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ON BIVARIATE GEOMETRIC DISTRIBUTION

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1. INTRODUCTION

Probability distributions of random sums of independently and identically distributed random variables are mainly applied in modeling practical problems that deal with certain phenomena in which the respective mathematical models are sums of random number of independent random variables. A lot of such situations arise in actuarial science, queuing theory and nuclear physics. Gnedenko and Korolev (1996) gave a number of situations where we usually come across random summation, especially geometric summation and describe the modeling of such situations with respective physical terminology.

Kozubowski and Panorska (1999) applied the distribution of geometric sums in financial portfolio modeling. Kozubowski and Rachev (1994) used geometric sums as an adequate device to model the foreign currency exchange rate data. In this paper our aim is to obtain characterizations of bivariate geometric distribution using geometric compounding. We introduce new bivariate geometric distributions using the geometric sums of independently and identically distributed random variables. These bivariate geometric distributions are closed under geometric summation. Therefore the probability distributions introduced in this study may be appropriate in modeling bivariate data sets which are closed under geometric summation.

It is well known that bivariate analogues of univariate distributions can be obtained by extending their generating functions appropriately. Consider a sequence of independent Bernoulli trials in which the probability of success in each trial is p, $0<p<1$. Let X be the number of failures preceding first success. Then X follows geometric distribution with probability generating function (pgf)

$$P(s)=\frac{1}{1+c(1-s)} \text{ where } c=\frac{1-p}{p}.$$

A natural extension of $P(s)$ gives the following bivariate geometric distribution. A non negative integer valued bivariate random variable (X,Y) has bivariate geometric distribution $(BGD(c_1,c_2,\theta^2))$ if its pgf is

$$\pi(s_1,s_2)=\frac{1}{(1+c_1(1-s_1))(1+c_2(1-s_2))-\theta^2 c_1 c_2(1-s_1)(1-s_2)} \tag{1}$$

where $c_1, c_2 > 0$, $0 \le \theta^2 \le 1$, $|s_1| \le 1$ and $|s_2| \le 1$. Note that the components of (X,Y) have univariate geometric distribution.

Phatak and Sreehari (1981) considered a bivariate geometric distribution which could be interpreted as a shock model. Assume that two components are affected by shocks, with probability p_1, the first component survives, with probability p_2, the second component survives and with probability p_0 both components fail. Let N_1 and N_2 be the number of shocks to the first and second components respectively before the first failure of the system. Then (N_1,N_2) has the following joint probability distribution

$$P(N_1=n_1,N_2=n_2)=\binom{n_1+n_2}{n_1} p_1^{n_1} p_2^{n_2} p_0,\ p_0+p_1+p_2=1,\ n_1,n_2=0,1,2,3,...$$

Its pgf is

$$\pi(s_1,s_2)=\frac{1}{1+\frac{p_1}{p_0}(1-s_1)+\frac{p_2}{p_0}(1-s_2)}.$$

Note that (N_1,N_2) has $BGD(c_1,c_2,1)$ where $c_1=\frac{p_1}{p_0}$ and $c_2=\frac{p_2}{p_0}$.

In order to obtain characterizations of $BGD(c_1,c_2,1)$ using geometric compounding, we make use of the operator $'\oplus'$ defined as follows:

Let X be random variable with pgf $\pi(s)$. $p\oplus X$ is defined (in distribution) by the pgf $\pi(1-p+ps)$ or $p\oplus X=\sum_{j=1}^{X} Z_j$ where $P(Z_j=1)=1-P(Z_j=0)=p$, all random variables Z_j being independent. A bivariate extension of this result is considered. If (X, Y) has pgf $\pi(s_1,s_2)$ then the distribution of $(p\oplus X, p\oplus Y)$ is defined by the pgf $\pi(1-p+ps_1, 1-p+ps_2)$.

Let $\{(X_i,Y_i), i\ge 1\}$ be a sequence of independently and identically distributed random variables with pgf $\pi(s_1,s_2)$. Define

$$U_N=X_1+X_2+...+X_N$$

and

$$V_N=Y_1+Y_2+...+Y_N \tag{2}$$

where N is independent of $(X_i, Y_i), i \geq 1$ and follows geometric distribution such that

$$\mathrm{P}(N = n)\ (1-p)^{n-1} p, n = 1, 2, 3, ..., \tag{3}$$

Then the pgf of (U_N, V_N) is given by

$$\begin{aligned}\eta(s_1, s_2) &= E(s_1^{U_N} s_2^{V_N}) \\ &= \sum_{n=1}^{\infty} (\pi(s_1, s_2))^n P(N = n) \\ &= \frac{p\pi(s_1, s_2)}{1-(1-p)\pi(s_1, s_2)}.\end{aligned} \tag{4}$$

Block (1977) considered a compounding scheme using bivariate geometric distribution. A random variable (N_1, N_2) has bivariate geometric distribution with parameters p_{00}, p_{10}, p_{01} and p_{11} if its survival function is

$$\begin{aligned}\overline{F}(n_1, n_2) &= P(N_1 > n_1, N_2 > n_2) \\ &= \begin{cases} p_{11}^{n_1}(p_{01} + p_{11})^{n_2 - n_1} & \text{if } n_1 \leq n_2 \\ p_{11}^{n_2}(p_{10} + p_{11})^{n_1 - n_2} & \text{if } n_2 \leq n_1 \end{cases}\end{aligned} \tag{5}$$

where $p_{00} + p_{10} + p_{01} + p_{11} = 1$, $p_{10} + p_{11} < 1$, $p_{01} + p_{11} < 1$ and $n_1, n_2 =$ 1, 2, 3, ...

