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BERNSTEIN-TYPE APPROXIMATION USING THE BETA-BINOMIAL DISTRIBUTION

Andrea Pallini

1. INTRODUCTION

The Bernstein polynomials are generally regarded as the most basic tools for the uniform approximation in the sense of Weierstrass of a continuous and real-valued function g on the closed interval $[0,1]$. The Bernstein polynomials are elegant linear positive operators. The Bernstein polynomials of order m are defined by the binomial distribution $p_m(k;t)$, for $k = 0,1,\ldots,m$, where $t \in [0,1]$ is the domain of g. The convergence of the Bernstein polynomials to g is uniform, as $m \to \infty$. Multivariate versions of the Bernstein polynomials can be defined by products of independent binomial distributions. See Korovkin (1960), chapter 1, Davis (1963), chapter 7, Feller (1968), chapter 6, Feller (1971), chapter 7, Rivlin (1981), chapter 1, Cheney (1982), chapters 1 to 4, Lorentz (1986), DeVore and Lorentz (1993), chapter 10, Phillips (2003), chapter 7.

The Bernstein-type approximations of order m in Pallini (2005) consider a convenient approximation coefficient in linear kernels and improve on the degree of approximation of the Bernstein polynomials. The convergence of these Bernstein-type approximations is uniform, as $m \to \infty$.

Here, following Pallini (2005), we study the Bernstein-type approximation of order m that can be defined by using the beta-binomial distribution. We obtain integral operators that approximate to a continuous and real-valued function g on any closed interval $D \subseteq R^1$. We also obtain their multivariate versions for a continuous and real-valued function g on any closed interval $\mathrm{D} \subseteq R^q$. The convergence of these Bernstein-type approximations is uniform, as $m \to \infty$.

In section 2, we overview the univariate and the multivariate Bernstein polynomials. In section 3, we present some basic notions for the use of the beta-binomial distribution in approximation. In section 4, we propose the univariate and multivariate Bernstein-type approximations that can be obtained by the beta-binomial distribution. We study the uniform convergence and the degree of approximation. We also compare these Bernstein-type approximations with the Bernstein polynomials. In section 5, we study the Bernstein-type estimators for smooth functions of the population means. In section 6, we discuss the results of a simulation study on some examples of smooth functions of means. Finally, in section 7, we conclude the contribution with comments and remarks.

We refer to Barndorff-Nielsen and Cox (1989), chapter 4, and Sen and Singer (1993), chapter 3, for more details on the smooth functions of means and their application to classical inferential problems.

2. BERNSTEIN POLYNOMIALS

Let g be a bounded and real-valued function defined on the closed interval $[0,1]$. The Bernstein polynomial $B_m(g;x)$ of order m for the function g is defined as

$$B_m(g;x)=\sum_{k=0}^{m} g(m^{-1}k)\binom{m}{k}x^k(1-x)^{m-k}\,, \tag{1}$$

where m is a positive integer number, and $x\in[0,1]$. If $g(x)$ is continuous on $x\in[0,1]$, then we have that $B_m(g;x)\to g(x)$, as $m\to\infty$, uniformly, at any point $x\in[0,1]$.

Let g be a bounded and real-valued function defined on the closed q-dimensional cube $[0,1]^q$. We let $\mathrm{x}=(x_1,\ldots,x_q)^T$, where $\mathrm{x}\in[0,1]^{\mathrm{q}}$. The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ for the function g is defined as

$$B_{\mathrm{m}}(g;x)=\sum_{k_1=0}^{m_1}\cdots\sum_{k_q=0}^{m_q} g(m_1^{-1}v_1,\ldots,m_q^{-1}v_q)$$

$$\cdot\binom{m_1}{v_1}\cdots\binom{m_q}{v_q}x_1^{v_1}(1-x_1)^{m_1-v_1}\cdots x_q^{v_q}(1-x_q)^{m_q-v_q}\,, \tag{2}$$

where $\mathrm{m}=(m_1,\ldots,m_q)^T$ are positive integer numbers, and $\mathrm{x}\in[0,1]^{\mathrm{q}}$. The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ is of order m, where $m=\sum_{i=1}^{q} m_i$, and $\mathrm{x}\in[0,1]^q$. The multivariate Bernstein polynomial $B_m(g;x)$ converges to $g(\mathrm{x})$ uniformly, at any q-dimensional point of continuity $\mathrm{x}\in[0,1]^q$, as $m_i\to\infty$, where $i=1,\ldots,q$.

