DUALS TO MOHANTY AND SAHOO’S ESTIMATORS

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1. INTRODUCTION

The use of an auxiliary variable x in the estimation of the finite population mean \( \bar{Y} \) of the study variate y is a common occurrence in practice. A good illustration of this is the ratio method of estimation. It is known that ratio estimator attains minimum variance when the regression of y on x passes through the origin. Mohanty and Das (1971) was first to introduce the use of transformation of the auxiliary variate x in sample surveys to reduce the bias and mean squared error (MSE). Later, Reddy (1974), Reddy and Rao (1977), Srivenkataramana (1978), Chaudhuri and Adhikari (1979) and others have carried out a great deal of work in this direction. It is to be noted that most of these methods use the knowledge of unknown parameters \( R = (\bar{Y} / \bar{X}) \) and \( \beta \) (population regression coefficient of y on x) in the process of suggested transformation and hence have limited applicability.

This led Mohanty and Sahoo (1995) to suggest two linear transformations using known minimum and maximum values of the auxiliary variable x.

Consider a finite population \( U = (U_1, U_2, U_3, ..., U_N) \) of size N. The variate of interest y and the auxiliary variate x positively correlated to y, assume real non-negative values \((y_i, x_i)\) on the unit \( U_i (i = 1, 2, ..., N) \). Let \( (\bar{Y}, \bar{X}) \) be the population means of y and x respectively. Assume that a simple random sample of size n is drawn without replacement (SRSWOR) from the population U. Then the traditional ratio estimator for the population mean \( \bar{Y} \) is defined by

\[
\bar{Y}_R = (\bar{Y} / \bar{X}) \bar{X},
\]

where \( \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} y_i / n, \bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i / n \) are the sample means of y and x respectively, and \( \bar{X} \) is the known population mean of the auxiliary variate x.

Employing the transformation \( x_i^* = (1 + g) \bar{X} - gx_i, i = 1, 2, ..., N \); with \( g = n / (N - n) \); Srivenkataramana (1980) and Bandyopadhyay (1980) suggested a dual to ratio estimator for \( \bar{Y} \) as
\[ \bar{Y}_a = \bar{\Xi}(\bar{\Xi}^* / \bar{\Xi}) \]  
(2)

where \( \bar{\Xi}^* = \{(1 + g)\bar{\Xi} - g\bar{\Xi}\} \) such that \( E(\bar{\Xi}^*) = \bar{\Xi} \).

Assuming that \( x_m \) (the minimum) and \( x_M \) (the maximum) of the auxiliary variate \( \Xi \) are known, Mohanty and Sahoo (1995) suggested the linear transformations

\[ z_i = \frac{x_i + x_m}{x_M + x_m}, \]  
(3)

and

\[ u_i = \frac{x_i + x_M}{x_M + x_m}, \]  
(4)

\( i = 1, 2, \ldots, N \) and consequently, suggested two ratio-type estimators for \( \bar{Y} \) as

\[ t_{1R} = \left( \frac{\bar{Y}}{\bar{\Xi}} \right) \bar{Z} \]  
(5)

\[ t_{2R} = \left( \frac{\bar{Y}}{\bar{\Xi}} \right) \bar{U} \]  
(6)

where \( \bar{Z} = \sum_{i=1}^{n} z_i / n \) and \( \bar{U} = \sum_{i=1}^{n} u_i / n \) such that \( E(\bar{Z}) = \bar{Z} \) and \( E(\bar{U}) = \bar{U} \).

Employing Taylor’s expansion under usual assumptions, the biases and mean squared errors (MSEs) of \( \bar{Y}_R, \bar{Y}_a, t_{1R}, t_{2R} \) to terms of order \( O(n^{-1}) \) are obtained as follows.

\[ B(\bar{Y}_R) = (\theta / \bar{\Xi}) (R_S^2 - \rho S_S S_X) \]  
(7)

\[ B(\bar{Y}_a) = -(\theta / \bar{\Xi}) g \rho S_S S_X \]  
(8)

\[ B(t_{1R}) = \{ \theta / (\bar{\Xi} C_1) \} (R_S^2 - \rho S_S S_X) \]  
(9)

\[ B(t_{2R}) = \{ \theta / (\bar{\Xi} C_2) \} (R_S^2 - \rho S_S S_X) \]  
(10)

\[ MSE(\bar{Y}_R) = \theta (S_S^2 + R^2 S_S^2 - 2R \rho S_S S_X) \]  
(11)

\[ MSE(\bar{Y}_a) = \theta (S_S^2 + g^2 R^2 S_X^2 - 2gR \rho S_S S_X) \]  
(12)
In this paper motivated by Srivenkataratamana (1980) and Bandypadhyaya (1980) we have suggested duals to Mohanty and Sahoo’s (1995) estimators \( t_{1R} \) and \( t_{2R} \) and discussed their properties. Numerical illustration is given in the support of the present study.

2. THE SUGGESTED ESTIMATORS

We consider the following estimators for \( \overline{Y} \) as

\[
t_{1a} = \overline{Y} \left( \frac{\overline{\xi}^*}{\overline{Z}} \right)
\]

and

\[
t_{2a} = \overline{Y} \left( \frac{\overline{u}^*}{\overline{U}} \right)
\]

where \( \overline{u}^* = \sum_{i=1}^{n} u_i^* / n \) and \( \overline{\xi}^* = \sum_{i=1}^{n} \zeta_i^* / n \) are unbiased estimators of \( \overline{U} \) and \( \overline{Z} \) respectively,

\[
u_i^* = (1 + g) \overline{U} - g u_i, \text{ and } \zeta_i^* = (1 + g) \overline{Z} - g \zeta_i, (i=1,2,\ldots,N).
\]

