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ASYMPTOTIC PROPERTIES OF MAXIMUM LIKELIHOOD ESTIMATOR FOR SOME DISCRETE DISTRIBUTIONS GENERATED BY CAUCHY STABLE LAW

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1. INTRODUCTION

Based on datasets for various large-scale biomolecular systems several common statistical facts have been discovered. From the mathematical point of view these are: skewness; regular variation at infinity; unimodality; continuity by parameters (stability), etc. of frequency distributions (Astola and Danielian, (2006)).

In large-scale biomolecular systems some frequency distributions are widely used. But the variety of such systems requires to generate new ones that satisfy the empirical facts above. The aim of this article is to prove the asymptotic properties of some distributions arising in large-scale biomolecular systems.

Taking into account statistical facts, new empirical frequency distributions so-called *Stable Laws* are suggested for the needs of large scale biomolecular systems. The *Stable Laws* are a rich class of probability distributions that allows skewness, heavy tails and have many intriguing mathematical properties. There are several equivalent definitions of stable distributions. One of them is as follows (Nolan, 2007; Zolotarev, 1986):

Definition 1.1. A random variable X is said to have a *stable* distribution if there is a sequence of independent, identically distributed random variables $X_1, X_2, \ldots$ and sequence of positive numbers $\{c_n\}$ and real numbers $\{a_n\}$, such that

$$\frac{X_1 + X_2 + \ldots + X_n}{c_n} + a_n \xrightarrow{d} X.$$

The notation $\xrightarrow{d}$ denotes convergence in distribution.

Any stable distribution $S(\alpha, \beta, \gamma, \delta)$ depends on the following parameters: the index exponent α, the skewness β, the scale parameter γ, and the location parameter δ. If $\alpha = 1$ and $\beta = 0$, that is, $S(1, 0, \gamma, \delta)$, then distribution is called *Cauchy* stable distribution which has the following density (Zolotarev, 1986).

$$s(x,\gamma,\delta)=\frac{\gamma}{2\left(\frac{\pi^2}{4}\gamma^2+(x-\delta)^2\right)}, \quad -\infty<x<+\infty. \tag{1}$$

In this article we suppose that

$$\Theta=\{(\gamma,\delta):\ 0<\gamma<+\infty,\ \delta=0\},$$

and F is an arbitrary open subset of Θ whose closure $\overline{F}$ is also contained in Θ.

Now, we construct the following discrete distribution

$$g(x,\gamma)=d_\gamma^{-1}s(x,\gamma), \quad x=0,1,2,... \quad \text{with } d_\gamma=\sum_{y=0}^{\infty}s(y,\gamma), \tag{2}$$

where $s(x,\gamma)$ is the *Cauchy* stable density.

2. ASYMPTOTIC PROPERTIES OF MAXIMUM LIKELIHOOD ESTIMATOR (M.L.E.)

Let $X^n=(X_1,X_2,...,X_n)$ be a finite sample from $g(x;\gamma),\ \gamma\in\Theta,$ and $x^n=(x_1,x_2,...,x_n)$ be the realization of X^n, and suppose $\hat{\gamma}_n$ is the M.L.E. of the scale parameter γ. Our goal is to demonstrate the strong consistency, asymptotic normality and asymptotic efficiency of the M.L.E. of the scale parameter γ. To do this, it is enough to verify that the family of $g(x,\gamma),\ \gamma\in\Theta,$ holds the regularity conditions (R.C.) 1-5 of the following theorem (Borovkov, 1998; DuMouchel, 1973; Lehmann, 1983).

Theorem 2.1. Let $\gamma_0\in\overline{F}$ be the true value of γ, and suppose that the following R.C. are met:

1. The function $g(x,\gamma)$ is continuous of γ for $\gamma\in\Theta$, and has continuous derivatives of first and second orders with respect to γ for $\gamma\in\overline{F}$.

2. For all $\gamma\in\Theta$, and for all $\gamma_0\in\overline{F}$ $(\gamma\neq\gamma_0)$, the condition $\sum_{x=0}^{\infty}\left|g(x,\gamma)-g(x,\gamma_0)\right|>0$ is met.

