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ON THE ESTIMATION OF RATIO AND PRODUCT OF TWO POPULATION MEANS USING SUPPLEMENTARY INFORMATION IN PRESENCE OF MEASUREMENT ERRORS

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1. INTRODUCTION

It is well known fact that in sample surveys supplementary information is often used for increasing the efficiency of estimators. Ratio, product and regression methods of estimation are good examples in this context. In many practical situations the estimation of the ratio (or product) of two population means may be of considerable interest, e.g. the crop production per hectare for different crops, ratio of male to female in working force, ratio of income to expenditure, the ratio of the liquid assets to total assets, profitability rate (Profit/Investment) etc. One may be interested in estimating the total value of sales from the prices and volume of sales. The theory of estimation of the ratio (or product) of two population means has been considered by Rao (1957), Singh (1965,67), Rao and Pereira (1968), Tripathi (1980), Singh (1982), Ray and Singh (1985), Upadhyaya and Singh (1985), Upadhyaya *et al.* (1985), Singh (1986, 1988), Okafor and Arnab (1987), Khare (1991), Singh *et al.* (1994 a, b), Prasad *et al.* (1996), Okafor (1992), Artes and Gracia (2001), Gracia and Artes (2002), Khare and Sinha (2004) and Garcia (2008).

The standard theory of survey sampling usually assumes that we observe "true values" when a data are collected. In reality the data may be contaminated with measurement errors. Such errors can distort the data in several ways, for example, see Sud and Srivastava (2000), Cochran (1963), Shalabh (1997) and Sahoo and Sahoo (1999). Measurement errors can result in serious misleading inferences; see Biemer *et al.* (1991).

In this paper we have considered the problem of estimation of the ratio and product of two population means using supplementary information on an auxiliary variable in the presence of measurement errors.

2. SUGGESTED ESTIMATORS FOR POPULATION RATIO AND PRODUCT OF TWO MEANS

For a simple random sample of size n, let (y_{0i}, y_{1i}, x_i) be the values instead of the true values (Y_{0i}, Y_{1i}, X_i) on three characteristics (Y_0, Y_1, X) respectively for the i^{th} $(i = 1, 2, ..., , n)$ unit in the sample. Let the observational or measurement errors be

$$u_{0i} = y_{0i} - Y_{0i}\,,\; u_{1i} = y_{1i} - Y_{1i}\,,\; v_i = x_i - X_i\,,$$

which are stochastic in nature and are uncorrelated with mean zero and variance σ^2_{U0}, σ^2_{U1} and σ^2_V respectively. Further, let the population means of (Y_0, Y_1, X) be $(\mu_{Y0}, \mu_{Y1}, \mu_X)$, population variances of (Y_0, Y_1, X) be σ^2_{Y0}, σ^2_{Y1}, and σ^2_X respectively and ρ_{01}, ρ_{0X} and ρ_{1X} be the population correlation coefficient between (Y_0 and Y_1), (Y_0 and X) and (Y_1, and X) respectively. We also assume that the measurement errors are independent of the true value of variables. It is desired to estimate population ratio $\left(\mathrm{R} = \dfrac{\mu_{Y0}}{\mu_{Y1}}, \mu_{Y1} \neq 0\right)$ and product $(P = \mu_{Y0}\mu_{Y1})$ of two population means. The conventional estimators (when the measurement errors are present) of R and P are respectively given by

$$\hat{\mathrm{R}} = \frac{\bar{y}_0}{\bar{y}_1}, \bar{y}_1 \neq 0, \tag{2.1}$$

$$\hat{P} = \bar{y}_0 \bar{y}_1\,, \tag{2.2}$$

where $\bar{y}_0 = \frac{1}{n}\sum_{i=1}^{n} y_{0i}$ *and* $\bar{y}_1 = \frac{1}{n}\sum_{i=1}^{n} y_{1i}$.

Assuming the population mean μ_X of the auxiliary variable X to be known for the estimation of population ratio R and product P, motivated by Singh (1965) we define the estimators

$$t_{1r} = \hat{\mathrm{R}}\left(\frac{\mu_X}{\bar{x}}\right) \tag{2.3}$$

$$t_{2r} = \hat{\mathrm{R}}\left(\frac{\bar{x}}{\mu_X}\right) \tag{2.4}$$

for ratio R, and

$$t_{1p} = \hat{P}\left(\frac{\mu_X}{\bar{x}}\right) \tag{2.5}$$

$$t_{2p} = \hat{P}\left(\frac{\bar{x}}{\mu_X}\right) \tag{2.6}$$

for product P, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

To obtain biases and mean squared errors of $\hat{R}$, $t_{1r}, t_{2r}, \hat{P}, t_{1p}$, and t_{2p} we introduce the following notations:

$$C_{Y0} = \sigma_{Y0}/\mu_{Y0}\,,\; C_{Y1} = \sigma_{Y1}/\mu_{Y1}\,,\; C_X = \sigma_X/\mu_X\,,\; W_{u0} = n^{-1/2}\sum_{i=1}^{n} u_{0i}\,,$$

$$W_{u1} = n^{-1/2}\sum_{i=1}^{n} u_{1i}\,,\; W_v = n^{-1/2}\sum_{i=1}^{n} v_i\,,\; W_{Y0} = n^{-1/2}\sum_{i=1}^{n}(Y_{0i} - \mu_{Y0})\,,$$

$$W_{Y1} = n^{-1/2}\sum_{i=1}^{n}(Y_{1i} - \mu_{Y1})\,,\; W_X = n^{-1/2}\sum_{i=1}^{n}(X_i - \mu_X)\,,$$

we have

$$\bar{y}_0 - \mu_{Y0} = \frac{1}{n}\sum_{i=1}^{n}[(Y_{0i} - \mu_{Y0}) + u_{0i}]$$

$$= n^{-1/2}(W_{Y0} + W_{u0}) \Rightarrow \bar{y}_0 = \mu_{Y0}(1 + \varepsilon_0)$$

$$\bar{y}_1 - \mu_{Y1} = \frac{1}{n}\sum_{i=1}^{n}[(Y_{1i} - \mu_{Y1}) + u_{1i}]$$