3. THE BETA-BINOMIAL DISTRIBUTION

More accurate versions of the Bernstein polynomials $B_m(g;x)$ and $B_{\mathrm{m}}(g;\mathrm{x})$, defined by (1) and (2), where $x \in [0,1]$ and $\mathrm{x} \in [0,1]^{\mathrm{q}}$, can be obtained by the beta-binomial distribution, that is reviewed and studied in Wilcox (1981) and Johnson, Kemp and Kotz (2005), chapter 6.

The standard beta distribution $p(t;a,b)$, with parameters $a > 0$ and $b > 0$, has probability density function (p.d.f.) $p(t;a,b) = \{B(a,b)\}^{-1} t^{a-1}(1-t)^{b-1}$, for every $t \in (0,1)$. We also recall that $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$, where $B(a,b) = (\Gamma(a+b))^{-1}\Gamma(a)\Gamma(b)$, and $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt$. See Balakrishnan and Nevzorov (2003), chapters 16 and 20.

The beta-binomial random variable (r.v.) Y, with parameters m, $a > 0$ and $b > 0$, has p.d.f. $p_m(k;a,b) = \Pr[Y = k]$, that is defined as

$$p_m(k;a,b) = \binom{m}{k}\{B(a,b)\}^{-1}\int_0^1 t^{a+k-1}(1-t)^{b+m-k-1}\,dt\,, \tag{3}$$

for every $k = 0,1,\ldots,m$. We can rewrite the definition (3) as

$$p_m(k;a,b) = \{B(a,b)\}^{-1}\int_0^1 p_m(k;t)\,t^{a-1}(1-t)^{b-1}\,dt\,,$$

where $p_m(k;t) = \binom{m}{k} t^k(1-t)^{m-k}$ is the binomial p.d.f., with parameters m and t, $t \in [0,1]$, for every $k = 0,1,\ldots,m$. Moments of the beta-binomial r.v. Y, are obtained by integrating the moments of the binomial p.d.f. $p_m(k;t)$, $t \in [0,1]$, $k = 0,1,\ldots,m$, that are functions of t, $t \in [0,1]$, through the definition (3) of the beta-binomial p.d.f. $p_m(k;a,b)$, $k = 0,1,\ldots,m$.

In particular, the first two moments about the origin, $\lambda_1'(a,b) \equiv \lambda_1'$ and $\lambda_2'(a,b) \equiv \lambda_2'$, of the beta p.d.f. $p(t;a,b)$, with values $t \in [0,1]$, are

$$\lambda_1'(a,b) = (a+b)^{-1}a\,, \tag{4}$$

$$\lambda_2'(a,b) = \{(a+b)(a+b+1)\}^{-1}a(a+1)\,, \tag{5}$$

and the third moment about the origin λ_3' is

$$\lambda_3' = \{(a+b)(a+b+1)(a+b+2)\}^{-1}a(a+1)(a+2)\,. \tag{6}$$

See Balakrishnan and Nevzorov (2003), chapters 5 and 16, and Johnson, Kemp and Kotz (2005), chapter 3.

The values of the parameters a and b, in the moments $\lambda_1^{'}(a,b)$ and $\lambda_2^{'}(a,b)$, given by (4) and (5), respectively, of the beta p.d.f. $p(t;a,b)$, with values $t \in (0,1)$, that yield a conveniently small quantity

$$\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b) = \{(a+b)(a+b+1)\}^{-1} ab , \tag{7}$$

can be regarded as constructive. More precisely, constructive values of a and b in (7) can directly help to improve the numerical performance of the Bernstein-type approximations that we are going to introduce in section 4. Constructive values of a and b in (7) can lower their uniform convergence rates, as $m \to \infty$.

The quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ given by (7) does not admit a minimizer, for $a > 0$ and $b > 0$. For further details and descriptions, see sections 6 and 7.

4. BERNSTEIN-TYPE APPROXIMATIONS

4.1. *Bernstein-type approximations*

Let g be a bounded and real-valued function defined on the closed interval $D \subseteq R^1$. The Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$ of order m for the function $g(x)$ is defined as

$$C_m^{(s)}(g;x,a,b) = \{B(a,b)\}^{-1} \int_0^1 \sum_{k=0}^{m} g(m^{-s}(m^{-1}k - t) + x) \cdot \binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1}\, dt , \tag{8}$$

where $s > -1/2$ is fixed, m is a positive integer number, and $x \in D$. Properties of the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$, given by (8), $x \in D$, are outlined in Appendix 8.1.