3. For all $\gamma_0\in\overline{F}$

$$E_{\gamma_0}\left[\sup_{\gamma\in\Theta-\overline{F}}\ln\frac{L(X^n,\gamma)}{L(X^n,\gamma_0)}\right]<\infty,$$

where $L(X^n,\gamma)=\prod_{i=1}^{n}g(X_i,\gamma)$ is likelihood function.

4. Suppose that $C(x,\gamma)=\dfrac{\partial^2 \ln g(x,\gamma)}{\partial\gamma^2}$. Then for $\gamma \in \overline{F}$, the function $|C(x,\gamma)|$ is restricted by a function $D(x)$, that is $|C(x,\gamma)| \le D(x)$, for which

$$\sum_{x=0}^{\infty} D(x) g(x,\gamma) < \infty. \tag{3}$$

5. For every $\gamma \in \overline{F}$, the Fisher's information quality $I(\gamma)=E_\gamma\left[\dfrac{\partial \ln g(X,\gamma)}{\partial\gamma}\right]^2$ satisfies $0 < I(\gamma) < \infty$ and is continuous with respect to γ.

Under these conditions, the M.L.E. $\hat{\gamma}_n$ is:

(a) with probability 1, the unique solution of the likelihood equation $\dfrac{\partial L(x^n,\gamma)}{\partial\gamma}=0$ in the region $|\hat{\gamma}_n-\gamma_0|<\zeta$, where ζ is some positive number independent of γ_0;

(b) strongly consistent, asymptotically normal and asymptotically efficient, i.e. as $n\to\infty$ then $\sqrt{n}(\hat{\gamma}_n-\gamma_0)\xrightarrow{d} N(0,I^{-1}(\gamma_0))$.

3. THE PROOFS OF THE REGULARITY CONDITIONS

The proofs of conditions 1 and 2 are trivial and omitted.

Proof of condition 3. For all $\gamma \in (\Theta-\overline{F})$, $L(x^n,\gamma)$ is uniformly bounded except for the cases $\gamma\to 0$ and $\gamma\to+\infty$.

From (2) we have

$$L(x^n,\gamma)=\prod_{i=1}^{n}\frac{s(x_i,\gamma)}{d_\gamma}.$$

If $\gamma\to+\infty$, then

$$L(x^n,\gamma)\to 0.$$

A similar proof is met for the case $\gamma\to 0$, which completes the proof of Condition 3.

Proof of condition 4. Taking into account that $\overline{F}$ is a compact set and $C(x,\gamma)$ is continuous with respect to $\gamma\in\overline{F}$, for x fixed and $\gamma\in\overline{F}$, $|C(x,\gamma)|$ is bounded by a function $D(x)$, which is itself bounded in any x fixed point. It remains to investigate the behavior of $D(x)$ as $x\to+\infty$.

Let us now denote

$$g_\gamma(x,\gamma)=\frac{\partial}{\partial\gamma}\left(\frac{s(x,\gamma)}{d_\gamma}\right),\quad g_{\gamma\gamma}(x,\gamma)=\frac{\partial^2}{\partial\gamma^2}\left(\frac{s(x,\gamma)}{d_\gamma}\right). \tag{4}$$

In addition, we have

$$s_\gamma(x,\gamma)=\frac{-\frac{\pi^2}{2}\gamma^2+2x^2}{4(\frac{\pi^2}{4}\gamma^2+x^2)^2} \tag{5}$$

and

$$s_{\gamma\gamma}(x,\gamma)=\frac{\pi^4\gamma^3-12\pi^2\gamma x^2}{16(\frac{\pi^2}{4}\gamma^2+x^2)^3}. \tag{6}$$

Now, using (4)-(6) and doing some calculations it is found that

$$D(x)=O(h(x)),\quad as\quad x\to+\infty,$$

where $h(x)=\frac{1}{x^2}+k,$ k is some positive constant.

In view of the value of $h(x),$ it is readily seen that (3) holds.