$$= n^{-1/2}(W_{Y1} + W_{u1}) \Rightarrow \bar{y}_1 = \mu_{Y1}(1 + \varepsilon_1)$$

$$\bar{x} - \mu_X = \frac{1}{n}\sum_{i=1}^{n}[(X_i - \mu_X) + v_i]$$

$$= n^{-1/2}(W_X + W_v) \Rightarrow \bar{x} = \mu_X(1 + \varepsilon_x)$$

where $\varepsilon_0 = \frac{(W_{Y0} - W_{u0})}{\sqrt{n}\mu_{Y0}}$, $\varepsilon_1 = \frac{(W_{Y1} - W_{u1})}{\sqrt{n}\mu_{Y1}}$, $\varepsilon_x = \frac{(W_X - W_v)}{\sqrt{n}\mu_X}$

such that $E(\varepsilon_0)=E(\varepsilon_1)=E(\varepsilon_x)=0$

and

$$\left.\begin{aligned} &E(\varepsilon_0^2)=\frac{C_{Y0}^2}{n}\left(1+\frac{\sigma_{U0}^2}{\sigma_{Y0}^2}\right), E(\varepsilon_1^2)=\frac{C_{Y1}^2}{n}\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}\right), E(\varepsilon_x^2)=\frac{C_X^2}{n}\left(1+\frac{\sigma_V^2}{\sigma_X^2}\right), \\ &E(\varepsilon_0\varepsilon_1)=K_{01}\frac{C_{Y1}^2}{n}, E(\varepsilon_0\varepsilon_x)=K_{0X}\frac{C_X^2}{n}, E(\varepsilon_1\varepsilon_x)=K_{1X}\frac{C_X^2}{n} \end{aligned}\right\}, \tag{2.7}$$

where $K_{01}=\rho_{01}\dfrac{C_{Y0}}{C_{Y1}}, K_{0X}=\rho_{01}\dfrac{C_{Y0}}{C_X}, K_{1X}=\rho_{1X}\dfrac{C_{Y1}}{C_X}$.

Expressing $\hat{\mathrm{R}}$, $t_{1r}, t_{2r}, \hat{P}, t_{1P}$, *and* t_{2P} in terms of $\varepsilon's$ we have

$$\hat{\mathrm{R}}=\frac{\mu_{Y0}(1+\varepsilon_0)}{\mu_{Y1}(1+\varepsilon_1)}=\mathrm{R}(1+\varepsilon_0)(1+\varepsilon_1)^{-1} \tag{2.8}$$

$$t_{1r}=\mathrm{R}(1+\varepsilon_0)(1+\varepsilon_1)^{-1}(1+\varepsilon_x)^{-1} \tag{2.9}$$

$$t_{2r}=\mathrm{R}(1+\varepsilon_0)(1+\varepsilon_1)^{-1}(1+\varepsilon_x) \tag{2.10}$$

$$\hat{P}=P(1+\varepsilon_0)(1+\varepsilon_1) \tag{2.11}$$

$$t_{1P}=P(1+\varepsilon_0)(1+\varepsilon_1)(1+\varepsilon_x)^{-1} \tag{2.12}$$

and

$$t_{2P}=P(1+\varepsilon_0)(1+\varepsilon_1)(1+\varepsilon_x) \tag{2.13}$$

It is assumed that the sample size is so large as to make $|\varepsilon_1|$ and $|\varepsilon_x|$ small ($i.e.$ $|\varepsilon_1|<1$ and $|\varepsilon_x|<1$) justifying the first degree of approximation wherein we ignore the terms involving $\varepsilon_i's\,(i=0,1)$ and /or ε_x in a degree greater than two. Thus expanding the right hand sides of (2.8)-(2.13), multiplying out and neglecting terms of having power greater than two, we have

$$\hat{\mathrm{R}}=\mathrm{R}(1+\varepsilon_0-\varepsilon_1-\varepsilon_0\varepsilon_1+\varepsilon_1^2) \tag{2.14}$$

$$t_{1r}=\mathrm{R}(1+\varepsilon_0-\varepsilon_1-\varepsilon_x-\varepsilon_0\varepsilon_1-\varepsilon_0\varepsilon_x+\varepsilon_1\varepsilon_x+\varepsilon_1^2+\varepsilon_x^2) \tag{2.15}$$

$$t_{2r}=\mathrm{R}(1+\varepsilon_0-\varepsilon_1+\varepsilon_x-\varepsilon_0\varepsilon_1+\varepsilon_0\varepsilon_x-\varepsilon_1\varepsilon_x+\varepsilon_1^2) \tag{2.16}$$

$$\hat{P} = P(1+\varepsilon_0+\varepsilon_1+\varepsilon_0\varepsilon_1) \tag{2.17}$$

$$t_{1P} = P(1+\varepsilon_0+\varepsilon_1-\varepsilon_x+\varepsilon_0\varepsilon_1-\varepsilon_0\varepsilon_x-\varepsilon_1\varepsilon_x+\varepsilon_x^2) \tag{2.18}$$

$$t_{2P} = P(1+\varepsilon_0+\varepsilon_1+\varepsilon_x+\varepsilon_0\varepsilon_1+\varepsilon_0\varepsilon_x+\varepsilon_1\varepsilon_x) \tag{2.19}$$

We further write (2.14)-(2.19) as

$$(\hat{\mathrm{R}}-\mathrm{R}) = \mathrm{R}(\varepsilon_0-\varepsilon_1-\varepsilon_0\varepsilon_1+\varepsilon_1^2) \tag{2.20}$$

$$(t_{1r}-\mathrm{R}) = \mathrm{R}(\varepsilon_0-\varepsilon_1-\varepsilon_x-\varepsilon_0\varepsilon_1-\varepsilon_0\varepsilon_x+\varepsilon_1\varepsilon_x+\varepsilon_1^2+\varepsilon_x^2) \tag{2.21}$$

$$(t_{2r}-\mathrm{R}) = \mathrm{R}(\varepsilon_0-\varepsilon_1+\varepsilon_x-\varepsilon_0\varepsilon_1+\varepsilon_0\varepsilon_x-\varepsilon_1\varepsilon_x+\varepsilon_1^2) \tag{2.22}$$

$$(\hat{P}-P) = P(\varepsilon_0+\varepsilon_1+\varepsilon_0\varepsilon_1) \tag{2.23}$$

$$(t_{1P}-P) = P(\varepsilon_0+\varepsilon_1-\varepsilon_x+\varepsilon_0\varepsilon_1-\varepsilon_0\varepsilon_x-\varepsilon_1\varepsilon_x+\varepsilon_x^2) \tag{2.24}$$

$$(t_{2P}-P) = P(\varepsilon_0+\varepsilon_1+\varepsilon_x+\varepsilon_0\varepsilon_1+\varepsilon_0\varepsilon_x+\varepsilon_1\varepsilon_x) \tag{2.25}$$