If the function $g(x)$ is continuous on $x \in D$, where $s > -1/2$, then $C_m^{(s)}(g;x,a,b) \to g(x)$, as $m \to \infty$, uniformly at any point $x \in D$. In Appendix 8.2, we provide a proof of this uniform convergence.

Let g be a bounded and real-valued function defined on the closed interval $\mathrm{D} \subseteq R^q$. The Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$ of order m for the function $g(\mathrm{x})$ is defined as

$$C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)=\{B(a,b)\}^{-q}\int_0^1\cdots\int_0^1\sum_{k_1=0}^{m_1}\cdots\sum_{k_q=0}^{m_q} g\begin{pmatrix} m_1^{-s}(m_1^{-1}k_1-t_1)+x_1\\ \vdots \\ m_q^{-s}(m_q^{-1}k_q-t_q)+x_q\end{pmatrix}$$

$$\cdot\binom{m_1}{k_1}\cdots\binom{m_q}{k_q} t_1^{a+k_1-1}(1-t_1)^{b+m_1-k_1-1}$$

$$\cdots t_q^{a+k_q-1}(1-t_q)^{m_q+b-k_q-1}\,dt_1\cdots dt_q\,, \qquad (9)$$

where $s>-1/2$ is fixed, $\mathrm{m}=(m_1,\ldots,m_q)^T$ are positive integer numbers, $m=\sum_{i=1}^{q} m_i$, and $\mathrm{x}\in\mathrm{D}$. Properties of the Bernstein-type approximations $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), $\mathrm{x}\in\mathrm{D}$, are outlined in Appendix 8.1.

If the function $g(\mathrm{x})$ is continuous on $\mathrm{x}\in\mathrm{D}$, where $s>-1/2$, then $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)\to g(\mathrm{x})$, as $m\to\infty$, uniformly at any q-dimensional point $\mathrm{x}\in\mathrm{D}$. In Appendix 8.2, we provide a proof of this uniform convergence.

4.2. *Degrees of approximation*

Let $\omega(\delta)$ be the modulus of continuity of the real-valued function g, for every $\delta>0$. The modulus of continuity $\omega(\delta)$ of the function $g(x)$, where $x\in D$, is defined as the maximum of $\left|g(x_0)-g(x)\right|$, for $\left|x_0-x\right|<\delta$, where $x_0,x\in D$. If the function g is continuous, then $\omega(\delta)\to 0$, as $\delta\to 0$.

Setting $\delta=m^{-1/2}$, for every $x\in D$, it can be shown that the Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$, given by (8), has degree of approximation

$$\left|C_m^{(s)}(g;x,a,b)-g(x)\right|$$

$$\leq [1+m^{-1}m^{-2s-1}\{\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)\}]\,\omega(m^{-1/2})\,, \qquad (10)$$

where the quantity $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$ is given by (7). See Appendix 8.3.

We let $|\mathrm{x}|=\left(\sum_{i=1}^{q}x_i^2\right)^{1/2}$, where $\mathrm{x}\in\mathrm{D}$. The modulus of continuity $\omega(\delta)$ of the real-valued function $g(\mathrm{x})$, $\mathrm{x}\in\mathrm{D}$, for every $\delta>0$, is defined as the maximum of $\left|g(\mathrm{x}_0)-g(\mathrm{x})\right|$, for $|\mathrm{x}_0\text{-}\mathrm{x}|<\delta$, where $\mathrm{x}_0,\mathrm{x}\in\mathrm{D}$. If the function g is continuous, then $\omega(\delta)\to 0$, as $\delta\to 0$.

Setting $\delta = m^{-1/2}$, for every $\mathrm{x} \in \mathrm{D}$, it can be shown that the multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), has degree of approximation

$$\left| C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b) - g(\mathrm{x}) \right|$$

$$\leq \left[1 + m^{-1} \left(\sum_{i=1}^{q} m_i^{-2s-1} \right) \{ \lambda_1^{'}(a,b) - \lambda_2^{'}(a,b) \} \right] \omega(m^{-1/2}), \tag{11}$$

where $m = \sum_{i=1}^{q} m_i$, and the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7). See Appendix 8.3.

4.3. *A comparison*

For a convenient value of the approximation coefficient s, the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), where $s > -1/2$, can typically outperform the Bernstein polynomials $B_m(g;x)$ and $B_{\mathrm{m}}(g;\mathrm{x})$, given by (1) and (2), for any function g to approximate, for every $x \in D$ and $\mathrm{x} \in \mathrm{D}$, respectively.