To the first degree of approximation, the biases and mean squared errors of \( t_{1a} \) and \( t_{2a} \) are respectively given by

\[
MSE(t_{1R}) = \theta \left( S_y^2 + R_1^2 S_x^2 - 2R_1 \rho S_y S_x \right) \tag{13}
\]

\[
MSE(t_{2R}) = \theta \left( S_y^2 + R_2^2 S_x^2 - 2R_2 \rho S_y S_x \right) \tag{14}
\]

and the variance of \( \overline{Y} \) under SRSWOR is given by

\[
Var(\overline{Y}) = \theta S_y^2 \tag{15}
\]

where

\[
p = \frac{S_y}{S_x}, \quad S_y^2 = \sum_{i=1}^{N} (y_i - \overline{Y})^2 / (N - 1), \quad S_x^2 = \sum_{i=1}^{N} (x_i - \overline{X})^2 / (N - 1),
\]

\[
R = \frac{\overline{Y}}{\overline{X}}, \quad S_{xy} = \sum_{i=1}^{N} (x_i - \overline{X}) (y_i - \overline{Y}) / (N - 1), \quad R_1 = R / C_1,
\]

\[
R_2 = R / C_2, \quad C_1 = 1 + \frac{X_u}{X}, \quad C_2 = 1 + \frac{X_M}{X}, \quad \theta = \left( \frac{1}{n} - \frac{1}{N} \right).
\]
\[ B(t_{1a}) = -\{ \theta/(\bar{X}_2) \} g \rho \theta_j \theta_x \]  
(18)

\[ B(t_{2a}) = -\{ \theta/(\bar{X}_2) \} g \rho \theta_j \theta_x \]  
(19)

\[ \text{MSE}(t_{1a}) = \theta(S_j^2 + g^2R_1^2S_x^2 - 2gR_1\rho \theta_j \theta_x) \]  
(20)

\[ \text{MSE}(t_{2a}) = \theta(S_j^2 + g^2R_2^2S_x^2 - 2gR_2\rho \theta_j \theta_x) \]  
(21)

We note that the exact formulae for the mean squared errors (MSEs) of the estimators \( t_{1a} \) and \( t_{2a} \) can be derived which is not the case for Mohanty and Sahoo’s (1995) estimators \( t_{1R} \) and \( t_{2R} \). As pointed out by Srivenkataramana (1980; p. 200) that terms involving \( 1/n^2 \) and \( 1/n^3 \) in the mean square errors can be neglected, provided \( N \) is large enough and \( n \) at least moderately large.

3. COMPARISON OF BIAS AND EFFICIENCY

3.1. Comparison of Bias

From (7), (8), (9), (10), (18) and (19), it follows that

(i) \[ |B(t_{1a})| < |B(\bar{X}_R)| \] if

\[ \left| \frac{\beta}{C_1} \right| < \frac{(N-n)}{n} |R - \beta| \]  
(22)

(ii) \[ |B(t_{1a})| < |B(\bar{X}_x)| \] if

\[ \left| \frac{\beta}{C_1} \right| < |\beta| \]  
(23)

which is always true.

(iii) \[ |B(t_{1a})| < |B(t_{1R})| \] if

\[ |\beta| < \frac{(N-n)}{n} |R_1 - \beta| \]  
(24)

(iv) \[ |B(t_{1a})| < |B(t_{2R})| \] if

\[ gC_2 |\beta| < C_1 |R_2 - \beta| \]  
(25)
(v) \[ |B(t_{2a})| < |B(t_{1R})| \text{ if } \]
\[
g |\beta| < C_2 |R - \beta| \tag{26}
\]

(vi) \[ |B(t_{2a})| < |B(t_{1R})| \text{ if } \]
\[
\left| \frac{\beta}{C_2} \right| < |\beta| \tag{27}
\]

which is always true.

(vii) \[ |B(t_{2a})| < |B(t_{1R})| \text{ if } \]
\[
g C_1 |\beta| < C_2 |R_1 - \beta| \tag{28}
\]

(viii) \[ |B(t_{2a})| < |B(t_{2R})| \text{ if } \]
\[
|\beta| < \frac{(N - n)}{n} |R_2 - \beta| \tag{29}
\]

(x) \[ |B(t_{2a})| < |B(t_{1u})| \text{ if } \]
\[
\alpha_m < X_M \tag{30}
\]

which is always true.

3.2. Comparison of Efficiency

From (11), (12), (13), (14), (15), (20) and (21) it follows that

(i) \[ \text{MSE}(t_{1u}) < V(\overline{y}) \text{ if } \]
\[
K > \frac{g}{2C_1} \tag{31}
\]

(ii) \[ \text{MSE}(t_{1a}) < \text{MSE}(\overline{y}_R) \text{ if } \]
\[
K < \frac{1}{2} \left( 1 + \frac{g}{C_1} \right) C_1 > g \tag{32}
\]

(iii) \[ \text{MSE}(t_{1d}) < \text{MSE}(\overline{y}_a) \text{ if } \]
\[
K < \frac{g(1 + C_1)}{2C_1} \tag{33}
\]
(iv) \( \text{MSE}(t_{1a}) < \text{MSE}(t_{1R}) \) if
\[
K < \frac{(1 + g)}{2C_1}, \quad f < \frac{1}{2}
\] (34)

(v) \( \text{MSE}(t_{1a}) < \text{MSE}(t_{2R}) \) if
\[
either \quad K < \frac{(C_1 + gC_2)}{2C_1C_2}, \quad f < \frac{C_1}{C_1 + C_2} \\
or \quad K > \frac{(C_1 + gC_2)}{2C_1C_2}, \quad f > \frac{C_1}{C_1 + C_2}
\] (35)

(vi) \( \text{MSE}(t_{2a}) < V(\overline{d}) \) if
\[
K > \frac{g}{2C_2}
\] (36)

(vii) \( \text{MSE}(t_{2a}) < \text{MSE}(\overline{d}_R) \) if
\[
K < \frac{1}{2} \left( 1 + \frac{g}{C_2} \right), \quad C_2 > g
\] (37)

(viii) \( \text{MSE}(t_{2a}) < \text{MSE}(\overline{d}_a) \) if
\[
K < \frac{g(1 + C_2)}{2C_2}
\] (38)

(ix) \( \text{MSE}(t_{2a}) < \text{MSE}(t_{1R}) \) if
\[
either \quad K < \frac{1}{2} \left( \frac{1}{C_1} + \frac{g}{C_2} \right), \quad C_2 > gC_1 \\
or \quad K > \frac{1}{2} \left( \frac{1}{C_1} + \frac{g}{C_2} \right), \quad C_2 < gC_1
\] (39)

(x) \( \text{MSE}(t_{2a}) < \text{MSE}(t_{2R}) \) if
\[
K < \frac{(1 + g)}{2C_2}, \quad f < \frac{1}{2}
\] (40)
(xi) $\text{MSE}(t_{2a}) < \text{MSE}(t_{1a})$ if
\[
K < \frac{g(C_1 + C_2)}{2C_1C_2},
\]

where
\[
K = \frac{(\rho C_y/C_x)}{\beta} = \frac{R}{\beta}
\]