Taking expectations of both sides of (2.20)-(2.25) we get biases of $\hat{\mathrm{R}}$, $t_{1r}, t_{2r}, \hat{P}, t_{1P}$ and t_{2P} to the first degree of approximation, respectively as

$$B(\hat{\mathrm{R}}) = \left(\frac{\mathrm{R}}{n}\right)C_{Y1}^2\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right) \tag{2.26}$$

$$B(t_{1r}) = B(\hat{\mathrm{R}})+\left(\frac{RC_X^2}{n}\right)\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0\right) \tag{2.27}$$

$$B(t_{2r}) = B(\hat{\mathrm{R}})+\left(\frac{RC_X^2}{n}\right)d_0 \tag{2.28}$$

$$B(\hat{P}) = \left(\frac{P}{n}\right)C_{Y1}^2K_{01} \tag{2.29}$$

$$B(t_{1P}) = B(\hat{P})+\left(\frac{PC_X^2}{n}\right)\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0^*\right) \tag{2.30}$$

$$B(t_{2P}) = B(\hat{P})+\left(\frac{PC_X^2}{n}\right)d_0^* \tag{2.31}$$

where $d_0 = (K_{0X}-K_{1X}), d_0^* = (K_{0X}+K_{1X})$.

Squaring both sides of (2.20)-(2.25) and neglecting terms of $\varepsilon's$ having power greater than two we have

$$(\hat{R}-R)^2 = R^2(\varepsilon_0^2+\varepsilon_1^2-2\varepsilon_0\varepsilon_1) \tag{2.32}$$

$$(t_{1r}-R)^2 = R^2(\varepsilon_0^2+\varepsilon_1^2+\varepsilon_x^2-2\varepsilon_0\varepsilon_1-2\varepsilon_0\varepsilon_x+2\varepsilon_1\varepsilon_x) \tag{2.33}$$

$$(t_{2r}-R)^2 = R^2(\varepsilon_0^2+\varepsilon_1^2+\varepsilon_x^2-2\varepsilon_0\varepsilon_1+2\varepsilon_0\varepsilon_x-2\varepsilon_1\varepsilon_x) \tag{2.34}$$

$$(\hat{P}-P)^2 = P^2(\varepsilon_0^2+\varepsilon_1^2+2\varepsilon_0\varepsilon_1) \tag{2.35}$$

$$(t_{1P}-P)^2 = P^2(\varepsilon_0^2+\varepsilon_1^2-\varepsilon_x^2+2\varepsilon_0\varepsilon_1-2\varepsilon_0\varepsilon_x-2\varepsilon_1\varepsilon_x) \tag{2.36}$$

$$(t_{2P}-P)^2 = P^2(\varepsilon_0^2+\varepsilon_1^2+\varepsilon_x^2+2\varepsilon_0\varepsilon_1+2\varepsilon_0\varepsilon_x+2\varepsilon_1\varepsilon_x) \tag{2.37}$$

Taking expectations of both sides of (2.32)-(2.37) and using the results cited in (2.7) we get the mean squared errors of $\hat{R}$, $t_{1r}, t_{2r}, \hat{P}, t_{1P}$ and t_{2P} to the first degree of approximation as

$$MSE(\hat{R}) = \left(\frac{R^2}{n}\right)\left[C_{Y0}^2\left(1+\frac{\sigma_{U0}^2}{\sigma_{Y0}^2}\right)+C_{Y1}^2\left(1-2K_{01}+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}\right)\right] \tag{2.38}$$

$$MSE(t_{1r}) = MSE(\hat{R})+\left(\frac{R^2C_X^2}{n}\right)\left(1-2d_0+\frac{\sigma_V^2}{\sigma_X^2}\right) \tag{2.39}$$

$$MSE(t_{2r}) = MSE(\hat{R})+\left(\frac{R^2C_X^2}{n}\right)\left(1+2d_0+\frac{\sigma_V^2}{\sigma_X^2}\right) \tag{2.40}$$

$$MSE(\hat{P}) = \left(\frac{P^2}{n}\right)\left[C_{Y0}^2\left(1+\frac{\sigma_{U0}^2}{\sigma_{Y0}^2}\right)+C_{Y1}^2\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}+2K_{01}\right)\right] \tag{2.41}$$

$$MSE(t_{1P}) = MSE(\hat{P})+\left(\frac{P^2C_X^2}{n}\right)\left(1-2d_0^*+\frac{\sigma_V^2}{\sigma_X^2}\right) \tag{2.42}$$

$$MSE(t_{2p}) = MSE(\hat{P})+\left(\frac{P^2C_X^2}{n}\right)\left(1+2d_0^*+\frac{\sigma_V^2}{\sigma_X^2}\right) \tag{2.43}$$

3. EFFICIENCY COMPARISON

From (2.26), (2.27), (2.28), (2.29), (2.30), (2.30) and (2.31) it is observed that

(i) the biases of $\hat{R}$ and t_{2r} are only affected by the measurement errors in study variable Y_1.

(ii) the measurement errors in study variable Y_1 as well as in auxiliary variable X have influence over the bias of the estimator t_{1r}.

(iii) the biases of $\hat{P}$ and t_{2P} are not affected by the measurement errors in study variable Y_1 as well as auxiliary variable X.

(iv) the bias of t_{1P} is affected only by the measurement errors in auxiliary variable X.

Examining the mean squared errors expressions (2.38)-(2.43) we observe that sampling variability in each case inflates when measurement errors are present. It is interesting to mention that the increase in variability attributable to measurement errors are small in case of $\hat{R}$ and $\hat{P}$ when compared with that of t_{1r}, t_{2r}, t_{1P}, and t_{2P}.