Choosing a value of s, where $s > -1/2$, can only modify the coefficients in the degrees of approximation (10) and (11), without affecting their modulus of continuity $\omega(m^{-1/2})$, for any fixed $m = \sum_{i=1}^{q} m_i$. Large values of s do not bring any advantage, with typical examples of application for the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, defined by (8) and (9), respectively, where $s > -1/2$, $x \in D$ and $\mathrm{x} \in \mathrm{D}$. The convergence to unity of the coefficients that distinguish the degrees of approximation (10) and (11) is rather fast, as s increases.

In Figure 1, we compare the Bernstein polynomial $B_m(g;x)$, given by (1), with the Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$, given by (8), for the approximation of the functions $g(x) = x^3 + x^2 + x$, and $g(x) = x^2 + x$, $x \in [0.25, 0.75]$, $m = 4$, $a = 1.5$, $b = 10$, $s = -0.1, -0.005, 0.05, 0.5, 1.5$. We also compare the Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$, given by (2), with the Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), for the approximation of the function $g(\mathrm{x}) = (x_2 + 1)^{-1}(x_1 + 1)$, $\mathrm{x} = (x_1, x_2)^T$, $x_1 \in [0.25, 0.75]$, $x_2 \in [0.45, 0.85]$, $m_1 = m_2 = 4$, $a = 1.5$, $b = 10$, $s = -0.1, -0.005, 0.05, 0.5, 1.5$. The values $m = 4$

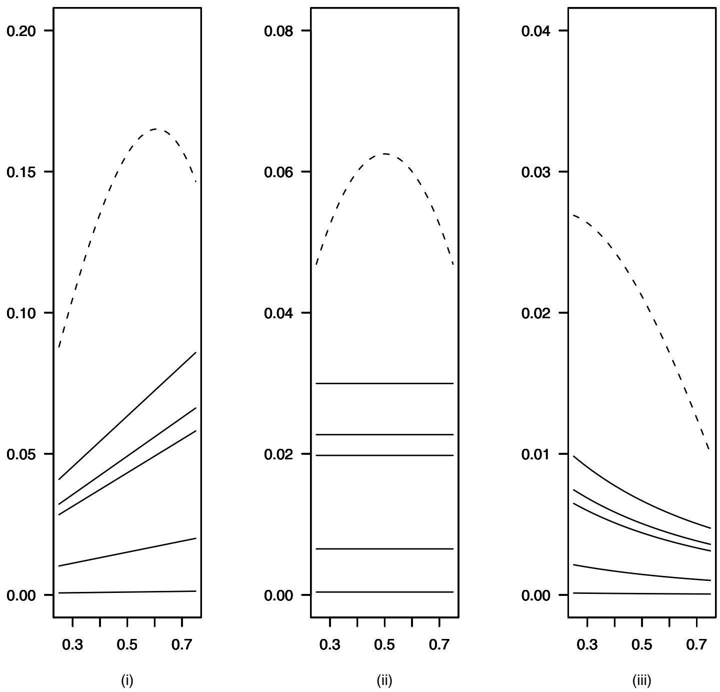

Figure 1 – Differences $B_m(g;x)-g(x)$, (hatched line), and $C_m^{(s)}(g;x,a,b)-g(x)$, (solid line), for the smooth function $g(x)=x^3+x^2+x$, where $x\in[0.25,0.75]$, where $m=4$, $a=1.5$, and $b=10$, with $s=-0.1$ (the worst performance), $s=-0.005,0.05,0.5$, and $s=1.5$ (the best performance) (panel (i)). Differences $B_m(g;x)-g(x)$, (hatched line), and $C_m^{(s)}(g;x,a,b)-g(x)$, (solid line), for $g(x)=x^2+x$, where $x\in[0.25,0.75]$, where $m=4$, $a=1.5$ and $b=10$, with $s=-0.1$ (the worst performance), $s=-0.005,0.05,0.5$, and $s=1.5$ (the best performance) (panel (ii)). The difference $B_{\mathrm{m}}(g;\mathrm{x})-g(\mathrm{x})$, (hatched line), and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)-g(\mathrm{x})$, (solid line), for $g(\mathrm{x})=(x_2+1)^{-1}(x_1+1)$, where $x_1\in[0.25,0.75]$, $x_2\in[0.45,0.85]$, where $m_1=m_2=4$, $a=1.5$, and $b=10$, with $s=-0.1$ (the worst performance), $s=-0.005,0.05,0.5$, and $s=1.5$ (the best performance) (panel (iii)).

and $m_1=m_2=4$ are very small, computationally. In any case, the numerical performances of the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$, $x\in[0.25,0.75]$, and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, $\mathrm{x}=(x_1,x_2)^T$, $x_1\in[0.25,0.75]$, $x_2\in[0.45,0.85]$, are always very effective.