Combining (31), (32), (33), (34), and (35) it is observed that

(i) $t_{1a}$ is preferred over $\overline{y}$ and $\overline{y}_R$, when
\[
\frac{g}{2C_1} < K < \frac{1}{2} \left(1 + \frac{g}{C_1}\right), \quad C_1 > g
\]

(ii) $t_{1a}$ is preferred over $\overline{y}$ and $\overline{y}_d$, when
\[
\frac{g}{2C_1} < K < \frac{g(1 + C_1)}{2C_1}
\]

(iii) $t_{1a}$ is preferred over $\overline{y}$ and $t_{1R}$, when
\[
\frac{g}{2C_1} < K < \frac{(1 + g)}{2C_1}
\]

(iv) $t_{1a}$ is preferred over $\overline{y}$ and $t_{2R}$, when

\[
either \quad \frac{g}{2C_1} < K < \frac{(C_1 + gC_2)}{2C_1C_2}, \quad f < \frac{C_1}{C_1 + C_2}
\]
or
\[
K > \frac{1}{2} \left(1 + \frac{g}{C_1}\right), \quad f > \frac{C_1}{C_1 + C_2}
\]

Further combining (42), (43) and (44) it is seen that $t_{1a}$ is preferred over $\overline{y}$, $\overline{y}_R$, $\overline{y}_d$ and $t_{1R}$, when

\[
either \quad \frac{g}{2C_1} < K < \frac{g(1 + C_1)}{2C_1}, \quad gC_1 < 1
\]
or
\[
\frac{g}{2C_1} < K < \frac{(1 + g)}{2C_1}, \quad gC_1 > 1
\]
Again combining (45) and (47) we find that the estimator $t_{1a}$ is better than $\bar{y}$, $\bar{y}_R$, $\bar{y}_L$, $t_{1R}$ and $t_{2R}$ if

\[
either \frac{g}{2C_1} < \frac{1}{C_1 + \frac{gC_2}{2C_1}}, \quad gC_1 > 1 \\
or \frac{g}{2C_1} < \frac{(1+C_1)}{2C_1}, \quad gC_1 < 1 \tag{48}
\]

Now combining (36), (37), (38), (39), (40), and (41) we find that

(i) $t_{2a}$ is preferred over $\bar{y}$ and $\bar{y}_R$, when

\[
\frac{g}{2C_2} < \frac{1}{2} \left( 1 + \frac{g}{C_2} \right) \quad \tag{49}
\]

(ii) $t_{2a}$ is preferred over $\bar{y}$ and $\bar{y}_L$, when

\[
\frac{g}{2C_2} < \frac{g(1+C_2)}{2C_2} \quad \tag{50}
\]

(iii) $t_{2a}$ is preferred over $\bar{y}$ and $t_{1R}$, when

\[
either \frac{g}{2C_2} < \frac{1}{2} \left( \frac{1}{C_1} + \frac{g}{C_2} \right), \quad C_2 > gC_1 \\
or \quad K > \frac{1}{2} \left( \frac{1}{C_1} + \frac{g}{C_2} \right), \quad C_2 < gC_1 \quad \tag{51}
\]

(iv) $t_{2a}$ is preferred over $\bar{y}$ and $t_{2R}$, when

\[
\frac{g}{2C_2} < \frac{(1+g)}{2C_2} \quad \tag{53}
\]

(v) $t_{2a}$ is preferred over $\bar{y}$ and $t_{1a}$, when

\[
\frac{g}{2C_2} < \frac{(C_1+C_2)g}{2C_1C_2} \quad \tag{54}
\]

Further combining (49), (50), (53) and (54) it is observed that $t_{2a}$ is better than $\bar{y}$, $\bar{y}_R$, $\bar{y}_L$, $t_{1R}$ and $t_{1a}$ when
Again, combining (51) and (55) we find that estimator \( t_{2a} \) is more efficient than \( \bar{Y}, \bar{Y}_R, \bar{Y}_a, t_{i1R}, t_{2R} \) and \( t_{1a} \) if

\[
\frac{g}{2C_2} < K < \frac{g(1 + C_2)}{2C_2}, \quad gC_2 < 1
\] (56)

or

\[
\frac{g}{2C_2} < K < \frac{(1 + g)}{2C_2}, \quad \frac{1}{C_2} < g < \frac{C_2}{C_1}
\] (57)

4. UNBIASED ESTIMATION

It is observed that the estimators \( t_{1a} \) and \( t_{2a} \) are biased. In some applications biasedness is disadvantageous. This led us to investigate unbiased estimators of \( \bar{Y} \).

If there is no correlation between the study variate \( y \) and the auxiliary variate \( x \), then \( B(t_{ji}) = 0, \ j = 1, 2; \) and hence the estimators \( t_{1a} \) and \( t_{2a} \) are unbiased. But this situation is not good since there will be an unacceptable increase in variance relative to the usual unbiased estimator \( \bar{Y} \). Owing to this we consider the following alternatives.

4.1. Unbiased Product Estimators For Interpenetrating Subsample Design

Consider the case of interpenetrating subsample discussed by Murthy (1964). Let \( y_i \) and \( x_i \) be unbiased estimates of the population totals \( Y \) and \( X \) respectively based on the \( i \)th independent interpenetrating subsample, \( i = 1, 2, \ldots, n \). Consider now the following estimators

\[
i_{1a}^{(1)} = \left( \frac{1}{n} \sum_{i=1}^{n} y_i \right) \left( \frac{1}{n} \sum_{i=1}^{n} x_i^* \right) / Z
\]

\[
= \bar{Y} \bar{X}^* / Z
\] (58)

\[
i_{1a}^{(2)} = \sum_{j=1}^{s} y_{ji} \bar{z}_i^* / (nZ)
\] (59)

\[
i_{2a}^{(1)} = \left( \frac{1}{n} \sum_{i=1}^{n} y_i \right) \left( \frac{1}{n} \sum_{i=1}^{n} u_i^* \right) / \bar{U}
\] (60)
\[ y = \frac{\bar{y}^*}{\bar{U}} \]

\[ t_{2a}^{(2)} = \sum_{i=1}^{n} y_i u_i^* / (n\bar{U}) \] \hspace{1cm} (61)

as estimators for the population mean \( \bar{Y} \). Following Murthy (1964), we can show that

\[ B(t_{1a}^{(2)}) = nB(t_{1a}^{(1)}) \] \hspace{1cm} (62)

and

\[ B(t_{2a}^{(2)}) = nB(t_{2a}^{(1)}) \] \hspace{1cm} (63)

and hence that

\[ t_{3a} = (nt_{1a}^{(1)} - t_{2a}^{(2)}) / (n - 1) \] \hspace{1cm} (64)

and

\[ t_{4a} = (nt_{2a}^{(1)} - t_{2a}^{(2)}) / (n - 1) \] \hspace{1cm} (65)

are unbiased estimators of the population mean \( \bar{Y} \).