Consider a sequence of independently and identically distributed random variables $\{(X_i, Y_i), i \geq 1\}$ which is also independent of (N_1, N_2) where (N_1, N_2) follows the bivariate geometric distribution in (5). Define

$$U_{N_1} = \sum_{i=1}^{N_1} X_i \text{ and } V_{N_2} = \sum_{i=1}^{N_2} Y_i \tag{6}$$

Block (1977) obtained the Laplace transform of (U_{N_1}, V_{N_2}). Its discrete analogue is

$$\eta(s_1, s_2) = \pi(s_1, s_2)(p_{00} + p_{10}\eta(s_1, 1) + p_{01}\eta(1, s_2) + p_{11}\eta(s_1, s_2)). \tag{7}$$

From (7) we get,

$$\eta(s_1, 1) = \frac{(p_{00} + p_{01})\pi(s_1, 1)}{1-(p_{10} + p_{11})\pi(s_1, 1)}$$

and

$$\eta(1,s_2)=\frac{(p_{00}+p_{10})\pi(1,s_2)}{1-(p_{01}+p_{11})\pi(1,s_2)} \tag{8}$$

Using geometric compounding, in section 2 we obtain characterizations of $BGD(c_1,c_2,1)$. In section 3, autoregressive models with $BGD(c_1,c_2,1)$ marginals are developed. Different bivariate geometric distributions are introduced in section 4 using bivariate geometric compounding.

2. CHARACTERIZATION OF BIVARIATE GEOMETRIC DISTRIBUTION

The following theorem gives a characterization of $BGD(c_1,c_2,1)$.

Theorem 2.1. Let $\{(X_i,Y_i),i\geq 1\}$ be a sequence of independently and identically distributed random variables and N be independent of $(X_i,Y_i),i\geq 1$. Suppose that N follows the geometric distribution in (3) and, U_N and V_N are as defined in (2). Then $(p\oplus U_N, p\oplus V_N)$ and $(X_i,Y_i),i\geq 1$ are identically distributed if and only if $(X_i,Y_i),i\geq 1$ follow $BGD(c_1,c_2,1)$.

Proof. Let $\pi(s_1,s_2)$ be the pgf of $(X_i,Y_i),i\geq 1$. Using (4), the pgf of $(p\oplus U_N, p\oplus V_N)$ is

$$\eta(s_1,s_2)=\frac{p\pi(1-p+ps_1,1-p+ps_2)}{1-(1-p)\pi(1-p+ps_1,1-p+ps_2)} \tag{9}$$

Assuming that $(X_i,Y_i),i\geq 1$ follow $BGD(c_1,c_2,1)$. Substituting $\pi(s_1,s_2)$ in (9) and simplifying, we get $\eta(s_1,s_2)=\dfrac{1}{1+c_1(1-s_1)+c_2(1-s_2)}$.

Conversely, assume that $(p\oplus U_N, p\oplus V_N)$ follows $BGD(c_1,c_2,1)$. Then from (9) we have

$$\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}=\frac{p\pi(1-p+ps_1,1-p+ps_2)}{1-(1-p)\pi(1-p+ps_1,1-p+ps_2)}.$$

Solving we get,

$$\pi(s_1,s_2)=\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}.$$

Now we obtain $\text{BGD}(c_1,c_2,\theta^2)$ as a geometric compound of independently and identically distributed random variables.

Theorem 2.2. Consider a sequence $\{(X_i, Y_i), i \geq 1\}$ of independently and identically distributed random variables, also independent of N which has the geometric distribution in (3). Let U_N and V_N be as defined in (2). Then $(p \oplus U_N, p \oplus V_N)$ follows $BGD(c_1, c_2, q)$ where $q = 1- p$ if and only if $(X_i, Y_i), i \geq 1$ have $BGD(c_1, c_2, 0)$.