5. ESTIMATION OF SMOOTH FUNCTIONS OF MEANS

5.1. *Bernstein-type estimators*

The Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(\mathrm{s})}(g;\mathrm{x},a,b)$, given by (8) and (9), where $x \in D \subseteq R^1$ and $\mathrm{x} \in \mathrm{D} \subseteq \mathrm{R}^{\mathrm{q}}$, can be used for estimating smooth functions of the population means in the statistical inference on a random sample of n independent and identically distributed (i.i.d.) observations.

Let X be a univariate random variable with values $x \in D$, distribution function F, and finite mean $\mu = E[X]$. We want to estimate a population characteristic $\theta = g(\mu)$, where g is a smooth function $g: D \to \mathrm{R}^1$. The natural estimator of θ is $\hat{\theta} = g(\bar{x})$, where $\bar{x} = n^{-1}\sum_{j=1}^{n} X_j$ is the sample mean, calculated on a random sample of n i.i.d. observations X_j, $j = 1,\ldots,n$, of X. An alternative estimator of $\theta = g(\mu)$ is the Bernstein-type estimator $C_m^{(s)}(g;\bar{x},a,b)$,

$$C_m^{(s)}(g;\bar{x},a,b) = \{B(a,b)\}^{-1} \int_0^1 \sum_{k=0}^{m} g(m^{-s}(m^{-1}k - t) + \bar{x}) \cdot \binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1}\, dt\,, \tag{12}$$

where $s > -1/2$ is fixed. The Bernstein-type estimator (12) follows from the definition (8) of $C_m^{(s)}(g;x,a,b)$, $s > -1/2$, by substituting $x \in D$ with the sample mean $\bar{x}$, where $\bar{x}$ ranges in D.

Let X be a q-variate random variable with values $\mathrm{x} \in \mathrm{D}$, where $\mathrm{X} = (X_1,\ldots,X_q)^T$, with distribution function F, and finite q-variate mean $\mu = E[\mathrm{X}]$, $\mu = (\mu_1,\ldots,\mu_q)^T$. We want to estimate $\theta = g(\mu)$, where $g: \mathrm{D} \to \mathrm{R}^1$. Its natural estimator is $\hat{\theta} = g(\bar{\mathrm{x}})$, where $\bar{\mathrm{x}} = (\bar{x}_1,\ldots,\bar{x}_q)^T$ is the sample mean on a random sample of n i.i.d. q-variate observations X_{i}, $i = 1,\ldots,n$, of X, $\bar{x}_i = n^{-1}\sum_{j=1}^{n} X_{ij}$, $i = 1,\ldots,q$. An alternative estimator of $\theta = g(\mu)$ is the multivariate Bernstein-type estimator $C_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}},a,b)$,

$$C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)=\{B(a,b)\}^{-q}\int_0^1\cdots\int_0^1\sum_{k_1=0}^{m_1}\cdots\sum_{k_q=0}^{m_q} g\begin{pmatrix} m_1^{-s}(m_1^{-1}k_1-t_1)+\overline{x}_1 \\ \vdots \\ m_q^{-s}(m_q^{-1}k_q-t_q)+\overline{x}_q \end{pmatrix}$$

$$\cdot\binom{m_1}{k_1}\cdots\binom{m_q}{k_q} t_1^{a+k_1-1}(1-t_1)^{b+m_1-k_1-1}$$

$$\cdots t_q^{a+k_q-1}(1-t_q)^{b+m_q-k_q-1}dt_1\cdots dt_q, \qquad (13)$$

where $s>-1/2$ is fixed. The multivariate Bernstein-type estimator (13) follows the definition (9) of $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, $s>-1/2$, by substituting $\mathrm{x}\in\mathrm{D}$ with the sample mean $\overline{\mathrm{x}}$, where $\overline{\mathrm{x}}$ ranges in D.