From (2.38)-(2.43) we note that

(i) $MSE(t_{1r}) < MSE(\hat{R})$ if

$$(K_{0X} - K_{1X}) > \frac{1}{2}\left(1 + \frac{\sigma_V^2}{\sigma_X^2}\right) \tag{3.1}$$

(ii) $MSE(t_{2r}) < MSE(\hat{R})$ if

$$(K_{0X} - K_{1X}) < -\frac{1}{2}\left(1 + \frac{\sigma_V^2}{\sigma_X^2}\right) \tag{3.2}$$

(iii) $MSE(t_{1P}) < MSE(\hat{P})$ if

$$(K_{0X} + K_{1X}) > \frac{1}{2}\left(1 + \frac{\sigma_V^2}{\sigma_X^2}\right) \tag{3.3}$$

(iv) $MSE(t_{2P}) < MSE(\hat{P})$ if

$$(K_{0X} + K_{1X}) < -\frac{1}{2}\left(1 + \frac{\sigma_V^2}{\sigma_X^2}\right) \tag{3.4}$$

In particular case where C_{Y0}, C_{Y1} and C_X are identical in magnitude, these conditions respectively reduce to the following

$$(\rho_{0X} - \rho_{1X}) > \frac{1}{2}\left(1 + \frac{\sigma_V^2}{\sigma_X^2}\right) \tag{3.5}$$

$$(\rho_{0X} - \rho_{1X}) < -\frac{1}{2}\left(1 + \frac{\sigma_V^2}{\sigma_X^2}\right) \tag{3.6}$$

$$(\rho_{0X} + \rho_{1X}) > \frac{1}{2}\left(1 + \frac{\sigma_V^2}{\sigma_X^2}\right) \tag{3.7}$$

$$(\rho_{0X} + \rho_{1X}) < -\frac{1}{2}\left(1 + \frac{\sigma_V^2}{\sigma_X^2}\right) \tag{3.8}$$

The above conditions (3.5)-(3.8) will not be satisfied if σ_V^2 exceeds σ_X^2. In other words, if the auxiliary variate is poorly measured that error variance σ_V^2 is greater than σ_X^2, then also $(t_{ir}, t_{ip}, i = 1, 2)$ are dominated by conventional estimators $\hat{\mathrm{R}}$ and $\hat{P}$ respectively besides the inequalities (3.5), (3.6), (3.7) and (3.8) satisfied with opposite signs.

We further note that the measurement errors in auxiliary variate X may change the preference ordering of $(\hat{\mathrm{R}}, \hat{P})$ and $(t_{ir}, t_{ip}, i = 1, 2)$ obtained under the supposition of absence of measurement errors. However, it is interesting to mention that the measurement errors in study variates (Y_0, Y_1) have no role to play in this kind of preference ordering (see, Shalabh (1997), p. 154).

4. COMBINED ESTIMATORS

Combining the estimators (t_{1r} with $\hat{\mathrm{R}}$), (t_{2r} with $\hat{\mathrm{R}}$), (t_{1P} with $\hat{P}$), (t_{2P} with $\hat{P}$) we define the following combined estimators:

$$t_\theta = \theta t_{1r} + (1 - \theta)\hat{\mathrm{R}} \tag{4.1}$$

$$t_\gamma = \gamma t_{2r} + (1 - \gamma)\hat{\mathrm{R}} \tag{4.2}$$

for ratio R, and

$$t_\eta = \eta t_{1P} + (1-\eta)\hat{P} \tag{4.3}$$

$$t_\delta = \delta t_{2P} + (1-\delta)\hat{P} \tag{4.4}$$

for product P, where θ, γ, η and δ are suitably chosen characterizing scalars. Following the method of derivation in section 2, we have expressions for bias and MSE of the estimators $t_\theta, t_\gamma, t_\eta$, and t_δ to the first degree of approximation, respectively as

$$B(t_\theta) = \left[B(\hat{R}) + \left(\frac{RC_X^2}{n}\right)\theta\left(1 - d_0 + \frac{\sigma_V^2}{\sigma_X^2}\right)\right] \tag{4.5}$$

$$B(t_\gamma) = \left[B(\hat{R}) + \left(\frac{RC_X^2}{n}\right)\gamma\, d_0 \right] \tag{4.6}$$

$$B(t_\eta) = \left[B(\hat{P}) + \left(\frac{PC_X^2}{n}\right)\eta\left(1 + \frac{\sigma_V^2}{\sigma_X^2} - d_0^*\right)\right] \tag{4.7}$$

$$B(t_\delta) = \left[B(\hat{P}) + \left(\frac{PC_X^2}{n}\right)\delta\, d_0^* \right] \tag{4.8}$$

$$MSE(t_\theta) = MSE(\hat{R}) + \left(\frac{R^2C_X^2\theta}{n}\right)\left[\theta\left\{1 + \frac{\sigma_V^2}{\sigma_X^2}\right\} - 2d_0\right] \tag{4.9}$$

$$MSE(t_\gamma) = MSE(\hat{R}) + \left(\frac{R^2C_X^2\gamma}{n}\right)\left[\gamma\left\{1 + \frac{\sigma_V^2}{\sigma_X^2}\right\} + 2d_0\right] \tag{4.10}$$

$$MSE(t_\eta) = MSE(\hat{P}) + \left(\frac{P^2C_X^2\eta}{n}\right)\left[\eta\left\{1 + \frac{\sigma_V^2}{\sigma_X^2}\right\} - 2d_0^*\right] \tag{4.11}$$

$$MSE(t_\delta) = MSE(\hat{P}) + \left(\frac{P^2C_X^2\delta}{n}\right)\left[\delta\left\{1 + \frac{\sigma_V^2}{\sigma_X^2}\right\} + 2d_0^*\right] \tag{4.12}$$

4.1 *Bias Comparisons*

It follows from (2.26), (2.27), (2.28), (2.29), (4.5), (4.6), (4.7) and (4.8) that

(i) $\left|B(t_\theta)\right| < \left|B(\hat{\mathrm{R}})\right|$ *if*

$$\min\left\{0, -\frac{2C_{Y1}^2\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right)}{C_X^2\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0\right)}\right\} < \theta < \max\left\{0, -\frac{2C_{Y1}^2\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right)}{C_x^2\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0\right)}\right\} \quad (4.13)$$