Proof. Suppose that $(X_i, Y_i), i \geq 1$ have pgf

$$\pi(s_1, s_2) = \frac{1}{(1 + c_1(1 - s_1))(1 + c_2(1 - s_2))}.$$

From (9), the pgf of $(p \oplus U_N, p \oplus V_N)$ is

$$\eta(s_1, s_2) = \frac{p}{(1 + c_1(1 - s_1))(1 + c_2(1 - s_2)) - 1 + p}$$

$$= \frac{1}{1 + c_1(1 - s_1) + c_2(1 - s_2) + pc_1c_2(1 - s_1)(1 - s_1)}.$$

Comparing with (1), we get $\theta^2 = q$. Hence $(p \oplus U_N, p \oplus V_N)$ follows $BGD(c_1, c_2, q)$.

To prove the converse of the theorem, substituting the pgf of $(p \oplus U_N, p \oplus V_N)$ in (9), we get

$$\frac{1}{1 + c_1(1 - s_1) + c_2(1 - s_2) + pc_1c_2(1 - s_1)(1 - s_2)} = \frac{p\pi(1 - p + ps_1, 1 - p + ps_2)}{1 - (1 - p)\pi(1 - p + ps_1, 1 - p + ps_2)}.$$

Solving, we obtain

$$\pi(s_1, s_2) = \frac{1}{(1 + c_1(1 - s_1))(1 + c_2(1 - s_2))}.$$

A characterization of geometric distribution is obtained in the following theorem.

Theorem 2.3. Suppose that $\{(X_i, Y_i), i \geq 1\}$ is a sequence of independently and identically distributed random variables according to $BGD(c_1, c_2, 1)$. Then $(p \oplus U_N, p \oplus V_N)$ and $(X_i, Y_i), i \geq 1$ are identically distributed if and only if N is geometric where U_N and V_N are as defined in (2).

Proof. The 'if' part of the theorem is proved in theorem 2.1. In order to prove the 'only if' part, assume that $(X_i, Y_i), i \geq 1$ and $(p \oplus U_N, p \oplus V_N)$ are identically distributed as $BGD(c_1, c_2, 1)$. Without loss of generality, taking $c_1 = c_2 = 1$. The pgf of $(p \oplus U_N, p \oplus V_N)$ is

$$\eta(s_1, s_2) = \sum_{n=1}^{\infty} (\pi(1-p+ps_1, 1-p+ps_2))^n P(N=n).$$

From the assumption, we get

$$\sum_{n=1}^{\infty} \left(\frac{1}{1+p(1-s_1)+p(1-s_2)} \right)^n P(N=n) = (1+(1-s_1)+(1-s_2))^{-1}.$$

Expanding with respect to n and comparing the coefficients of $((1-s_1)+(1-s_2))^j$, we get

$$\sum_{n=1}^{\infty} n(n+1)(n+2)...(n+j-1)P(N=n) = \frac{j!}{p^j}, \text{ for } j = 1,2,3,...$$

Consider

$$E(1-t)^{-N} = 1 + \frac{t}{1!}E(N) + \frac{t^2}{2!}E(N(N+1)) + \frac{t^3}{3!}E(N(N+1)(N+2)) + ...$$

$$= \frac{p}{p-t}$$

$$= p\sum_{n=1}^{\infty} (1-t)^{-n}(1-p)^{n-1}.$$

But $E(1-t)^{-N} = \sum_{n=1}^{\infty} (1-t)^{-n} P(N=n)$.

Therefore,

$$P(N=n) = (1-p)^{n-1} p, \text{ for } n = 1, 2, 3,...$$

Theorem 2.4. Let N_1 and N_2 be two independent random variables following geometric distribution given in (3) with parameters a and b respectively and $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random variables which are also independent of N_1 and N_2. Let $U_{N_i} = X_1 + X_2 + ... + X_{N_i}$ and $V_{N_i} = Y_1 + Y_2 + ... + Y_{N_i}$, for $i = 1,2$. Then

$(a\oplus U_{N_1}, a\oplus V_{N_1})$ and $(b\oplus U_{N_2}, b\oplus V_{N_2})$ are identically distributed if $(X_i, Y_i), i\ge 1$ have $BGD(c_1, c_2, 1)$.

Proof. Suppose that $(X_i, Y_i), i\ge 1$ have $BGD(c_1, c_2, 1)$. Therefore from (9) we obtain the pgf of $(a\oplus U_{N_1}, a\oplus V_{N_1})$ as $\eta(s_1,s_2)=\dfrac{1}{1+c_1(1-s_1)+c_2(1-s_2)}$. Similarly, we can show that $(b\oplus U_{N_2}, b\oplus V_{N_2})$ has also the same pgf. Hence the proof.

In the next theorem we obtain characterization of $BGD(c_1, c_2, \theta^2)$ using bivariate geometric compounding.

Theorem 2.5. Let $\{(X_i, Y_i), i\ge 1\}$ be a sequence of independently and identically distributed random variables. Suppose that (N_1, N_2) has bivariate geometric distribution given in (5) and independent of $(X_i, Y_i), i\ge 1$. Take $p_{00}=0$, $p_{01}=\mu_1$, $p_{10}=\mu_2$ and $p_{11}=1-(\mu_1+\mu_2)$. Then (U_{N_1}, V_{N_2}) follows $BGD\left(\dfrac{c_1}{\mu_1}, \dfrac{c_2}{\mu_2}, 0\right)$ if and only if $(X_i, Y_i), i\ge 1$ follow $BGD(c_1, c_2, 1)$ where U_{N_1} and V_{N_2} are as given in (6).