The conditions for \( t_{3a} \) and \( t_{4a} \) to be more efficient than \( t_{1a}^{(1)} \) and \( t_{2a}^{(1)} \) respectively are similar to those given in Murthy and Nanjamma (1959) in the case of obtaining an almost unbiased ratio estimator.

Motivated by the approach illustrated in Rao (1981), we consider more generally, the estimator

\[ T_{1p} = \delta t_{1a}^{(1)} + \{1 - E(f(\delta))\} t_{1a}^{(2)} \] \hspace{1cm} (66)

where \( \delta \) is random and \( f(\delta) \) is a function of \( \delta \).

Then \( T_{1p} \) is unbiased for \( \bar{Y} \) if

\[ E(T_{1p}) = \bar{Y} \]

i.e. if \( E(\delta t_{1a}^{(1)} - E(f(\delta))t_{1a}^{(2)}) = E(\bar{Y} - t_{1a}^{(2)}) \) \hspace{1cm} (67)

for which

\[ \delta = \frac{Z}{\xi} \quad \text{and} \quad f(\delta) = \frac{\delta Z^*}{Z} \]

is a solution.
We further write (67) as

\[ E[\delta \hat{t}_{1a}^{(1)} - E(f(\delta)) \hat{t}_{1a}^{(2)}] = E[\delta \bar{Y} - \hat{t}_{1a}^{(2)} + \alpha \hat{t}_{1a}^{(1)} - \alpha \hat{t}_{1a}^{(1)}] \]  

(68)

where \( \alpha \) is a scalar,

From the relation

\[ B(\hat{t}_{1a}^{(1)}) = \frac{1}{n} B(\hat{t}_{1a}^{(2)}) \]

\[ \Rightarrow E(\hat{t}_{1a}^{(1)} - \bar{Y}) = \frac{1}{n} B(\hat{t}_{1a}^{(2)}) \]

\[ \Rightarrow E(\hat{t}_{1a}^{(1)}) = \bar{Y} + \frac{1}{n} B(\hat{t}_{1a}^{(2)}) \]

\[ = \bar{Y} + \frac{1}{n} E(\hat{t}_{1a}^{(2)} - \bar{Y}) \]

\[ = (1 - \frac{1}{n}) \bar{Y} + \frac{1}{n} E(\hat{t}_{1a}^{(2)}) \]

\[ = \left( \frac{n-1}{n} \right) \bar{Y} + \frac{1}{n} E(\hat{t}_{1a}^{(2)}) \]

i.e.

\[ E(\hat{t}_{1a}^{(1)}) = C \bar{Y} + (1-C) E(\hat{t}_{1a}^{(2)}) \]  

(69)

where \( C = \frac{n-1}{n} \).

Putting (69) in (67) we have

\[ E[\delta \hat{t}_{1a}^{(1)} - E(f(\delta)) \hat{t}_{1a}^{(2)}] = E \left[ (1-\alpha C) \bar{Y} + \alpha \bar{Y} \frac{\bar{Z}}{\bar{X}} - \{1 + \alpha (1-C)\} \hat{t}_{1a}^{(2)} \right] \]

\[ = E \left[ \alpha + (1-\alpha C) \frac{\bar{Z}}{\bar{X}} \right] \hat{t}_{1a}^{(1)} - \{1 + \alpha (1-C)\} \hat{t}_{1a}^{(2)} \]

which gives a solution

\[ \delta = \alpha + (1-\alpha C) \frac{\bar{Z}}{\bar{X}} \quad \text{and} \quad f(\delta) = \delta \frac{\bar{X}}{\bar{Z}} \]
for which
\[ E(f(\delta)) = \{ \alpha + (1 - \alpha C) \} = \{ 1 + (1 - C)\alpha \} \]

Thus putting \( \delta = \left\{ \alpha + (1 - \alpha) \frac{Z}{\xi} \right\} \) and \( E(f(\delta)) = \{ 1 + (1 - C)\alpha \} \) in (66) we get a general class of estimators of population mean \( \bar{Y} \) as

\[ T_{1P} = \left[ \left\{ \alpha + (1 - \alpha C) \frac{Z}{\xi} \right\} t_{1a}^{(1)} - \alpha (1 - C) t_{1a}^{(2)} \right] \]

or

\[ T_{1P} = [(1 - \alpha C) \bar{Y} + \alpha t_{1a}^{(1)} - (1 - C)\alpha t_{1a}^{(2)}] \quad (70) \]

**Remark 4.1.** For \( \alpha = 0 \), \( T_{1P} \) reduces to the usual unbiased estimator \( \bar{Y} \) while \( \alpha = C^{-1} \) gives the estimator \( t_{3a} \) in (64) when \( \alpha = (1 - C)^{-1} \). We get another estimator

\[ T_{1P}^{(1)} = (2 - n) \bar{Y} + n t_{1a}^{(1)} - t_{1a}^{(2)} \quad (71) \]

Many other unbiased estimators can be generated from \( T_{1P} \) at (70) just by putting suitable value of \( \alpha \).