(ii) $\left|B(t_\gamma)\right| < \left|B(\hat{\mathrm{R}})\right|$ *if*

$$\min\left\{0, -\frac{2C_{Y1}^2}{C_X^2 d_0}\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right)\right\} < \gamma < \max\left\{0, -\frac{2C_{Y1}^2}{C_X^2 d_0}\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right)\right\} \quad (4.14)$$

(iii) $\left|B(t_\eta)\right| < \left|B(\hat{P})\right|$ *if*

$$\min\left\{0, -\frac{2C_{Y1}^2 K_{01}}{C_X^2(1-d_0^*)}\right\} < \eta < \max\left\{0, -\frac{2C_{Y1}^2 K_{01}}{C_X^2(1-d_0^*)}\right\} \quad (4.15)$$

(iv) $\left|B(t_\delta)\right| < \left|B(\hat{P})\right|$ *if*

$$\min\left\{0, -\frac{2C_{Y1}^2 K_{01}}{C_X^2 K*}\right\} < \delta < \max\left\{0, -\frac{2C_{Y1}^2 K_{01}}{C_X^2 K^*}\right\} \quad (4.16)$$

(v) $\left|B(t_\theta)\right| < \left|B(t_{1r})\right|$ *if*

$$\min\left\{1, -1-\frac{2C_{Y1}^2\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right)}{C_X^2\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0\right)}\right\} < \theta < \max\left\{1, -1-\frac{2C_{Y1}^2\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right)}{C_X^2\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0\right)}\right\} \quad (4.17)$$

(vi) $\left|B(t_\gamma)\right| < \left|B(t_{2r})\right|$ *if*

$$\min\left\{1, -1-\frac{2C_{Y1}^2\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right)}{C_X^2\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0\right)}\right\} < \gamma < \max\left\{1, -1-\frac{2C_{Y1}^2\left(1+\frac{\sigma_{U1}^2}{\sigma_{Y1}^2}-K_{01}\right)}{C_X^2\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0\right)}\right\} \quad (4.18)$$

(vii) $\left|B(t_\eta)\right| < \left|B(t_{1P})\right|$ *if*

$$\min\left\{1,-1-\frac{2K_{01}C_{Y1}^2}{C_X^2\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0^*\right)}\right\}<\eta<\min\left\{1,-1-\frac{2K_{01}C_{Y1}^2}{C_X^2\left(1+\frac{\sigma_V^2}{\sigma_X^2}-d_0^*\right)}\right\} \tag{4.19}$$

(viii) $\left|B(t_\delta)\right| < \left|B(t_{2P})\right|$ *if*

$$\min\left\{1,-1-\frac{2K_{01}C_{Y1}^2}{d_0^*C_X^2}\right\}<\delta<\max\left\{1,-1-\frac{2K_{01}C_{Y1}^2}{d_0^*C_X^2}\right\} \tag{4.20}$$

4.2 *Efficiency Comparisons*

From (2.39), (2.40), (2.41), (2.42), (2.43), (2.44), (4.9), (4.10), (4.11) and (4.12)

(i) $MSE(t_\theta) < MSE(\hat{R})$ *if*

$$\left.\begin{array}{l} \textit{either } 0<\theta<2d_0\left(\dfrac{\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right) \\ \textit{or } 2d_0\left(\dfrac{\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)<\theta<0 \end{array}\right\}$$

or equivalently

$$\min.\left\{0,\ 2d_0\left(\frac{\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right\}<\theta<\max.\left\{0,\ 2d_0\left(\frac{\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right\} \tag{4.21}$$

(ii) $MSE(t_\theta) < MSE(t_{1r})$ *if*

$$\left.\begin{array}{l} \textit{either } 1<\theta<\left(\dfrac{2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}-1\right) \\ \textit{or}\left(\dfrac{2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}-1\right)<\theta<1 \end{array}\right\}$$

or equivalently

$$\min\left\{1,\left(\frac{2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}-1\right)\right\}<\theta<\max\left\{1,\left(\frac{2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}-1\right)\right\} \tag{4.22}$$

(iii) $MSE(t_\gamma) < MSE(\hat{R})$ *if*

$$\left.\begin{array}{l} either\ 0<\gamma<\left(\dfrac{-2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right) \\ or\left(\dfrac{-2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)<\gamma<0 \end{array}\right\}$$

or equivalently

$$\min\left\{0,\left(\frac{-2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right\}<\gamma<\max\left\{0,\left(\frac{-2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right\} \tag{4.23}$$

(iv) $MSE(t_\gamma) < MSE(t_{2r})$ *if*

$$\left.\begin{array}{l} either\ 1<\gamma<\left(\dfrac{-2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}-1\right) \\ or\left(\dfrac{-2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}-1\right)<\gamma<0 \end{array}\right\}$$

or equivalently

$$\min\left\{1,\left(\frac{-2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}-1\right)\right\}<\gamma<\max\left\{1,\left(\frac{-2d_0\sigma_X^2}{\sigma_X^2+\sigma_V^2}-1\right)\right\} \tag{4.24}$$

(v) $MSE(t_\eta) < MSE(\hat{P})$ *if*

$$\left.\begin{array}{l} either\ 0<\eta<\left(\dfrac{-2d_0^*\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right) \\ or\left(\dfrac{-2d_0^*\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)<\eta<0 \end{array}\right\}$$

or equivalently

$$\min\left\{0,\left(\frac{-2d_0^*\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right\}<\eta<\max\left\{0,\left(\frac{-2d_0^*\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right\} \tag{4.25}$$

(vi) $MSE(t_\eta) < MSE(t_{1P})$ *if*

$$\left.\begin{array}{l} \textit{either } 1 < \eta < \left(\dfrac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2} - 1\right) \\ or \left(\dfrac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2} - 1\right) < \eta < 1 \end{array}\right\}$$

or equivalently

$$\min\left\{1, \left(\frac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2} - 1\right)\right\} < \eta < \max\left\{1, \left(\frac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2} - 1\right)\right\} \tag{4.26}$$