Proof. Assume that $(X_i, Y_i), i\ge 1$ follow $BGD(c_1, c_2, 1)$. Substituting its pgf and the values of p_{00}, p_{10}, p_{01}, p_{11} in (7). The pgf of (U_{N_1}, V_{N_2}) is

$$\eta(s_1,s_2)=\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}$$

$$\left(\mu_2\left(1+\frac{c_1(1-s_1)}{\mu_1}\right)^{-1}+\mu_1\left(1+\frac{c_2(1-s_2)}{\mu_2}\right)^{-1}+(1-\mu_1-\mu_2)\eta(s_1,s_2)\right).$$

Solving we get,

$$\eta(s_1,s_2)=\frac{1}{\left(1+\dfrac{c_1(1-s_1)}{\mu_1}\right)\left(1+\dfrac{c_2(1-s_2)}{\mu_2}\right)}.$$

Conversely, assume that

$$\eta(s_1,s_2)=\frac{1}{\left(1+\dfrac{c_1(1-s_1)}{\mu_1}\right)\left(1+\dfrac{c_2(1-s_2)}{\mu_2}\right)}.$$

Therefore from (7)

$$\frac{1}{\left(1+\frac{c_1(1-s_1)}{\mu_1}\right)\left(1+\frac{c_2(1-s_2)}{\mu_2}\right)}=\pi(s_1,s_2)$$

$$\left(p_{00}+p_{10}\frac{1}{\left(1+\frac{c_1(1-s_1)}{\mu_1}\right)}+p_{01}\frac{1}{\left(1+\frac{c_2(1-s_2)}{\mu_2}\right)}+p_{11}\frac{1}{\left(1+\frac{c_1(1-s_1)}{\mu_1}\right)\left(1+\frac{c_2(1-s_2)}{\mu_2}\right)}\right)$$

Substituting the values of p_{00}, p_{10}, p_{01} and p_{11} we get

$$\pi(s_1,s_2)=\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}.$$

Theorem 2.6. Consider a sequence of independently and identically distributed random variables $\{(X_i,Y_i), i\geq 1\}$. Suppose that (N_1,N_2) is independent of $(X_i,Y_i), i\geq 1$ and follows the bivariate geometric distribution in (5) such that $p_{11}=p, p_{10}+p_{11}=p_1, p_{01}+p_{11}=p_2$ and $p_{00}=0$. Then $(q_1\oplus U_{N_1}, q_2\oplus V_{N_2})$ has $BGD(c_1,c_2,0)$ if and only if $(X_i,Y_i), i\geq 1$ have $BGD(c_1,c_2,1)$ where $q_i=1-p_i$.

Proof. Using (7) the pgf of $(q_1\oplus U_{N_1}, q_2\oplus V_{N_2})$ is

$$\eta(s_1,s_2)=\frac{\pi(s_1,s_2)\left(\frac{p_{10}(p_{00}+p_{01})\pi(s_1,1)}{1-(p_{10}+p_{11})\pi(s_1,1)}+\frac{p_{01}(p_{00}+p_{10})\pi(1,s_2)}{1-(p_{01}+p_{11})\pi(1,s_2)}\right)}{1-p_{11}\pi(s_1,s_2)}$$

where $\pi(s_1,s_2)$ is the pgf of $(X_i,Y_i), i\geq 1$. Assume that $(X_i,Y_i), i\geq 1$ have $BGD(c_1,c_2,1)$. Then we get

$$\eta(s_1,s_2)=\frac{q_1q_2}{1+c_1q_1(1-s_1)+c_2q_2(1-s_2)-p}$$

$$\left(\frac{1}{1+c_1q_1(1-s_1)-p_1}+\frac{1}{1+c_2q_2(1-s_2)-p_2}\right).$$

On simplification, we get

$$\eta(s_1,s_2)=\frac{1}{(1+c_1(1-s_1))(1+c_2(1-s_2))}.$$

Conversely suppose that $(q_1 \oplus U_{N_1}, q_2 \oplus V_{N_2})$ follows $BGD(c_1,c_2,0)$. Then from (7) we have

$$\frac{1}{(1+c_1(1-s_1))(1+c_2(1-s_2))}=\frac{\pi(1-q_1+q_1s_1,1-q_2+q_2s_2)}{1-p\pi(1-q_1+q_1s_1,1-q_2+q_2s_2)}$$

$$\left(\frac{q_2}{1+c_1(1-s_1)}+\frac{q_1}{1+c_2(1-s_2)}\right).$$

Solving, we get

$$\pi(s_1,s_2)=\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}.$$

Theorem 2.7. Let $\{(X_i,Y_i), i\geq 1\}$ be a sequence of independently and identically distributed random variables. Suppose that (N_1,N_2) is independent of $(X_i,Y_i), i\geq 1$ and has bivariate geometric distribution, as stated in theorem 2.6. Then $(q\oplus U_{N_1}, q\oplus V_{N_2})$ has pgf $BGD(q,q,0)$ if and only if $(X_i,Y_i), i\geq 1$ have pgf $BGD(q_1,q_2,1)$ where q = 1 - p.