**Remark 4.2.** Proceeding in a similar way we obtain another class of estimators of \( \bar{Y} \) as

\[ T_{2P} = [(1 - \alpha_1 C) \bar{Y} + \alpha_1 t_{2a}^{(1)} - (1 - C)\alpha_1 t_{2a}^{(2)}] \quad (72) \]

where \( \alpha_1 \) is a suitable chosen scalar, for \( \alpha_1 = 0 \), \( T_{2P} \) boils down to the estimator \( \bar{Y} \) while for \( \alpha_1 = C^{-1} \) it reduces to the estimator

\[ T_{2P}^{(1)} = \frac{1}{C} t_{2a}^{(1)} - \frac{(1 - C)}{C} t_{2a}^{(2)} \]

\[ = t_{4a} \quad (73) \]

when \( \alpha_1 = (1 - C)^{-1} \) we get another unbiased estimator of \( \bar{Y} \) as

\[ T_{2P}^{(2)} = [(2 - n) \bar{Y} + n t_{2a}^{(1)} - t_{2a}^{(2)}] \quad (74) \]

many other unbiased estimators can be generated from (72) just by putting suitable value of \( \alpha_1 \).
4.2. Optimum Estimator in the Class $T_{1p}$

From (70) we have

$$Var(T_{1p}) = (1 - \alpha C)^2 Var(\overline{y}) + \alpha^2 Var(t_{1a}^{(1)}) + (1 - C)^2 \alpha^2 Var(t_{1a}^{(2)})$$

$$+ 2\alpha(1 - \alpha C) Cov(\overline{y}, t_{1a}^{(1)}) - 2\alpha(1 - C)(1 - \alpha C) Cov(\overline{y}, t_{1a}^{(2)}) - 2\alpha^2 (1 - C) Cov(t_{1a}^{(1)}, t_{1a}^{(2)})$$

$$= Var(\overline{y}) + \alpha^2 [C^2 Var(\overline{y}) + Var(t_{1a}^{(2)})] + (1 - C)^2 Var(t_{1a}^{(2)})$$

$$- 2C Cov(\overline{y}, t_{1a}^{(1)}) + 2C (1 - C) Cov(\overline{y}, t_{1a}^{(2)}) - 2(1 - C) Cov(t_{1a}^{(1)}, t_{1a}^{(2)})$$

$$- 2\alpha[C Var(\overline{y}) - Cov(\overline{y}, t_{1a}^{(1)}) + 2(1 - C) Cov(\overline{y}, t_{1a}^{(2)})]$$

$$= Var(\overline{y}) + \alpha^2 B - 2\alpha A$$

(75)

where

$$A = [C Var(\overline{y}) - Cov(\overline{y}, t_{1a}^{(1)}) + 2(1 - C) Cov(\overline{y}, t_{1a}^{(2)})]$$

$$B = [C^2 Var(\overline{y}) + Var(t_{1a}^{(1)}) + (1 - C)^2 Var(t_{1a}^{(2)}) - 2C Cov(\overline{y}, t_{1a}^{(1)})$$

$$+ 2C (1 - C) Cov(\overline{y}, t_{1a}^{(2)}) - 2(1 - C) Cov(t_{1a}^{(1)}, t_{1a}^{(2)})]$$

The variance of $T_{1p}$ at (75) is minimized for

$$\alpha_{opt} = \frac{A}{B}$$

$$= \frac{Cov(\overline{y}, t)}{Var(t)}$$

(76)

where

$$t = (C \overline{y} - t^*), \quad t^* = [t_{1a}^{(1)} - (1 - C)t_{1a}^{(2)}].$$

Thus the resulting minimum variance of $T_{1p}$ is given by

$$\text{min.} Var(T_{1p}) = Var(\overline{y})(1 - \rho^2)$$

(77)

where $\rho^* = \frac{Cov(\overline{y}, t)}{\sqrt{Var(\overline{y}) Var(t)}}$ is the correlation between $\overline{y}$ and $t$.

It follows from (77) that $\text{min.} Var(T_{1p}) < Var(\overline{y})$.

Further from (20) and (77) that $\text{min.} Var(T_{1p}) < Var(t_{1a}^{(1)})$. 
If

\[ \begin{align*}
\text{Var}(\bar{y})(1 - \rho^2) & < \text{Var}(\bar{y}) + g^2R^2\text{Var}(\bar{z}) - 2gR_1\rho\sigma(\bar{y})\sigma(\bar{z}) \\
(gR_1\sigma(\bar{y}) - \rho\sigma(\bar{z}))^2 + \text{Var}(\bar{y})(\rho^2 - \rho^2) & > 0
\end{align*} \]  

(78)

Thus a sufficient condition for the estimator \( T_{1p(\theta)} \) (i.e. Optimum estimator in the class \( T_{1p} \)) to be more efficient than \( t_{12}^{(1)} \) is that \( \rho^2 > \rho^2 \). It may be noted here that the sufficient conditions can be examined in practice for their estimators which are obtained based on the subsamples [see Rao(1983)].

**Remark 4.3.** Similar studies can be carried out for the class of unbiased estimators \( T_{2p} \) in (72).

The variance of \( T_{2p} \) is given by

\[ \text{Var}(T_{2p}) = \text{Var}(\bar{y}) + \alpha_1^2B^* - 2\alpha_1A^* \]  

(79)

where

\[ A^* = \text{Cov}(\bar{y}, t_1), \quad B^* = \text{Var}(t_1), \quad t_1 = (C\bar{y} - \bar{t}_1) \]

and

\[ \bar{t}_1 = [t_{2a}^{(1)} - (1 - C)t_{2a}^{(2)}] \]

The variance of \( T_{2p} \) at (79) is minimized for

\[ \alpha_1 = \frac{\text{Cov}(\bar{y}, t_1)}{\text{Var}(t_1)} = \frac{A^*}{B^*} = \alpha_{opt} \text{ (say)} \]  

(80)

Thus the resulting minimum variance of \( T_{2p} \) is given by

\[ \text{min. Var}(T_{2p}) = \text{Var}(\bar{y})(1 - \rho^2) \]  

(81)

\[ \rho_1^* = \frac{\text{Cov}(\bar{y}, t_1)}{\sqrt{\text{Var}(\bar{y})\text{Var}(t_1)}} \]  

is the correlation coefficient between \( \bar{y} \) and \( t_1 \).

The ‘Optimum’ estimator \( T_{2p(\theta)} \) is more efficient than \( t_{12}^{(1)} \) if

\[ (gR_2\sigma(\bar{y}) - \rho\sigma(\bar{z}))^2 + \text{Var}(\bar{y})(\rho_1^* - \rho^2) > 0 \]  

(82)

which is always true if \( \rho_1^* > \rho^2 \).
It follows from (81) that the ‘Optimum’ estimator $T_{2P(q)}$ is better than usual unbiased estimator $\bar{y}$.