5.2. *Orders of error in probability of the Bernstein-type estimators*

We know that $\overline{x}=\mu+O_p(n^{-1/2})$, as $n\to\infty$. We also know that

$$g(\overline{x})=g(\mu)+O_p(n^{-1/2}),$$

as $n\to\infty$. It is shown that the Bernstein-type estimator $C_m^{(s)}(g;\overline{x},a,b)$, given by (12), for $s>-1/2$, is a consistent estimator of $g(\mu)$, as $m\to\infty$ and $n\to\infty$. In particular, it is shown that

$$C_m^{(s)}(g;\overline{x},a,b)=g(\mu)+O(m^{-2s-1})+O_p(n^{-1/2}), \qquad (14)$$

for $s>-1/2$, as $m\to\infty$ and $n\to\infty$. See Appendix 8.4.

We know that $\overline{\mathrm{x}}=\mu+\mathrm{O}_{\mathrm{p}}(n^{-1/2})$, where $\overline{x}_i=\mu_i+O_p(n^{-1/2})$, $i=1,\ldots,q$, as $n\to\infty$. We also know that

$$g(\overline{\mathrm{x}})=g(\mu)+O_p(n^{-1/2}),$$

as $n\to\infty$. It is shown that the multivariate Bernstein-type estimator $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$, given by (13), where $\mathrm{m}=(m_1,\ldots,m_q)^T$, for $s>-1/2$, is a consistent estimator of $g(\mu)$, as $m_i\to\infty$, $i=1,\ldots,q$, and $n\to\infty$. In particular, it is shown that

$$C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)=g(\mu)+\sum_{i=1}^{q}O(m_i^{-2s-1})+O_p(n^{-1/2}), \qquad (15)$$

for $s>-1/2$, as $m_i\to\infty$, $i=1,\ldots,q$, and $n\to\infty$. See Appendix 8.4.

5.3. *Asymptotic normality of Bernstein-type estimators*

The Bernstein-type estimator $C_m^{(s)}(g;\overline{x},a,b)$ is defined by (12), where $s>-1/2$. We denote by σ^2 the asymptotic variance of $n^{1/2}g(\overline{x})$, as $n\to\infty$. That is,

$$\sigma^2=\{g'(\mu)\}^2E[(X-\mu)^2],$$

where $g'(x)=(dx)^{-1}dg(x)$, and $x\in D$. The distribution of the Bernstein-type estimator $C_m^{(s)}(g;\overline{x},a,b)$ is asymptotically normal,

$$n^{1/2}\{C_m^{(s)}(g;\overline{x},a,b)-g(\mu)\}\xrightarrow{d}N(0,\sigma^2), \tag{16}$$

for $s>-1/2$, as $m\to\infty$ and $n\to\infty$. See Appendix 8.5.

The Bernstein-type estimator $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$ is defined by (13), where $s>-1/2$. We denote by σ^2 the asymptotic variance of $n^{1/2}g(\overline{\mathrm{x}})$, as $n\to\infty$. That is,

$$\sigma^2=\sum_{i=1}^{q}\sum_{j=1}^{q}(\partial x_i)^{-1}\partial g(x_1,\dots,x_i,\dots,x_q)\Big|_{\mathrm{x}=\mu}$$

$$\cdot(\partial x_j)^{-1}\partial g(x_1,\dots,x_j,\dots,x_q)\Big|_{\mathrm{x}=\mu}E\Big[(X_i-\mu_i)(X_j-\mu_j)\Big]$$

The distribution of the Bernstein-type estimator $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$ is asymptotically normal,

$$n^{1/2}\{C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)-g(\mu)\}\xrightarrow{d}N(0,\sigma^2), \tag{17}$$

for $s>-1/2$, as $m_i\to\infty$, $i=1,\dots,q$, and $n\to\infty$. See Appendix 8.5.

6. SIMULATION STUDY

Following subsection 4.3, we report on a small Monte Carlo experiment concerning with the empirical behaviour of the Bernstein-type estimators $C_m^{(s)}(g;\overline{x},a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$, given by (12) and (13).

We applied the Bernstein-type estimators $C_m^{(s)}(g;\overline{x},a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$, given by (12) and (13), to the approximation of the smooth functions of means $g(\overline{x})=\overline{x}^3+\overline{x}^2+\overline{x}$, $g(\overline{x})=\overline{x}^2+\overline{x}$, where $\overline{x}=n^{-1}\sum_{j=1}^{n}X_j$, and

$g(\overline{\mathrm{x}}) = (\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1)$, where $\overline{\mathrm{x}} = (\overline{x}_1, \overline{x}_2)^T$, $\overline{x}_i = n^{-1}\sum_{j=1}^{n} X_{ij}$, $i = 1,2$. Random samples of different size n, of i.i.d. observations, were always drawn from the univariate folded normal distribution $|N(0,1)|$ and from the bivariate folded normal distribution with independent marginals $|N(0,1)|$. We always considered the values $a = 1.5$ and $b = 10$.