Proof. Suppose that $(X_i,Y_i), i\geq 1$ have $BGD(q_1,q_2,1)$. Substituting in (7), we get the pgf of $(q\oplus U_{N_1}, q\oplus V_{N_2})$ as

$$\eta(s_1,s_2)=\frac{q_1q_2}{1-q_1q(1-s_1)+q_2q(1-s_2)-p}\left(\frac{1}{1+q_1q(1-s_1)-p_1}+\frac{1}{1+q_2q(1-s_2)-p_2}\right)$$

On simplification we get

$$\eta(s_1,s_2)=\frac{1}{(1+q(1-s_1))(1+q(1-s_2))}.$$

Conversely, take that $(q\oplus U_{N_1}, q\oplus V_{N_2})$ follows $BGD(q,q,0)$. From (9), we have

$$\frac{1}{(1+q(1-s_1))(1+q(1-s_2))}=$$

$$\frac{\pi(1-q+qs_1,1-q+qs_2)}{1-p\pi(1-q+qs_1,1-q+qs_2)}\left(\frac{q_2}{1+q(1-s_1)}+\frac{q_1}{1+q(1-s_2)}\right).$$

Simplifying, we get

$$\pi(s_1,s_2)=\frac{1}{1+q_1(1-s_1)+q_2(1-s_2)}.$$

3. BIVARIATE AUTOREGRESSIVE GEOMETRIC PROCESS

There are many situations in which discrete time series arise, often as counts of events, objects or individuals in consecutive intervals. For example, number of road accidents that occur on national highways of a country on a day, number of customers waiting in a ticket booking counter that recorded in every one hour duration, number of calls received in a fire rescue centre in a week, etc. Moreover such data can also be obtained by discretization of continuous variate time series. Recently, much focus is given on developing integer valued stationary autoregressive processes. (see, Jayakumar (1995)). Here we develop first order autoregressive process with marginals as bivariate geometric distribution.
Consider an autoregressive process $(X_n,Y_n),n,\geq 1$ with structure

$(X_0,Y_0)\underline{\underline{d}}(U_1,V_1)$ and for n=1, 2, 3,

$$(X_n,Y_n)=(\rho\oplus X_{n-1}+U_n,\rho\oplus Y_{n-1}+V_n),\ 0<\rho<1 \tag{10}$$

where $(U_n,V_n),n\geq 1$ are independently and identically distributed random variables satisfying $(U_n,V_n)\underline{\underline{d}}(I_n\varepsilon_n,I_n\psi_n)$. $\{I_n,n\geq 1\}$ and $\{(\varepsilon_n,\psi_n),n\geq 1\}$ are two independent sequences of independently and identically distributed random variables and $(I_n,n\geq 1)$ have Bernoulli distribution with $P(I_n=0)=1-P(I_n=1)=\rho$. The following theorem gives a necessary and sufficient condition for a stationary autoregressive process to have $BGD(c_1,c_2,1)$ marginals.

Theorem 3.1. Let $(X_n,Y_n),n\geq 1$ be a first order bivariate autoregressive process with structure in (10). The process is stationary with $BGD(c_1,c_2,1)$ marginals if and only if $(\varepsilon_n,\psi_n),n\geq 1$ follow $BGD(c_1,c_2,1)$
Proof. The pgf of (10) is

$$\pi_{X_n,Y_n}(s_1,s_2)=\pi_{X_{n-1},Y_{n-1}}(1-\rho+\rho s_1,1-\rho+\rho s_2)\pi_{\varepsilon_n,\psi_n}(s_1,s_2). \tag{11}$$

Suppose that the process is stationary with $BGD(c_1,c_2,1)$ marginals, then we have

$$\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}=\frac{1}{1+c_1\rho(1-s_1)+c_2\rho(1-s_2)}\pi_{U,V}(s_1,s_2).$$

Solving we get,

$$\pi_{U,V}(s_1,s_2)=\rho+\frac{1-\rho}{1+c_1(1-s_1)+c_2(1-s_2)}.$$

To prove the converse assume that $(\varepsilon_n,\psi_n,n\geq 1)$ follow $BGD(c_1,c_2,1)$. Taking $n=1$ in (11), we get

$$\pi_{X_1,Y_1}(s_1,s_2)=\pi_{X_0,Y_0}(1-\rho+\rho s_1,1-\rho+\rho s_2)\pi_{U_1,V_1}(s_1,s_2).$$

Under the assumption,

$$\pi_{X_1,Y_1}(s_1,s_2)=\frac{1}{1+c_1\rho(1-s_1)+c_2\rho(1-s_2)}\left(\rho+\frac{1-\rho}{1+c_1(1-s_1)+c_2(1-s_2)}\right)$$

$$=\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}.$$

Hence the process is stationary with marginals follow $BGD(c_1,c_2,1)$.