### 4.3. Unbiased Product Estimators for SRSWOR Design

For the case of simple random sampling without replacement (SRSWOR), let $y_i$ and $x_i$ denote respectively the $y$ and $x$ values of the $i$th sample unit, $i=1,2,\ldots,n$. Then we consider the following estimators of $\bar{Y}$ as

\[
T^{(1)}_{1d} = \frac{\bar{y}^*}{Z}, \quad T^{(2)}_{1d} = \frac{1}{nZ} \sum_{i=1}^{n} y_j z^*_j
\]

\[
T^{(1)}_{2d} = \frac{\bar{y}^*}{U}, \quad T^{(2)}_{2d} = \frac{1}{nU} \sum_{i=1}^{n} y_j u^*_j
\]

and

\[
B(j_{\mu}) = (1 - C^*)^{-1} B(j_{\mu}^{(1)}), \quad j = 1,2
\]

where $C^* = N(n-1)/\{n(N-1)\}$

Following the procedure as outlined in section 4.2, we get the following class of unbiased estimators of $\bar{Y}$ as

\[
T^{*}_{j\mu} = [(1 - C^* \gamma_j) \bar{y} + \gamma_j T^{(1)}_{j\mu} - (1 - C^*) \gamma_j T^{(2)}_{j\mu}]
\]

where $\gamma_j (j = 1,2)$ is the suitable chosen scalar.

**Remark 4.4.** Notice that $\gamma_j = 0 \quad j = 1,2$; gives the conventional unbiased estimator $\bar{y}$ and $\gamma_j = (C^*)^{-1}$ yields the estimator

\[
T^{*(1)}_{j\mu} = \frac{1}{C^*} T^{(1)}_{j\mu} - \frac{(1 - C^*)}{C^*} T^{(2)}_{j\mu} ; \quad j = 1,2;
\]

For $j=1$, we have

\[
T^{(1)}_{1\mu} = \frac{n(N-1)}{N(n-1)} \frac{\bar{y}^*}{Z} - \frac{(N-n)}{N(n-1)} \left( \frac{1}{n} \sum_{j=1}^{n} y_j z^*_j / Z \right)
\]

\[
= \frac{\bar{y}^*}{Z} - \left( \frac{1}{n} - \frac{1}{N} \right) \frac{s^*_j}{Z}
\]
and for \( j=2 \), we have

\[
T_{2j}^{(2)} = \frac{n(N-n)}{N(N-n-1)} \frac{\overline{u^*}}{\overline{U}} - \left( \frac{N-n}{n} \right) \left( \frac{1}{n} \sum_{i=1}^{n} y_{i}^{*} / \overline{U} \right)
\]

\[
= \frac{\overline{u^*}}{\overline{U}} - \left( \frac{1}{n} - \frac{1}{N} \right) \frac{\sum_{i=1}^{n} s_{y_{i}^{*}}}{\overline{U}}
\]

where

\[
s_{y_{i}^{*}} = \frac{1}{(n-1)} \sum_{i=1}^{n} (y_{i}^{*} - \overline{y}^{*})(y_{i} - \overline{y})
\]

and

\[
s_{u_{i}^{*}} = \frac{1}{(n-1)} \sum_{i=1}^{n} (u_{i}^{*} - \overline{u}^{*})(y_{i} - \overline{y})
\]

Many other unbiased estimators of \( \overline{Y} \) can be obtained from (84) just by putting suitable values of scalars \( \gamma_{j} (j=1,2) \).

As in the case of interpenetrating subsample, it is easy to obtain the optimum value of \( \gamma_{j} (j=1,2) \).

4.4. Quenouille’s Jackknife Method

In this method we take a sample \( n = 2m \) and split it at random into two subsamples of \( m \) units each. Let \( \overline{y}_{i}, \overline{x}_{i}, (i=1,2) \) be unbiased estimators of \( \overline{Y} \) and \( \overline{X} \) based on the subsamples and \( \overline{Y}, \overline{X} \) those based on the entire sample. Take

\[
\overline{y}_{i}^{*} = (1 + g)\overline{Z} - g\overline{x}_{i}, \quad \overline{x}_{i}^{*} = (1 + g)\overline{U} - g\overline{u}_{i};
\]

\[
\overline{x}_{i}^{*} = (1 + g)\overline{Z} - g\overline{y}_{i}, \quad \overline{u}_{i}^{*} = (1 + g)\overline{U} - g\overline{u};
\]

\[
\overline{y}_{i} = \frac{\overline{x}_{i} + \overline{x}_{m}}{\overline{x}_{m} + \overline{x}_{m}}, \quad \overline{x}_{i} = \frac{\overline{y}_{i} + \overline{y}_{M}}{\overline{y}_{M} + \overline{y}_{M}}; \quad (i=1,2)
\]

Consider the product–type estimators

\[
t_{1a}^{(i)} = \frac{\overline{y}_{i}^{*}}{\overline{Z}}, \quad t_{1a} = \frac{\overline{y}^{*}}{\overline{Z}}
\]

\[
t_{2a}^{(i)} = \frac{\overline{y}_{i}^{*}}{\overline{U}}, \quad t_{2a} = \frac{\overline{y}^{*}}{\overline{U}}
\]
Then it can be easily shown that

\[ t_{5a} = \frac{(2N-n)}{N} t_{1a} - \frac{(N-n)}{2N} (t_{1a}^{(1)} + t_{1a}^{(2)}) \]  

(88)

and

\[ t_{6a} = \frac{(2N-n)}{N} t_{2a} - \frac{(N-n)}{2N} (t_{2a}^{(1)} + t_{2a}^{(2)}) \]  

(89)

Further it can be shown to the first degree of approximation that

\[ MSE(t_{5a}) = MSE(t_{1a}) \]  

(90)

\[ MSE(t_{6a}) = MSE(t_{2a}) \]  

(91)

Since \( t_{5a} (t_{6a}) \) is unbiased while \( t_{1a} (t_{2a}) \) is not the former \( t_{5a} (t_{6a}) \) is to be preferred to the latter \( t_{1a} (t_{2a}) \).