From the definition (12) of $C_m^{(s)}(g;\overline{x},a,b)$, we have the Bernstein-type estimator

$$C_m^{(s)}(\overline{x}^3 + \overline{x}^2 + \overline{x};\overline{x},a,b) = \overline{x}^3 + \overline{x}^2 + \overline{x}$$
$$+ m^{-2s-1}(3\overline{x} + 1)\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\}$$
$$+ m^{-3s-2}\{2\lambda_3^{'} - 3\lambda_2^{'}(a,b) + \lambda_1^{'}(a,b)\},$$

where $s > -1/2$, and the moments $\lambda_1^{'}(a,b)$, $\lambda_2^{'}(a,b)$, and $\lambda_3^{'}$ are given by (4), (5), and (6), respectively.

In Figure 2, we compare the Monte Carlo variances of $g(\overline{x}) = \overline{x}^3 + \overline{x}^2 + \overline{x}$ and $C_m^{(s)}(\overline{x}^3 + \overline{x}^2 + \overline{x};\overline{x},a,b)$, for $s = 0.5, 2$, $m = 4,5$, and the sample sizes $n = 2,4,6,\ldots,28,30$. The Monte Carlo variances were based on 10000 independent replications from the folded normal distribution. The empirical results were equivalent.

From the definition (12) of $C_m^{(s)}(g;\overline{x},a,b)$, we have the Bernstein-type estimator

$$C_m^{(s)}(\overline{x}^2 + \overline{x};\overline{x},a,b) = \overline{x}^2 + \overline{x} + m^{-2s-1}\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\},$$

where $s > -1/2$, and the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7).

We have a constant difference $C_m^{(s)}(\overline{x}^2 + \overline{x};\overline{x},a,b) - \overline{x}^2 - \overline{x}$. We had the value $C_m^{(s)}(\overline{x}^2 + \overline{x};\overline{x},a,b) - \overline{x}^2 - \overline{x} = 0.006522$, for $s = 0.5$, $m = 4$, and $n = 6$, $C_m^{(s)}(\overline{x}^2 + \overline{x};\overline{x},a,b) - \overline{x}^2 - \overline{x} = 0.000339$, for $s = 2$, $m = 5$, and $n = 16$.

From the definition (13) of $C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)$, in order to approximate the integral in the Bernstein-type estimator $C_{\mathrm{m}}^{(s)}((\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1);\overline{\mathrm{x}},a,b)$, we obtained $\tilde{C}_{\mathrm{m}}^{(s)}((\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1);\overline{\mathrm{x}},a,b)$,