Now consider a first order random coefficient autoregressive model with following structure.

$$(X_0,Y_0)\underline{\underline{d}}(\varepsilon_0,\psi_0)$$

$$(X_n,Y_n)=(V_n\oplus X_{n-1}+V_n\oplus\varepsilon_n,V_n\oplus Y_{n-1}+V_n\oplus\psi_n),\ n\geq 1 \tag{12}$$

where $\{(\varepsilon_n,\psi_n,n\geq 1)\}$ and $\{V_n,n\geq 1\}$ are two independent sequences of independently and identically distributed random variables such that $V_n,n\geq 1$ have uniform distribution on (0, 1). The following theorem gives a necessary and sufficient condition for the process in (12) to be stationary.

Theorem 3.2. Consider an autoregressive process $(X_n,Y_n),n\geq 1$ given in (12). It is a stationary first order autoregressive process with $BGD(c_1,c_2,1)$ marginals if and only if $(\varepsilon_0,\psi_0)\underline{\underline{d}}\ BGD(c_1,c_2,1)$.

Proof. The pgf of (12) is

$$\pi_{X_n,Y_n}(s_1,s_2)=E(s_1^{V_n\oplus X_{n-1}+V_n\oplus\varepsilon_n}s_2^{V_n\oplus Y_{n-1}+V_n\oplus\psi_n})$$

$$=\int_0^1\pi_{X_{n-1},Y_{n-1}}(1-v_n+v_ns_1,1-v_n+v_ns_2)\pi_{\varepsilon_n,\psi_n}(1-v_n+v_ns_1,1-v_n+v_ns_2)dv.$$

If $(X_n,Y_n),n\geq 1$ is stationary, we have

$$\pi_{X,Y}(s_1,s_2)=\int_0^1 \pi^2_{X,Y}(1-v+vs_1,1-v+vs_2)\,dv\,.$$

Let $\pi_{X,Y}(1-s_1,1-s_2)=\gamma_{X,Y}(s_1,s_2)$. Then we get

$$\gamma_{X,Y}(s_1,s_2)=\int_0^1 \gamma^2_{X,Y}(vs_1,vs_2)dv\,. \tag{13}$$

Taking $s_j=\delta_j s$ for j = 1, 2.

$$\gamma_{X,Y}((\delta_1,\delta_2)s)=\int_0^1 \gamma^2_{X,Y}((\delta_1,\delta_2)sv)dv\,.$$

If $sv=t$, then $s\gamma_{X,Y}((\delta_1,\delta_2)s)=\int_0^s \gamma^2_{X,Y}((\delta_1,\delta_2)t)dt$.

Differentiating with respect to s and then dividing by $\gamma^2_{X,Y}((\delta_1,\delta_2)s)$, we get

$$\frac{s\gamma'_{X,Y}((\delta_1,\delta_2)s)}{\gamma^2_{X,Y}((\delta_1,\delta_2)s)}+\frac{1}{\gamma_{X,Y}((\delta_1,\delta_2)s)}=1\,.$$

Writing $\gamma_{X,Y}((\delta_1,\delta_2)s)=\dfrac{1}{1+\omega((\delta_1,\delta_2)s)}$, we get

$$\omega((\delta_1,\delta_2)s)=\mu^* s \text{ where } \mu^* \text{ is a function of } (\delta_1,\delta_2)$$

That is,

$$\gamma_{X,Y}((\delta_1,\delta_2)s)=\frac{1}{1+\mu^* s}$$

$$=\frac{1}{1+c_1s_1+c_2s_2}\,.$$

Thus $\pi_{X,Y}(s_1,s_2)=\dfrac{1}{1+c_1(1-s_1)+c_2(1-s_2)}$ and hence we obtain

$$(\varepsilon_0,\psi_0)\underline{\underline{d}}\ BGD(c_1,c_2,1)\,.$$

Conversely, suppose that $(X_0,Y_0)\underline{\underline{d}}(\varepsilon_0,\psi_0)$ and (ε_0,ψ_0) has $BGD(c_1,c_2,1)$. From (13), we have

$$\gamma_{X_1,Y_1}(s_1,s_2)=\int_0^1 \gamma^2_{\varepsilon_0,\psi_0}(vs_1,vs_2)dv$$

$$=\frac{1}{1+c_1s_1+c_2s_2}.$$

Therefore,

$$\pi_{X_1,Y_1}(s_1,s_2)=\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}.$$

Thus (X_1,Y_1) is distributed as bivariate geometric distribution. By induction we get $(X_n,Y_n), n\geq 1$ follow $BGD(c_1,c_2,1)$. Hence the process is stationary.