Now we define a class of product–type estimators for \( \bar{Y} \) as

\[ t_{w}^{(j)} = \omega_{1j} \bar{Y} + \omega_{2j} t_{ja} + \omega_{3j} t_{ja}^{*}, \quad (j = 1, 2) \]  

(92)

where \( \omega_{i}^{*} s \ (i = 1, 2, 3; \ j = 1, 2) \) are suitable chosen constants such that

\[ \omega_{1j} + \omega_{2j} + \omega_{3j} = 1, \ j = 1, 2 \]  

(93)

and

\[ t_{ja}^{*} = \frac{1}{2} \sum_{j=1}^{2} t_{ja}^{(l)} \]  

(94)

we have

\[ B(t_{1a}) = -\frac{(N-n)}{nN} \left( \frac{g}{XC_{1}} \right) \rho S_{y} S_{x} \]  

\[ B(t_{2a}) = -\frac{(N-n)}{nN} \left( \frac{g}{XC_{2}} \right) \rho S_{y} S_{x} \]  

\[ B(t_{1a}^{*}) = -\frac{(2N-n)}{nN} \left( \frac{g}{XC_{1}} \right) \rho S_{y} S_{x} \]  

(95)

\[ B(t_{2a}^{*}) = -\frac{(2N-n)}{nN} \left( \frac{g}{XC_{2}} \right) \rho S_{y} S_{x} \]
It follows from (95) that
\[
\frac{B(t_{ja})}{B(t_{ja}^*)} = \frac{(N-n)}{(2N-n)} \ ; \ j = 1, 2 ; \quad (96)
\]

The class of estimators \( t_{α}^{(j)} \) is unbiased if
\[
\omega_{2j} B(t_{ja}) + \omega_{3j} B(t_{ja}^*) = 0 \quad (j = 1, 2)
\]

Thus from (96) and (97) we get
\[
\omega_{3j} = -\omega_{2j} \frac{B(t_{ja})}{B(t_{ja}^*)} = -\omega_{2j} \frac{(N-n)}{(2N-n)} \quad (98)
\]

With \( \omega_{2j} = \omega^{(j)} \) (a constant) and from (92), (93) and (98) we have
\[
t_{α}^{(j)} = \left(1 - C \omega^{(j)}\right) \bar{Y} + \omega^{(j)} t_{ja} - \omega^{(j)}(1-C)t_{ja}^* \quad (j = 1, 2)
\]

Thus for \( j=1 \), we get
\[
t_{α}^{(1)} = \left[1 - C \omega^{(1)}\right] \bar{Y} + \omega^{(1)} t_{1a} - \omega^{(1)}(1-C)t_{1a}^* \]
\[
= \left(1 - C \omega^{(1)}\right) \bar{Y} + \omega^{(1)} \frac{\bar{x}^*}{\bar{Z}} - \omega^{(1)}(1-C) \left\{ \frac{1}{2\bar{Z}} \sum_{i=1}^{3} \bar{y}_j \bar{x}_j^* \right\}
\]

a class of unbiased product–type estimators of \( \bar{Y} \).

For \( j=2 \), we get another class of unbiased product–type estimators for \( \bar{Y} \) as
\[
t_{α}^{(2)} = \left[1 - C \omega^{(2)}\right] \bar{Y} + \omega^{(2)} t_{2a} - \omega^{(2)}(1-C)t_{2a}^* \]
\[
= \left(1 - C \omega^{(2)}\right) \bar{Y} + \omega^{(2)} \frac{\bar{u}^*}{\bar{U}} - \omega^{(2)}(1-C) \left\{ \frac{1}{2\bar{U}} \sum_{i=1}^{3} \bar{y}_j \bar{u}_j^* \right\}
\]

Remark 4.5. For \( \omega^{(j)} = 0 \) , \( t_{α}^{(j)} \) reduce to the usual unbiased estimator \( \bar{Y} \) while for \( \omega^{(j)} = C^{-1} \) it boils down to the estimator \( t_{5a}(j=1) \) and \( t_{6a}(j=2) \).
4.5. Optimum Estimator in the Class $t^{(j)}$

From (99) we have

$$\text{Var}(t^{(j)}) = [\text{Var}(\bar{y}) + \omega^{(j)}] \{C^2 \text{Var}(\bar{y}) + \text{Var}(t_{ja}) + (1 - C^2)\text{Var}(t_{ja}^*)$$

$$- 2C \text{Cov}(\bar{y}, t_{ja}) + 2C(1 - C)\text{Cov}(\bar{y}, t_{ja}^*) - 2C(1 - C)\text{Cov}(t_{ja}, t_{ja}^*)\}$$

$$- 2\omega^{(j)} [C \text{Var}(\bar{y}) - \text{Cov}(\bar{y}, t_{ja}) + (1 - C)\text{Cov}(\bar{y}, t_{ja}^*)]\}$$

$$j = 1, 2.$$  

To the first degree of approximation, it is easy to see that

$$\text{Var}(t_{ja}) = \text{Var}(t_{ja}^*) = \text{Cov}(t_{ja}, t_{ja}^*) = \theta[S_j^2 + C^2R_j^2S_x^2 - 2gR_j\rho S_jS_x]$$

$$\text{Cov}(\bar{y}, t_{ja}) = \text{Cov}(\bar{y}, t_{ja}^*) = \theta(S_j^2 - gR_j\rho S_jS_x)$$

$$\text{Var}(\bar{y}) = \theta S_j^2$$

$$j = 1, 2 \quad R_j = R/C_j$$

Putting (101) in (100) we get the variance of $t^{(j)}$, $j = 1, 2$, to terms of order $n^{-1}$ as

$$\text{Var}(t^{(j)}) = \theta[S_j^2 + \omega^2g^2C^2R_j^2S_x^2 - 2\omega CgR_j\rho S_jS_x]$$

which is minimized for

$$\omega^{(j)} = \left(\frac{C_j}{C_g}\right) \frac{\beta}{R} = \omega^{(j)}_{opt}$$

$$j = 1, 2.$$  

Thus the resulting minimum variance of $t^{(j)}$ is given by

$$\text{min.}\text{Var}(t^{(j)}) = \theta S_j^2(1 - \rho^2)$$

which is equal to the approximate variance of the usual biased regression estimator

$$\bar{y}_p = \bar{y} + \hat{y}(\bar{x} - \bar{x})$$

where $\hat{y}$ is the sample regression coefficient of $y$ on $x$.

Substituting the value of $\omega^{(j)}_{opt}$ for $\omega^{(j)}$ in (99) we get the asymptotically optimum unbiased product-type estimator in the class (4.42) as
\[ t^{(j)}_{\mu(0)} = \left[ 1 - \left( \frac{C_j}{g} \right) \frac{\beta}{R} \right] \overline{y} + \left( \frac{C_1}{gC} \right) \frac{\beta}{R} t_\mu - \left( \frac{C_1}{gC} \right) \frac{\beta}{R} (1 - C_j) t_\mu^* \], \quad j = 1, 2. \quad (106)

with the variance as given in (104).

5. EMPIRICAL STUDY

To illustrate the performance of the suggested estimators \( t_{1a} \) and \( t_{2a} \) over \( \overline{y}, \overline{y}_R, \overline{y}_a, t_{1R}, \) and \( t_{2R} \), we have considered four populations whose descriptions are given in the Table 1.

