\bar{Y}$ as

\[ t_{RPV(a)}^1 = \left[ \frac{(N - n + m)}{N} \bar{Y} \exp \left\{ \frac{2(\bar{X}_1 \bar{X}_2 - \bar{X}_1 \bar{X}_2)}{\bar{X}_1 + \bar{X}_1}(\bar{X}_2 + \bar{X}_2) \right\} \right. \]
\[ - \frac{(N - n)(g - 1)}{Ng} \sum_{j=1}^{g} \bar{y}_j \exp \left\{ \frac{2(\bar{X}_1 \bar{X}_2 - \bar{X}_1 \bar{X}_2)}{\bar{X}_1 + \bar{X}_1}(\bar{X}_2 + \bar{X}_2) \right\} \] \hspace{1cm} (4.8)

which is jackknifed version of the suggested estimator $t_{RPV}$.

A large number of almost unbiased ratio-cum-product type exponential estimators from (4.8) can be generated by substituting the suitable values of the scalar $\lambda$.

5. SEARCH OF ASYMPTOTICALLY OPTIMUM ALMOST UNBIASED RATIO-CUM-PRODUCT TYPE EXPONENTIAL ESTIMATOR IN THE CLASS OF ESTIMATORS $t_{RPV(a)}$ AT (4.7)

We write the estimator $t_{RPV(a)}$ at (4.8) as
\[ t_{\text{RP}(u)} = [\bar{y} \{1 - \lambda(1 - \delta)\} + \lambda t_{\text{RP}, u} - \lambda \delta t_{\text{RP}, f}] \]  

(5.1)

The variance of \( t_{\text{RP}(u)} \) is given by

\[
V(t_{\text{RP}(u)}) = \left[ \lambda^2 \{(1 - \delta)^2 \text{Var}(\bar{y}) + \text{Var}(t_{\text{RP}, u}) + \delta^2 \text{Var}(t_{\text{RP}, f}) \right. \\
- 2\delta \text{Cov}(t_{\text{RP}, u}, t_{\text{RP}, f}) - 2(1 - \delta) \text{Cov}(\bar{y}, t_{\text{RP}, u}) + 2\delta(1 - \delta) \text{Cov}(\bar{y}, t_{\text{RP}, f}) \} \\
\left. - 2\lambda^2 \{(1 - \delta) \text{Var}(\bar{y}) - \text{Cov}(\bar{y}, t_{\text{RP}, u}) + \delta \text{Cov}(\bar{y}, t_{\text{RP}, f}) \} \right] + \text{Var}(\bar{y}) \\
\]  

(5.2)

To the first degree of approximation, it can be easily shown that

\[
\text{Var}(t_{\text{RP}, f}) = \text{Var}(t_{\text{RP}, u}) = \text{Cov}(t_{\text{RP}, u}, t_{\text{RP}, f}) = \text{MSE}(t_{\text{RP}, u}) \\
\]  

(5.3)

and

\[
\text{Cov}(\bar{y}, t_{\text{RP}, u}) = \text{Cov}(\bar{y}, t_{\text{RP}, f}) = (\theta / 2) \left[ 2C_0^2 + K_{02}C_2^2 - K_{01}C_1^2 \right], \\
\]  

(5.4)

where MSE\( (t_{\text{RP}, u}) \) is given by (2.2).

Substitution of (1.4), (5.3) and (5.4) in (5.2), we get the variance of \( t_{\text{RP}(u)} \) to the first degree of approximation as

\[
\text{Var}(t_{\text{RP}(u)}) = (\theta^2) \bar{y}^2 [C_0^2 + (\lambda^2 / 4)(1 - \delta)^2(C_1^2 + C_2^2 - 2K_{12}C_2^2) \\
+ \lambda(1 - \delta)(K_{02}C_2^2 - K_{01}C_1^2)] \\
\]  

(5.5)

which is minimized for

\[
\lambda = \frac{2(K_{00}C_1^2 - K_{02}C_2^2)}{(1 - \delta)(C_1^2 + C_2^2 - 2K_{12}C_2^2)} = \lambda_0 \text{ (say)} \\
\]  

(5.6)

Thus the resulting minimum variance of \( t_{\text{RP}(u)} \) is given by

\[
\text{Var}\min(t_{\text{RP}(u)}) = \theta \bar{y}^2 \left[ C_0^2 - \frac{(K_{00}C_1^2 - K_{02}C_2^2)^2}{(C_1^2 + C_2^2 - 2K_{12}C_2^2)} \right] \\
\]  

(5.7)