(vii) $MSE(t_\delta) < MSE(\hat{P})$ *if*

$$\left.\begin{array}{l} \textit{either } 0 < \delta < \left(\dfrac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2}\right) \\ or \left(\dfrac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2}\right) < \delta < 0 \end{array}\right\}$$

or equivalently

$$\min\left\{0, \left(\frac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2}\right)\right\} < \delta < \max\left\{0, \left(\frac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2}\right)\right\} \tag{4.27}$$

(viii) $MSE(t_\delta) < MSE(t_{2P})$ *if*

$$\left.\begin{array}{l} \textit{either } 1 < \delta < \left(\dfrac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2} - 1\right) \\ or \left(\dfrac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2} - 1\right) < \delta < 0 \end{array}\right\}$$

or equivalently

$$\min\left\{1, \left(\frac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2} - 1\right)\right\} < \delta < \max\left\{1, \left(\frac{-2d_0^*\sigma_X^2}{\sigma_X^2 + \sigma_V^2} - 1\right)\right\} \tag{4.28}$$

4.3 *Asymptotically Optimum Estimators (AOE's)*

Minimizing the mean squared error expressions of $t_\theta, t_\gamma, t_\eta$ *and* t_δ given by (4.9), (4.10), (4.11) and (4.12) respectively, we get the optimum values of constants θ, γ, η and δ as

$$\theta = \frac{d_0 \sigma_X^2}{(\sigma_X^2 + \sigma_V^2)} = \theta_0 (say) \tag{4.29}$$

$$\gamma = -\frac{d_0 \sigma_X^2}{(\sigma_X^2 + \sigma_V^2)} = \gamma_0 (say) \tag{4.30}$$

$$\eta = \frac{d_0^* \sigma_X^2}{(\sigma_X^2 + \sigma_V^2)} = \eta_0 (say) \tag{4.31}$$

$$\delta = -\frac{d_0^* \sigma_X^2}{(\sigma_X^2 + \sigma_V^2)} = \delta_0 (say) \tag{4.32}$$

Thus the resulting asymptotically optimum estimators (AOE's) in the classes $t_\theta, t_\gamma, t_\eta$ *and* t_δ are respectively given by

$$t_{\theta_0} = \theta_0 t_{1r} + (1-\theta_0)\hat{\mathrm{R}} \tag{4.33}$$

$$t_{\gamma_0} = \gamma_0 t_{2r} + (1-\gamma_0)\hat{\mathrm{R}} \tag{4.34}$$

$$t_{\eta_0} = \eta_0 t_{1r} + (1-\eta_0)\hat{P} \tag{4.35}$$

$$t_{\delta_0} = \delta_0 t_{2r} + (1-\delta_0)\hat{P} \tag{4.36}$$

Putting $\theta_0, \gamma_0, \eta_0$ *and* δ_0 respectively from (4.29), (4.30), (4.31) and (4.32) in (4.9), (4.10), (4.11) and (4.12) we get the resulting minimum mean squared errors of $t_\theta, t_\gamma, t_\eta$ *and* t_δ (or the mean squared error of AOE's $t_{\theta_0}, t_{\gamma_0}, t_{\eta_0}$ *and* t_{δ_0}) respectively as

$$\min MSE(t_\theta) = MSE(\hat{\mathrm{R}}) - \left(\frac{\mathrm{R}^2 C_X^2}{n}\right) \frac{d_0^2 \sigma_X^2}{(\sigma_X^2 + \sigma_V^2)}$$

$$= \min MSE(t_\gamma) = MSE(t_{\theta_0}) = MSE(t_{\gamma_0}) \tag{4.37}$$

$$\min MSE(t_\eta) = MSE(\hat{P}) - \left(\frac{P^2 C_X^2}{n}\right)\frac{d_0^{*^2}\sigma_X^2}{(\sigma_X^2+\sigma_V^2)}$$

$$= \min MSE(t_\delta) = MSE(t_{\eta_0}) = MSE(t_{\delta_0}) \tag{4.38}$$

From (2.38), (2.39), (2.40), (2.41), (2.42), (2.43) and (4.37), (4.37) it can be shown that the proposed classes of estimators $t_\theta, t_\gamma, t_\eta$ *and* t_δ (or AOE's $t_{\theta_0}, t_{\gamma_0}, t_{\eta_0}$ *and* t_{δ_0}) are better than $\hat{R}, \hat{P}, t_{1r}, t_{1p}, t_{2p}$ *and* t_{2r} at their optimum conditions.

Remark 4.1

It is to be mentioned that the asymptotically optimum estimators (AOE's) $t_{\theta_0}, t_{\gamma_0}, t_{\eta_0}$ *and* t_{δ_0} depend on the optimum values $(\theta_0, \gamma_0, \eta_0, \delta_0)$ respectively of constants $(\theta, \gamma, \eta, \delta)$, which are function of unknown parameters $d_0 = (K_{0X} - K_{1X})$, $d_0^* = (K_{0X} + K_{1X})$, reliability ratio $r = \sigma_X^2 / (\sigma_X^2 + \sigma_V^2)$, $\rho_{0X}, \rho_{1X}, C_{Y0}, C_{Y1}$ and C_X. The AOE's $t_{\theta_0}, t_{\gamma_0}, t_{\eta_0}$ *and* t_{δ_0} can be used in practice only when the values of K_{0X}, K_{1X} and *r* are known in advance. But specifying precisely the values of K_{0X}, K_{1X} and *r* are difficult in practice. However, in repeated surveys or studies based on multiphase sampling, when information regarding the same variates is collected on several occasions, it is possible to guess (or estimate) quite precisely the values of certain parameters $\rho_{iX}, C_{Yi}\ (i = 0,1)$ and C_X for instance, see Murthy (1967, pp. 96-99), Reddy (1978), Srivenkataramana and Tracy (1980) and Singh and Ruiz Espejo (2003). In many scientific investigations, the value of the reliability ratio *r* is correctly known. Such knowledge may arise from some theoretical considerations or from empirical experience. Or a reasonably accurate estimate of *r* may be available from some other independent studies, see Srivastava and Shalabh (1996). Moreover, the reliability ratio is not difficult to find in many applications and there are various methods to do it; see e.g. Ashley and Vaughan (1986), Lord and Novik (1968) and Marquis *et al.* (1986). Thus the value of K_{0X}, K_{1X} and $r = \sigma_X^2 / (\sigma_X^2 + \sigma_V^2)$ can be guessed (or estimated) precisely, and thus the estimators $t_{\theta_0}, t_{\gamma_0}, t_{\eta_0}$ *and* t_{δ_0} can be used in practice.