4. BIVARIATE GEOMETRIC DISTRIBUTION

In this section we construct various bivariate geometric distributions using the bivariate geometric compounding. We discuss the discrete analogues of many important bivariate exponential distributions like, Marshall-Olkin's (1967) bivariate exponential, Downton's (1970) bivariate exponential and Hawkes' (1972) bivariate exponential.

Theorem 4.1. Let $\{(X_i,Y_i), i\geq 1\}$ be a sequence of independently and identically distributed random variables according to $BGD(c_1,c_2,1)$ and (N_1,N_2) follow, independent of $(X_i,Y_i), i\geq 1,$ the bivariate geometric distribution given in (5). Choose $p_{00}=\mu_{12}, p_{10}=\mu_2, p_{01}=\mu_1 \quad p_{11}=1-\mu$ and $\mu=\mu_1+\mu_2+\mu_{12}$. Define U_{N_1} and V_{N_2} as given in (6). Then (U_{N_1},V_{N_2}) follows bivariate geometric distribution which is the discrete analogue of Marshall-Olkin's bivariate exponential.
Proof. Let $(X_i,Y_i), i\geq 1$ have pgf $\pi(s_1,s_2)$. Substituting the values of $p_{00}, p_{10}, p_{01}, p_{11}$ and $\pi(s_1,s_2)$ in (7), we get

$$\eta(s_1,s_2)=\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}(\mu_{12}+\mu_2\eta(s_1,1)+\mu_1\eta(1,s_2)+(1-\mu)\eta(s_1,s_2)). \tag{14}$$

Solving for $\eta(s_1,1)$ and $\eta(1,s_2)$, we get

$$\eta(s_1,1)=\frac{\mu_2+\mu_{12}}{\mu_2+\mu_{12}+c_1(1-s_1)} \text{ and } \eta(1,s_2)=\frac{\mu_1+\mu_{12}}{\mu_1+\mu_{12}+c_2(1-s_2)}.$$

Substituting $\eta(s_1,1)$ and $\eta(1,s_2)$ in (14), we get

$$\eta(s_1,s_2)=\frac{1}{1+c_1(1-s_1)+c_2(1-s_2)}$$

$$\left(\mu_{12}+\frac{\mu_1(\mu_2+\mu_{12})}{\mu_2+\mu_{12}+c_1(1-s_1)}+\frac{\mu_2(\mu_1+\mu_{12})}{\mu_1+\mu_{12}+c_2(1-s_2)}+(1-\mu)\eta(s_1,s_2)\right)$$

On simplification,

$$\eta(s_1,s_2)=\frac{1}{\mu+c_1(1-s_1)+c_2(1-s_2)}\left(\mu_{12}+\mu_1\left(1+\frac{c_1(1-s_1)}{1-\mu_1}\right)^{-1}+\mu_2\left(1+\frac{c_2(1-s_2)}{1-\mu_2}\right)^{-1}\right)$$

In the next theorem, we give the bivariate geometric distribution which is analogues to Downton's bivariate exponential distribution.

Theorem 4.2. Suppose that $(X_i,Y_i), i\geq 1$ are independently and identically distributed random variables with pgf $\pi(s_1,s_2)=\left(1+\frac{c_1(1-s_1)}{1+\mu}\right)^{-1}\left(1+\frac{c_2(1-s_2)}{1+\mu}\right)^{-1}$, $\mu>0$ and independent of (N_1,N_2) where (N_1,N_2) follows the bivariate geometric distribution given in (5). Taking $p_{00}=(1+\mu)^{-1}$, $p_{10}=p_{01}=0$ and $p_{11}=\mu(1+\mu)^{-1}$. Then (U_{N_1},V_{N_2}) has bivariate geometric distribution which is the discrete analogue of Downton's bivariate exponential distribution where U_{N_1} and V_{N_2} are as defined in (6).

Proof. Supposing that $(X_i,Y_i), i\geq 1$ have pgf $\pi(s_1,s_2)$. From (7) the pgf of (U_{N_1},V_{N_2}) is

$$\eta(s_1,s_2)=\left(1+\frac{c_1(1-s_1)}{1+\mu}\right)^{-1}\left(1+\frac{c_2(1-s_2)}{1+\mu}\right)^{-1}((1+\mu)^{-1}+\mu(1+\mu)^{-1}\eta(s_1,s_2))$$

$$=\frac{1}{(1+\mu)\left(1+\frac{c_1(1-s_1)}{1+\mu}\right)\left(1+\frac{c_2(1-s_2)}{1+\mu}\right)-\mu}$$

$$=\frac{1+\mu}{(1+\mu)^2+(1+\mu)c_1(1-s_1)+(1+\mu)c_2(1-s_2)+c_1c_2(1-s_1)(1-s_2)-\mu(1+\mu)}$$

$$=\frac{1}{(1+c_1(1-s_1))(1+c_2(1-s_2))-\frac{\mu}{1+\mu}c_1c_2(1-s_1)(1-s_2)}.$$

By comparing with (1), we note that (U_{N_1}, V_{N_2}) follows $BGD(c_1, c_2, \theta^2)$ where $\theta^2 = \dfrac{\mu}{1+\mu}$.