From (1.4), (2.2) and (5.7) we have

\[
\text{Var}(\bar{y}) - \text{Var}\min(t_{\text{RP}(u)}) = \theta \bar{y}^2 \left[ \frac{(K_{01}C_1^2 - K_{02}C_2^2)^2}{(C_1^2 + C_2^2 - 2K_{12}C_2^2)} \right] \geq 0 \\
\]  

(5.8)
\[ V\text{ar}(t_{RPe}) - V\text{ar} \min(t_{RPe(a)}) = \left[ \frac{\theta \bar{Y}^2 \{ C_1^2 (1 - 2K_{01}) + C_2^2 (1 + 2K_{02} - 2K_{12}) \}}{4(C_1^2 + C_2^2 - 2K_{12} C_2^2)} \right]^2 \geq 0 \] 

(5.9)

It follows from (5.8) and (5.9) that the proposed class of estimators \( t_{RPe(a)} \) is more efficient than usual unbiased estimator \( \bar{Y} \) and the ratio-cum-product type exponential estimator \( t_{RPe} \) at its optimum condition (i.e. when \( \lambda \) coincides exactly with its optimum value \( \lambda_0 \) given by (5.6)). The optimum value \( \lambda_0 \) of \( \lambda \) can be obtained quite accurately either through past data or experience gathered in due course of time.

6. A GENERALIZED VERSION OF THE SUGGESTED RATIO-CUM-PRODUCT TYPE EXPONENTIAL ESTIMATOR \( t_{RPe} \)

We define the following class of ratio-cum-product type exponential estimator for the population mean \( \bar{Y} \) as

\[
t^{(a_1, a_2)}_{RPe} = \bar{Y} \exp \left( \frac{a_1(\bar{x}_1 - \bar{x})}{(\bar{x}_1 - \bar{x})} \right) \exp \left( \frac{a_2(\bar{x}_2 - \bar{x})}{(\bar{x}_2 - \bar{x})} \right) = \bar{Y} \exp \left[ \frac{a_1(\bar{x}_1 - \bar{x})}{(\bar{x}_1 - \bar{x})} + \frac{a_2(\bar{x}_2 - \bar{x})}{(\bar{x}_2 - \bar{x})} \right],
\]

(6.1)

where \( a_1 \) and \( a_2 \) are suitably chosen constants. For \((a_1, a_2) = (0,0), (1,1), (1,0), \) and \((0,1), t^{(a_1, a_2)}_{RPe} \) respectively reduce to \( \bar{Y}, t_{RPe}, t_{Re} \) and \( t_{Pe} \). To the first degree of approximation the bias and mean squared error of \( t^{(a_1, a_2)}_{RPe} \) are respectively given by

\[
B(t^{(a_1, a_2)}_{RPe}) = \theta \bar{Y} \left[ \frac{a_1 C_2}{2} K_{02} + \frac{(2 - a_2)}{4} - \frac{a_1 C_1}{2} \left( K_{01} - \frac{(2 + a_1)}{4} \right) - \frac{a_1 a_2 \rho_{12} C_1 C_2}{4} \right],
\]

(6.2)

\[
MSE(t^{(a_1, a_2)}_{RPe}) = \theta \bar{Y}^2 \left[ \frac{a_1 C_1^2}{4} C_0 + \frac{a_1 C_1^2}{4} (a_1 - 4K_{01}) + \frac{a_2 C_2^2}{4} (a_2 + 4K_{02}) - \frac{a_1 a_2 \rho_{12} C_1 C_2}{2} \right].
\]

(6.3)

We mention that to the first degree of approximation the biases and MSEs of the estimators \( \bar{Y}, t_{RPe}, t_{Re} \) and \( t_{Pe} \) can be easily obtained from (6.2) and (6.3) just by putting \((a_1, a_2) = (0,0), (1,1), (1,0), \) and \((0,1) \) respectively.

The MSE \( t^{(a_1, a_2)}_{RPe} \) at (6.3) is minimized for
\[ a_1 = \frac{2(K_{01} - K_{02}K_{21})}{(1 - K_{12}K_{21})} = a_{10} \text{(say)}, \quad a_2 = \frac{2(K_{02} - K_{01}K_{12})}{(1 - K_{12}K_{21})} = a_{20} \text{(say)}. \quad (6.4) \]

Substitution of (6.4) in (6.1) yields asymptotically optimum estimator (AOE) in the class of estimators \( \hat{f}(\hat{a}_{10}, \hat{a}_{20}) \) as

\[
\hat{f}(\hat{a}_{10}, \hat{a}_{20}) = \overline{y} \exp \left[ \frac{a_{10}(\bar{x}_1 - \bar{X}_1)}{\bar{x}_1 + \bar{X}_1} \right] + \exp \left[ \frac{a_{20}(\bar{x}_2 - \bar{X}_2)}{\bar{x}_2 + \bar{X}_2} \right]. \quad (6.5)
\]