The condition 4 is established.

Proof of condition 5. We have

$$I(\gamma)=E_\gamma\left[\frac{\partial\ln g(X,\gamma)}{\partial\gamma}\right]^2=\sum_{x=0}^{\infty}\frac{(g_\gamma(x,\gamma))^2}{g(x,\gamma)}. \tag{7}$$

Obviously, the continuity condition of $I(\gamma)$ with respect to $\gamma\in\overline{F}$ is met.

In order to demonstrate $I(\gamma)>0$, it is enough to show that there exists an x such that $g_\gamma(x,\gamma)\neq 0.$ This is equivalent to saying that

$$s_\gamma(x,\gamma)\sum_{y=0}^{\infty}s(y,\gamma)\neq s(x,\gamma)\sum_{y=0}^{\infty}s_\gamma(y,\gamma). \tag{8}$$

Substituting $x=0$ into (8) and with the help of (1) and (5) the proof is completed.

To satisfy $I(\gamma)<\infty,$ from (7) we obtain

$$I(\gamma)=\frac{1}{d_\gamma}\sum_{x=0}^{\infty}\frac{(s_\gamma(x,\gamma))^2}{s(x,\gamma)}-2\frac{(\frac{\partial d_\gamma}{\partial\gamma})}{(d_\gamma)^2}\sum_{x=0}^{\infty}s_\gamma(x,\gamma)+\frac{(\frac{\partial d_\gamma}{\partial\gamma})^2}{(d_\gamma)^3}\sum_{x=0}^{\infty}s(x,\gamma),$$

implies that

$$I(\gamma)=\frac{1}{8d_\gamma}\sum_{x=0}^{\infty}\frac{(-\frac{\pi^2}{2}\gamma^2+2x^2)^2}{\gamma(\frac{\pi^2}{4}\gamma^2+x^2)^3}$$

$$-\frac{(\frac{\partial d_\gamma}{\partial\gamma})}{2(d_\gamma)^2}\sum_{x=0}^{\infty}\frac{(-\frac{\pi^2}{2}\gamma^2+2x^2)}{(\frac{\pi^2}{4}\gamma^2+x^2)^2}$$

$$+\frac{(\frac{\partial d_\gamma}{\partial\gamma})^2}{2(d_\gamma)^3}\sum_{x=0}^{\infty}\frac{\gamma}{(\frac{\pi^2}{4}\gamma^2+x^2)}.$$

The proof of this statement after doing some simplifications, and with regards to the forms d_γ and $\frac{\partial d_\gamma}{\partial\gamma}$, is not difficult.

The condition 5 is demonstrated.

Finally, in accordance with Theorem 2.1 and the proofs of R.C., the following theorem is proved:

Theorem 3.1. When sampling from $g(x,\gamma),\gamma\in\Theta,$ the M.L.E. $\hat{\gamma}_n$ for γ based on the first n observations, is strongly consistent, asymptotically normal and asymptotically efficient if γ_0 (the true value of γ), is in the interior of the parameter space Θ.

4. CONCLUDING REMARKS

In this paper, we studied the large-sample properties of the M.L.E. for the scale parameter γ of some discrete distributions generated by *Cauchy* stable law. To prove this statement we showed that the family of probability functions $g(x,\gamma),\gamma\in\Theta,$ satisfies Conditions 1-5 given in Section 2.

This results can be applied for obtaining the respective statistical inferences in evolutionary large-scale biomolecular systems, including biomolecular networks with growing size over the time.

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SUMMARY

Asymptotic properties of maximum likelihood estimator for some discrete distributions generated by Cauchy stable law

In large-scale biomolecular sysrems there are frequency distribuions with properties like Stable Laws. It is of interest to construct such frequency distributions. In the present article we consider Cauchy stable law. The large-sample distribution of the Maximum Likelihood Estimator (M.L.E.) of the scale parameter for some discrete distributions generated by Cauchy stable law are investigated. The existence, strong consistency, asymptotic normality and asymptotic efficiency of that is established.