5. CONCLUDING REMARKS

(i) If we set $\theta = \gamma = \eta = \delta = 0$ and $Y_1 \equiv 1$ (*i.e.* study variate Y_1 takes value unity only) in (4.1) – (4.4), $t_\theta, t_\gamma, t_\eta$ and t_δ reduce to the usual unbiased estimator

$$t_0 = \bar{y}_0 \tag{5.1}$$

of the population mean μ_{Y0}, while for $\theta = \gamma = \eta = \delta = 1$ and $Y_1 \equiv 1$, the estimators t_θ *and* t_η reduce to conventional ratio estimator for population mean μ_{Y0} as

$$t_R = \bar{y}_0 \left(\frac{\mu_X}{\bar{x}} \right) \tag{5.2}$$

and the estimators t_γ and t_δ reduce to the usual product estimator

$$t_P = \bar{y}_0 \left(\frac{\bar{x}}{\mu_X} \right) \tag{5.3}$$

of the population mean μ_{Y0}.

Thus we infer that the results obtained by Shalabh (1997) are the special case of the present study for $\theta = \gamma = \eta = \delta = 0,1$ and $Y_1 \equiv 1$ when the observations are subject to measurement errors.

(ii) For $Y_1 \equiv 1$ (i.e. the study variate Y_1 takes value unity only) and $\theta = \gamma = \eta = \delta = \theta^*$ (say) the estimators $t_\theta, t_\gamma, t_\eta$ and t_δ reduce to the estimator.

$$t_{\theta^*} = \theta^* t_R + (1 - \theta^*) \bar{y}_0 \tag{5.4}$$

of the population mean μ_{Y0} and θ^* is the characterizing scalar to be chosen suitably. It is to be mentioned that the estimator t_{θ^*} is due to Manisha and Singh (2001). Thus the work of Manisha and Singh (2001) are special case of the present investigation when the observations are subject to measurement errors.

(iii) In practice, the usual expression for the mean squared error of $\hat{R}, (\hat{P})$ is

$$(R^2/n)[C_{Y0}^2 + C_{Y1}^2(1 - 2K_{01})]\ ((P^2/n)[C_{Y0}^2 + C_{Y1}^2(1 + 2K_{01})]).$$

From (2.38) ((2.41)) it is observed that the use of

$$(R^2/n)[C_{Y0}^2 + C_{Y1}^2(1 - 2K_{01})]\ ((P^2/n)[C_{Y0}^2 + C_{Y1}^2(1 + 2K_{01})])$$

will lead to an under-reporting of true standard error. Similar is the case when we verify the formulae (2.39)((2.42)) and (2.40)((2.43)) for the mean squared errors of $t_{1r}(t_{1p})$ and $t_{2r}(t_{2p})$ to our order of approximation. It is interesting to mention that the under-reporting in case of $\hat{R}(\hat{P})$ is an amount

$$\frac{R^2}{n}\left[\frac{\sigma_{U0}^2}{\mu_{Y0}^2}+\frac{\sigma_{U1}^2}{\mu_{Y1}^2}\right]\left(\frac{P^2}{n}\left[\frac{\sigma_{U0}^2}{\mu_{Y0}^2}+\frac{\sigma_{U1}^2}{\mu_{Y1}^2}\right]\right)$$

which is smaller than the corresponding quantity

$$\frac{R^2}{n}\left[\frac{\sigma_{U0}^2}{\mu_{Y0}^2}+\frac{\sigma_{U1}^2}{\mu_{Y1}^2}+\frac{\sigma_{V}^2}{\mu_{X}^2}\right]\left(\frac{P^2}{n}\left[\frac{\sigma_{U0}^2}{\mu_{Y0}^2}+\frac{\sigma_{U1}^2}{\mu_{Y1}^2}+\frac{\sigma_{V}^2}{\mu_{X}^2}\right]\right)$$

for $t_{ir}(t_{ip})$ (i=1,2).

As reported by Shalabh (1997) the consequence of under-reporting of variability can be clearly appreciated. For example, it may mislead the practitioner about the precision of the estimate. It may give shorter but incorrect confidence intervals for the population ratio(product) $R(P)$.
(iv) From (4.9)-(4.12) we have,

$$MSE(t_\theta) < MSE(\hat{R})\, if\, (K_{0X} - K_{1X}) > \frac{\theta}{2}\left(1+\frac{\sigma_V^2}{\sigma_X^2}\right), \theta > 0 \tag{5.5}$$

$$MSE(t_\gamma) < MSE(\hat{R})\, if\, (K_{0X} - K_{1X}) < -\frac{\gamma}{2}\left(1+\frac{\sigma_V^2}{\sigma_X^2}\right), \gamma > 0 \tag{5.6}$$

$$MSE(t_\eta) < MSE(\hat{P})\, if\, (K_{0X} + K_{1X}) > \frac{\eta}{2}\left(1+\frac{\sigma_V^2}{\sigma_X^2}\right), \eta > 0 \tag{5.7}$$

and

$$MSE(t_\delta) < MSE(\hat{P})\, if\, (K_{0X} + K_{1X}) < -\frac{\delta}{2}\left(1+\frac{\sigma_V^2}{\sigma_X^2}\right), \delta > 0 \tag{5.8}$$

For $0 < a < 1$, $(a = \theta, \gamma, \eta, \delta)$ comparing (3.1) {(3.2), (3.3), (3.4)} and (5.5) {(5.6), (5.7), (5.8)} it is the observed that the efficiency condition (5.1) for $t_\theta\{t_\gamma, t_\eta, t_\delta\}$ to be more efficient than the usual estimator $\hat{R}\{\hat{R}, \hat{P}, \hat{P}\}$ is wider than the efficiency condition (3.1) {(3.2), (3.3), (3.4)} for the estimator $t_{1r}\{t_{2r}, t_{1p}, t_{2p}\}$ to be more efficient than the conventional estimators $\hat{R}\{\hat{R}, \hat{P}, \hat{P}\}$. Thus in the extended range of the efficiency condition (5.5) {(5.6), (5.7), (5.8)} over that of the efficiency condition (3.1) {(3.2), (3.3), (3.4)}, the estimator $t_\theta\{t_\gamma, t_\eta, t_\delta\}$ is better than both the estimators $(t_{1r}\ and\ \hat{R})\{(t_{2r}\ and\ \hat{R}), (t_{1p}\ and\ \hat{P}), (t_{2p}\ and\ \hat{P})\}$.