The MSE of AOE \( \hat{f}(\hat{a}_{10}, \hat{a}_{20}) \) is given by

\[
MSE(\hat{f}(\hat{a}_{10}, \hat{a}_{20})) = (\theta / 2) \sigma^2 \left[ 1 - (\rho_{01}^2 + \rho_{02}^2 - 2\rho_{01}\rho_{02}\rho_{12}) / (1 - \rho_{12}^2) \right]. \quad (6.6)
\]

It is to be noted that the AOE \( \hat{f}(\hat{a}_{10}, \hat{a}_{20}) \) can be used in practice only when the optimum values \( a_{10} \) and \( a_{20} \) of the scalars \( a_1 \) and \( a_2 \) respectively are known. However it may happen in some practical situations that the optimum values \( a_{10} \) and \( a_{20} \) are not known. In such situation it is worth advisable to estimate them from the sample data at hand, let

\[
\hat{a}_{10} = \frac{2(\hat{K}_{01} - \hat{K}_{02}\hat{K}_{21})}{(1 - \hat{K}_{12}\hat{K}_{21})}, \quad \hat{a}_{20} = \frac{-2(\hat{K}_{02} - \hat{K}_{01}\hat{K}_{12})}{(1 - \hat{K}_{12}\hat{K}_{21})}; \quad (6.7)
\]

be consistent estimators of \( a_{10} \) and \( a_{20} \) respectively, where

\[
\hat{K}_{01} = \hat{\rho}_{01} \frac{\hat{C}_0}{\hat{C}_1}; \quad \hat{K}_{21} = \hat{\rho}_{12} \frac{\hat{C}_2}{\hat{C}_1}; \quad \hat{K}_{02} = \hat{\rho}_{02} \frac{\hat{C}_0}{\hat{C}_2}; \quad \hat{K}_{12} = \hat{\rho}_{12} \frac{\hat{C}_1}{\hat{C}_2}; \quad \hat{\rho}_{0i} = \frac{s_{0i}}{s_{1i}s_{2i}} \quad \text{and} \quad \hat{\rho}_{12} = \frac{s_{12}}{s_{1i}s_{2i}}.
\]

Replacing \( a_{10} \) and \( a_{20} \) by their consistent estimators \( \hat{a}_{10} \) and \( \hat{a}_{20} \) respectively in (6.5) we get a ratio-cum-product type exponential estimator \( \hat{f}(\hat{a}_{10}, \hat{a}_{20}) \) (based estimated optimum values) of the population mean \( \bar{Y} \) as

\[
\hat{f}(\hat{a}_{10}, \hat{a}_{20}) = \overline{y} \exp \left[ \frac{\hat{a}_{10}(\bar{x}_1 - \bar{X}_1)}{\bar{x}_1 + \bar{X}_1} \right] \exp \left[ \frac{\hat{a}_{20}(\bar{x}_2 - \bar{X}_2)}{\bar{x}_2 + \bar{X}_2} \right]. \quad (6.8)
\]

The MSE of AOE \( \hat{f}(\hat{a}_{10}, \hat{a}_{20}) \) is given by

\[
MSE(\hat{f}(\hat{a}_{10}, \hat{a}_{20})) = (\theta / 2) \sigma^2 \left[ (\rho_{01}^2 + \rho_{02}^2 - 2\rho_{01}\rho_{02}\rho_{12}) / (1 - \rho_{12}^2) \right] = MSE(\hat{f}(\hat{a}_{10}, \hat{a}_{20})) \quad (6.9)
\]
Thus if the optimum values $a_{10}$ and $a_{20}$ of $a_1$ and $a_2$ respectively are not known, then it is worth advisable to prefer the estimator $\hat{t}_{RPe}^{(a_{10}, a_{20})}$ (based on estimated optimum value) over the AOE $\hat{t}_{RPe}^{(a_{10}, a_{20})}$ in practical research.

7. EMPIRICAL STUDY

To have tangible idea about the performance of the various estimators of $\bar{Y}$, we consider the two natural population data.