<table>
<thead>
<tr>
<th>Population</th>
<th>Source</th>
<th>( N )</th>
<th>( n )</th>
<th>( Y )</th>
<th>( X )</th>
<th>( \rho )</th>
<th>( C_x )</th>
<th>( C_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Panse and Sukhatme (1967) p.118</td>
<td>25</td>
<td>10</td>
<td>Parental plot mean</td>
<td>Parental plant value</td>
<td>0.53</td>
<td>0.07</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(mm)</td>
<td>(mm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>Panse and Sukhatme (1967) p.118 (1-20)</td>
<td>20</td>
<td>8</td>
<td>Parental plot mean</td>
<td>Parental plant value</td>
<td>0.56</td>
<td>0.07</td>
<td>0.04</td>
</tr>
<tr>
<td></td>
<td>(mm)</td>
<td>(mm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>Panse and Sukhatme (1967) p.118 (1-10)</td>
<td>10</td>
<td>4</td>
<td>Porgeny mean</td>
<td>Parental plant value</td>
<td>0.44</td>
<td>0.07</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>(mm)</td>
<td>(mm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>Sampford (1962) p.61 (1-9)</td>
<td>9</td>
<td>3</td>
<td>Acreage under oats</td>
<td>Acreage of crops and grass in 1957</td>
<td>0.07</td>
<td>0.10</td>
<td>0.29</td>
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<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Where \( \rho \) is the correlation coefficient between \( x \) and \( y \), and \( C_x = S_x / \overline{X} \) and \( C_y = S_y / \overline{Y} \) are the coefficients of variation of \( x \) and \( y \) respectively.

(Figures in (.) indicate values of \( C_1, C_2 \) and \( K \) respectively and in { } show the values of \( R, \overline{X} \) and \( \overline{Y} \) respectively. (See. Mohanty and Sahoo (1995))).

We have computed the absolute biases of the estimators \( \overline{y}_R, \overline{y}_a, t_{1R}, t_{2R}, t_{1a} \) and \( t_{2a} \) and presented in Table-5.2. Percent relative efficiency (%) of these estimators \( (\overline{y}_R, \overline{y}_a, t_{1R}, t_{2R}, t_{1a} \) and \( t_{2a} \) have also been computed and compiled in Table-5.3. Formulae for absolute biases and percent relative efficiencies are given below.

\[ |B(\overline{y}_R)| = \left| (\theta/\overline{X})(RS^2_x - \rho S_y S_x) \right| \quad (107) \]

\[ |B(\overline{y}_a)| = \left| \theta/\overline{X} \right| g \rho S_y S_x \quad (108) \]
\[ |B(t_{1R})| = \left| \left\{ \theta/(\bar{X} C_1) \right\} \left( R_1 S_x^2 - \rho S_y S_x \right) \right| \] (109)

\[ |B(t_{2R})| = \left| \left\{ \theta/(\bar{X} C_2) \right\} \left( R_2 S_x^2 - \rho S_y S_x \right) \right| \] (110)

\[ |B(t_{1a})| = \left| \left\{ \theta/(\bar{X} C_1) \right\} g \rho S_y S_x \right| \] (111)

\[ |B(t_{2a})| = \left| \left\{ \theta/(\bar{X} C_2) \right\} g \rho S_y S_x \right| \] (112)

\[ PRE(\bar{Y}_R, \bar{Y}) = \frac{S_y^2}{(S_y^2 + R^2 S_x^2 - 2R \rho S_y S_x)} \times 100 \] (113)

\[ PRE(\bar{Y}_a, \bar{Y}) = \frac{S_y^2}{(S_y^2 + \gamma^2 R^2 S_x^2 - 2\gamma R \rho S_y S_x)} \times 100 \] (114)

\[ PRE(t_{1R}, \bar{Y}) = \frac{S_y^2}{(S_y^2 + R_1^2 S_x^2 - 2R_1 \rho S_y S_x)} \times 100 \] (115)

\[ PRE(t_{2R}, \bar{Y}) = \frac{S_y^2}{(S_y^2 + R_2^2 S_x^2 - 2R_2 \rho S_y S_x)} \times 100 \] (116)

\[ PRE(t_{1a}, \bar{Y}) = \frac{S_y^2}{(S_y^2 + \gamma^2 R_1^2 S_x^2 - 2\gamma R_1 \rho S_y S_x)} \times 100 \] (117)

\[ PRE(t_{2a}, \bar{Y}) = \frac{S_y^2}{(S_y^2 + \gamma^2 R_2^2 S_x^2 - 2\gamma R_2 \rho S_y S_x)} \times 100 \] (118)

**TABLE 2**

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Absolute Biases</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Population I</td>
</tr>
<tr>
<td>( \bar{Y}_R )</td>
<td>0.0054</td>
</tr>
<tr>
<td>( \bar{Y}_a )</td>
<td>0.0013</td>
</tr>
<tr>
<td>( t_{1R} )</td>
<td>0.0011</td>
</tr>
<tr>
<td>( t_{2R} )</td>
<td>0.0007</td>
</tr>
<tr>
<td>( t_{1a} )</td>
<td>0.0007</td>
</tr>
<tr>
<td>( t_{2a} )</td>
<td>0.0006</td>
</tr>
</tbody>
</table>
It is observed from Table-2 that the estimator $t_{2a}$ has least bias followed by $t_{1a}$ and $t_{2R}$ for population I while in population II it is at par with $t_{2R}$ and has less bias than $\bar{y}, \bar{y}_R, t_{1R}$ and $t_{1a}$. However in population III, the estimator $t_{2a}$ has less bias than $\bar{y}_R, \bar{y}_a, t_{1R}$ and $t_{1a}$ but has marginally more bias than $t_{2R}$. Table 3 exhibits that the estimator $t_{2a}$ has largest efficiency for all the population data sets I-IV followed by $t_{1a}$. Thus the proposed estimators $t_{1a}$ and $t_{2a}$ are to be preferred in practice.

**ACKNOWLEDGEMENT**

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**REFERENCES**


**SUMMARY**

*Duals to Mohanty and Sahoo’s estimators*

This paper proposes duals to Mohanty and Sahoo’s (1995) estimators and analyzes their properties. Unbiased estimators have also been obtained for interpenetrating subsample design and by using Jackknife technique given by Quenouille (1956). An empirical study is carried out to demonstrate the performances of the suggested estimators over other estimators.