(v) The asymptotically optimum estimators (AOE's) $t_{\theta_0}, t_{\gamma_0}, t_{\eta_0}\ and\ t_{\delta_0}$ require prior knowledge about K_{0X}, K_{1X} and the reliability ratio $r = \sigma_X^2 / (\sigma_X^2 + \sigma_V^2)$ in advance. In practice it may be hard to obtain the exact values of K_{0X}, K_{1X} and r as these contain unknown parameters $\rho_{0X}, \rho_{1X}, C_{Y0}, C_{Y1}$, C_X , σ_X^2 and σ_V^2 . In practical samples surveys, prior guessed (or estimates) of $\rho_{0X}, \rho_{1X}, C_{Y0}, C_{Y1}$, C_X , σ_X^2 and σ_V^2 and hence K_{0X}, K_{1X} and r can be obtained with reasonable accuracy either from a pilot survey, or past data, or experience, or even from expert guesses by specialist in the field concerned, see Singh and Ruiz Espejo (2003). However in some practical situations it may not be possible at all to have good estimates (or guessed) values of the parameters $\rho_{0X}, \rho_{1X}, C_{Y0}, C_{Y1}, C_X$, and σ_X^2 , and in such situations it is worth advisable to replace these parameters by their estimates based on the sample data at hand. We note that usually practitioner has the prior information quite accurately about the error variance σ_V^2 associated with auxiliary variable X, see Schneeweiss (1976) and Srivastava and Shalabh (1997). Thus we assume that the variance σ_V^2 associated with auxiliary variable X is known. These lead authors to define estimators based on 'estimated optimum' as

$$\hat{t}_{\theta_0} = \hat{\theta}_0 t_{1r} + (1 - \hat{\theta}_0)\hat{\mathrm{R}} \tag{5.9}$$

$$\hat{t}_{\gamma_0} = \hat{\gamma}_0 t_{2r} + (1 - \hat{\gamma}_0)\hat{\mathrm{R}} \tag{5.10}$$

$$\hat{t}_{\eta_0} = \hat{\eta}_0 t_{1r} + (1 - \hat{\eta}_0)\hat{P} \tag{5.11}$$

and

$$\hat{t}_{\delta_0} = \hat{\delta}_0 t_{2r} + (1 - \hat{\delta}_0)\hat{P} , \tag{5.12}$$

where

$$\hat{\theta}_0 = \frac{\hat{d}_0 s_x^2}{(s_x^2 + \sigma_V^2)} \qquad \hat{\gamma}_0 = -\frac{\hat{d}_0 s_x^2}{(s_x^2 + \sigma_V^2)}$$

$$\hat{\eta}_0 = \frac{\hat{d}_0^* s_x^2}{(s_x^2 + \sigma_V^2)} \qquad \hat{\delta}_0 = -\frac{\hat{d}_0^* s_x^2}{(s_x^2 + \sigma_V^2)}$$

where $\hat{d}_0 = (\hat{K}_{0X} - \hat{K}_{1X}), \hat{d}_0^* = (\hat{K}_{0X} + \hat{K}_{1X}), \hat{K}_{0X} = \frac{s_{xy0}}{s_x^2}\left(\frac{\bar{X}}{\bar{y}_0}\right), \hat{K}_{1X} = \frac{s_{xy1}}{s_x^2}\left(\frac{\bar{X}}{\bar{y}_1}\right)$

$s_{xy0}=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(y_{0i}-\bar{y}_0), s_{xy1}=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})(y_{1i}-\bar{y}_1)$, and the error variance σ_V^2 associated with auxiliary variable X and the population mean $\bar{X}$ of the auxiliary variable X are known.

To the first degree of approximation it can be shown that

$$MSE(\hat{t}_{\theta_0})=\left[MSE(\hat{\mathrm{R}})-\left(\frac{\mathrm{R}^2C_X^2}{n}\right)\left(\frac{d_0^2\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right]$$
$$=\min MSE(t_\theta)=MSE(t_{\theta_0})$$

$$MSE(\hat{t}_{\gamma_0})=\left[MSE(\hat{\mathrm{R}})-\left(\frac{\mathrm{R}^2C_X^2}{n}\right)\left(\frac{d_0^2\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right]$$
$$=\min MSE(t_\gamma)=MSE(t_{\gamma_0})$$

$$MSE(\hat{t}_{\eta_0})=\left[MSE(\hat{P})-\left(\frac{P^2C_X^2}{n}\right)\left(\frac{d_0^{*2}\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right]$$
$$=\min MSE(t_\eta)=MSE(t_{\eta_0})$$

$$MSE(\hat{t}_{\delta_0})=\left[MSE(\hat{P})-\left(\frac{P^2C_X^2}{n}\right)\left(\frac{d_0^{*2}\sigma_X^2}{\sigma_X^2+\sigma_V^2}\right)\right]$$
$$=\min MSE(t_\delta)=MSE(t_{\delta_0})$$

Thus in practice if the prior information regarding the optimum values $(\theta_0,\gamma_0,\eta_0,\delta_0)$ of the constants $(\theta,\gamma,\eta,\delta)$ are not available to the practitioner that mars the AOE's $t_{\theta_0}, t_{\gamma_0}, t_{\eta_0}$ *and* t_{δ_0} utilities. In such a situation the estimators $\hat{t}_{\theta_0}, \hat{t}_{\gamma_0}, \hat{t}_{\eta_0}$ *and* $\hat{t}_{\delta_0}$ based on 'estimated optimum' are recommended for their use in practice.

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SUMMARY

On the estimation of ratio and product of two population means using supplementary information in presence of measurement errors

This paper proposes some estimators for estimating the ratio and product of two population means using auxiliary information in presence of measurement errors and analyzes their properties.