Population –I: [Source: Steel and Torrie (1960, p. 282)]

$Y$: Log of leaf burn in sec., $X_1$: Potassium percentage, $X_2$: Chlorine percentage.

\[
\begin{align*}
\bar{Y} &= 0.6860, \quad \bar{X}_1 = 4.6537, \quad \bar{X}_2 = 0.8077, \quad C_0 = 0.4803, \quad C_1 = 0.2295, \\
C_2 &= 0.7493, \quad \rho_{01} = 0.1794, \quad \rho_{02} = -0.4996 \text{ and } \rho_{12} = 0.4074
\end{align*}
\]

Population –II: [Source: Singh (1965, p. 325)]

$Y$: Females employed, $X_1$: Females in service, $X_2$: Educated females

\[
\begin{align*}
\bar{Y} &= 7.46, \quad \bar{X}_1 = 5.31, \quad \bar{X}_2 = 179.00, \quad C_0 = 0.5046, \quad C_1 = 0.5757, \quad C_2 = 0.0633, \\
\rho_{01} &= 0.7737, \quad \rho_{02} = -0.2070 \text{ and } \rho_{12} = -0.0033
\end{align*}
\]

We have computed the percent relative efficiencies (PRE(s)) of different estimators of population mean $\bar{Y}$ with respect to usual unbiased estimator $\bar{Y}$ and findings are complied in Table 7.1.

<table>
<thead>
<tr>
<th>Estimators</th>
<th>PRE (*, $\bar{Y}$)</th>
<th>Population –I</th>
<th>Population –II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{Y}$</td>
<td>100.00</td>
<td>100.00</td>
<td></td>
</tr>
<tr>
<td>$t_R$</td>
<td>94.62</td>
<td>208.23</td>
<td></td>
</tr>
<tr>
<td>$t_P$</td>
<td>53.34</td>
<td>102.16</td>
<td></td>
</tr>
<tr>
<td>$t_{RP}$</td>
<td>75.50</td>
<td>216.66</td>
<td></td>
</tr>
<tr>
<td>$t_{Re}$</td>
<td>102.95</td>
<td>217.95</td>
<td></td>
</tr>
<tr>
<td>$t_{Pe}$</td>
<td>121.25</td>
<td>104.38</td>
<td></td>
</tr>
<tr>
<td>$t_{Re}+\rho$ or $\hat{t}_{RP}^{(\rho)}$</td>
<td>155.23</td>
<td>241.81</td>
<td></td>
</tr>
<tr>
<td>$t_{Re(a)}$</td>
<td>157.30</td>
<td>276.25</td>
<td></td>
</tr>
<tr>
<td>$t_{Re(b)}$</td>
<td>174.04</td>
<td>278.10</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.1 exhibits that the proposed ratio-cum-product type exponential estimator $\hat{t}_{RPe}^{(a_{10}, a_{20})}$ is more efficient than all the estimators $\bar{Y}$, $t_R$, $t_P$, $t_{Re}$, $t_{Pe}$, $t_{RP}$,
Ratio-cum-product type exponential estimator

It is interesting to note that the ratio-cum-product type exponential estimator \( t_{RP} \) or \( t_{RP(\alpha)} \) is more efficient than Singh’s (1967) ratio-cum-product estimator \( t_{RP} , \bar{f} , \bar{t}_R , \bar{t}_P , \bar{t}_{Re} , \bar{t}_{Pe} \) with substantial gain in efficiency in both the populations I and II. Thus the proposed ratio-cum-product type exponential estimators \( t_{RP} , t_{RP(\alpha)} , t_{RP(\alpha)} \) and \( t_{RP(\alpha \beta \gamma \delta \epsilon \zeta \eta \theta \iota \kappa \lambda \mu \nu \xi \omicron \pi \rho \sigma \tau \upsilon \phi \chi \psi \omega \alpha \beta \gamma \delta \epsilon \eta \theta \iota \kappa \lambda \mu \nu \xi \omicron \pi \rho \sigma \tau \upsilon \phi \chi \psi \omega \) are to be preferred over Singh’s (1967) estimator \( t_{RP} \) in practical research.

ACKNOWLEDGEMENT

Authors are thankful to the referee for his valuable suggestions regarding improvement of this paper. The research of the second author was supported by the grant of U.G.C., New Delhi.

REFERENCES


SUMMARY

Ratio-cum-product type exponential estimator

This paper addresses the problem of estimating the population mean \( \bar{Y} \) of the study variate \( Y \) using information on two auxiliary variables \( X_1 \) and \( X_2 \). A ratio-cum-product
type exponential estimator has been suggested and its bias and mean squared error have been derived under large sample approximation. An almost unbiased ratio-cum-product type exponential estimator has also been derived by using Jackknife technique envisaged by Quenouille (1956). A generalized version of the ratio-cum-product exponential estimator has also been given along with its properties. Numerical illustration is given in support of the present study.