The bivariate geometric form of Hawkes' bivariate exponential distribution is given in the following theorem.

Theorem 4.3. Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of independently and identically distributed random variables with pgf is $\pi(s_1, s_2) = \dfrac{1}{(1+\mu_1(1-s_1))(1+\mu_2(1-s_2))}$. Assume that (N_1, N_2) follow bivariate geometric distribution given in (5) and independent of $(X_i, Y_i), i \geq 1$. Choose $p_{10} = p_{01}$, $\gamma_1 = p_{01} + p_{00}$ and $\gamma_2 = p_{10} + p_{00}$. Then the distribution of (U_{N_1}, V_{N_2}) is the bivariate geometric form of Hawkes' bivariate exponential distribution.

Proof. From (8), we have

$$\eta(s_1, 1) = \pi(s_1, 1)(\gamma_1 + (1-\gamma_1)\pi(s_1, 1)).$$

Substituting $\pi(s_1, 1)$ and solving $\eta(s_1, 1)$, we get

$$\eta(s_1, 1) = \frac{1}{1 + \dfrac{\mu_1(1-s_1)}{\gamma_1}}.$$

Similarly

$$\eta(1, s_2) = \frac{1}{1 + \dfrac{\mu_2(1-s_2)}{\gamma_2}}.$$

Again from (7)

$$\eta(s_1, s_2) = \frac{1}{(1+\mu_1(1-s_1))(1+\mu_2(1-s_2))}$$

$$\left(p_{00} + p_{10}\frac{1}{1+\dfrac{\mu_1(1-s_1)}{\gamma_1}} + p_{01}\frac{1}{1+\dfrac{\mu_2(1-s_2)}{\gamma_2}} + p_{11}\pi(s_1, s_2) \right)$$

$$= \frac{p_{00}\left(1+\dfrac{\mu_1(1-s_1)}{\gamma_1}\right)\left(1+\dfrac{\mu_2(1-s_2)}{\gamma_2}\right) + p_{10}\left(1+\dfrac{\mu_2}{\gamma_2}(1-s_2)\right) + p_{01}\left(1+\dfrac{\mu_1}{\gamma_1}(1-s_1)\right)}{\left(1+\dfrac{\mu_1}{\gamma_1}(1-s_1)\right)\left(1+\dfrac{\mu_2}{\gamma_2}(1-s_2)\right)((1+\mu_1(1-s_1))(1+\mu_2(1-s_2)) - p_{11})}$$

Thus (U_{N_1}, V_{N_2}) have the bivariate geometric form of Hawkes' bivariate exponential distribution.

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REFERENCES

H. W. BLOCK, (1977), *A family of bivariate life distributions*. "Theory and Applications of Reliability: With Emphasis on Bayesian and Nonparametric Methods", C. P. Tsokos and I.N. Shimi, eds, Academic Press, pp. 349-372.

F. DOWNTON, (1970), *Bivariate exponential distributions in reliability theory*. "Journal of Royal Statistical Society", Series B, 32, pp. 408-417.

B. V. GNEDENKO, V. KOROLEV, (1996), *Random summation: Limit theorems and applications.*CRC Press, New York.

A. G. HAWKES, (1972), *A bivariate exponential distribution with applications to reliability*. "Journal of Royal Statistical Society", Series B, 34, pp. 129-131.

K. JAYAKUMAR, (1995), *The stationary solution of a first order integer valued autoregressive processes.* "Statistica", LV, pp. 221-228.

T. J. KOZUBOWSKI, A. K. PANORSKA, (1999), *Multivariate geometric stable distribution in financial applications*. "Mathematical and Computer Modeling", 29, pp. 83-92.

T. J. KOZUBOWSKI, S. T. RACHEV, (1994), *The theory of geometric stable laws and its use in modeling financial data*. "European Journal of Operations Research", 74, pp. 310-324.

A. W. MARSHALL, I. OLKIN, (1967), *A multivariate exponential distribution*. "Journal of the American Statistical Association", 62, pp. 30-44.

A. G. PHATAK, M. SREEHARI, (1981), *Some characterizations of bivariate geometric distribution*, "Journal of Indian Statistical Association", 19, pp. 141-146.

SUMMARY

On bivariate geometric distribution

Characterizations of bivariate geometric distribution using univariate and bivariate geometric compounding are obtained. Autoregressive models with marginals as bivariate geometric distribution are developed. Various bivariate geometric distributions analogous to important bivariate exponential distributions like, Marshall-Olkin's bivariate exponential, Downton's bivariate exponential and Hawkes' bivariate exponential are presented.