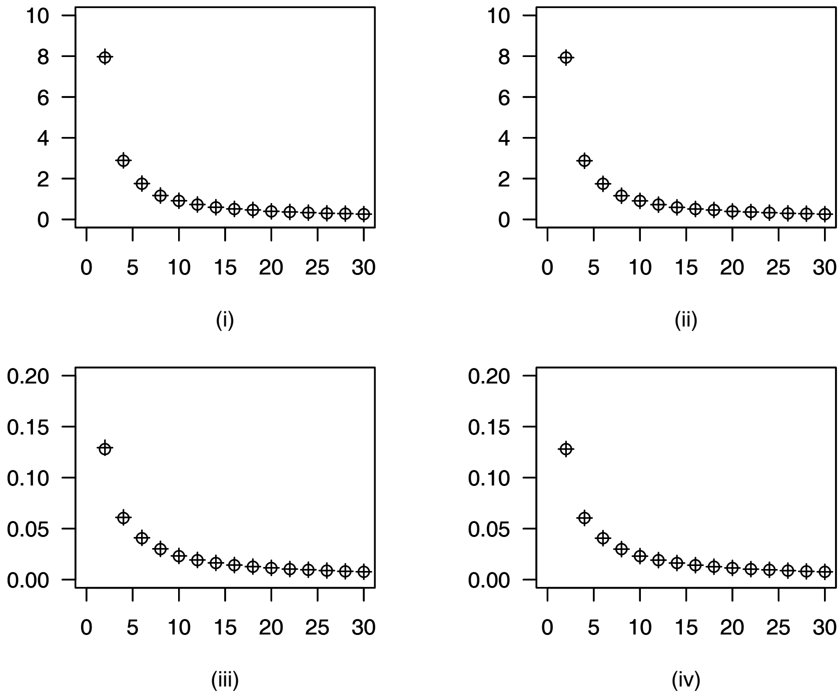

Figure 2 – Monte Carlo variances of $g(\overline{x}) = \overline{x}^3 + \overline{x}^2 + \overline{x}$, $(\circ)$, and $C_m^{(s)}(\overline{x}^3 + \overline{x}^2 + \overline{x}; \overline{x}, a, b)$, $(+)$, where $a = 1.5$ and $b = 10$, for random samples of size $n = 2, 4, 6, \ldots, 28, 30$, from the folded normal distribution; $s = 0.5$ and $m = 4$, in panel (i), $s = 2$ and $m = 5$, in panel (ii). Monte Carlo variances of $g(\overline{\mathrm{x}}) = (\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1)$, $(\circ)$, and $\tilde{C}_{\mathrm{m}}^{(s)}((\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1); \overline{\mathrm{x}}, a, b)$, $(+)$, where $a = 1.5$ and $b = 10$, for random samples of size $n = 2, 4, 6, \ldots, 28, 30$, from the bivariate folded normal distribution; $s = 0.5$ and $m_1 = m_2 = 4$, in panel (iii), $s = 2$ and $m_1 = m_2 = 5$, in panel (iv).

$$\tilde{C}_{\mathrm{m}}^{(s)}((\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1); \overline{\mathrm{x}}, a, b) = (\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1) + m_2^{-2s-1}(\overline{x}_2 + 1)^{-3}(\overline{x}_1 + 1) \cdot \{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\},$$

where $s > -1/2$, and the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7). We have

$$\tilde{C}_{\mathrm{m}}^{(s)}((\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1); \overline{\mathrm{x}}, a, b) = C_{\mathrm{m}}^{(s)}((\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1); \overline{\mathrm{x}}, a, b) + O(m_2^{-3s-2}),$$

$s > -1/2$, as $m_1 \to \infty$ and $m_2 \to \infty$. The approximate Bernstein-type estimator $\tilde{C}_{\mathrm{m}}^{(s)}((\overline{x}_2 + 1)^{-1}(\overline{x}_1 + 1); \overline{\mathrm{x}}, a, b)$ was obtained by calculating the integral in

$C_{\mathrm{m}}^{(s)}((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b)$ with the denominator of the function that was replaced by its three-term Taylor expansion around $\bar{x}_2+1$. See Wong (2001), chapter 5, for further details about this procedure.

In Figure 2, we compare the Monte Carlo variances of $g(\bar{\mathrm{x}})=(\bar{x}_2+1)^{-1}(\bar{x}_1+1)$ and $\tilde{C}_{\mathrm{m}}^{(s)}((\bar{x}_2+1)^{-1}(\bar{x}_1+1);\bar{\mathrm{x}},a,b)$, for $s=0.5,2$, $m=4,5$, and the sample sizes $n=2,4,6,\ldots,28,30$. The Monte Carlo variances were based on 10000 independent replications from the folded normal distribution. The empirical results were equivalent.

7. CONCLUDING REMARKS

1). The quantity $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$, given by (7), is crucial for the numerical performance of the univariate Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$, defined as (8), where $s>-1/2$, and $x\in D\subseteq \mathrm{R}^1$, and for the numerical performance of the multivariate Bernstein-type approximations $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, defined as (9), where $s>-1/2$, and $\mathrm{x}\in \mathrm{D}\subseteq R^q$. The function $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$, given by (7), does not admit a minimizer, for $a>0$ and $b>0$. See Chong and Żak (1996), chapter 6. Space curves $(a(t),b(t),t)$, where $t\in E$, and $E\subseteq \mathrm{R}^1$, can be easily drawn in order to determine specific degrees of approximation. See Montiel and Ros (2005), chapter 1. The degrees (10) and (11) of approximation of the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), respectively, where $s>-1/2$, $x\in D\subseteq \mathrm{R}^1$, and $\mathrm{x}\in \mathrm{D}\subseteq R^q$, can be better than the degrees of approximation of the Bernstein-type approximations, that are proposed in Pallini (2005), for values of a and b such that $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)<1/4$.

2). More efficient results for the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, defined as (8) and (9), respectively, where $s>-1/2$, $x\in D\subseteq \mathrm{R}^1$, and $\mathrm{x}\in \mathrm{D}\subseteq R^q$, can be obtained by over-skewing the beta p.d.f. $beta(a,b)$, and the moments $\lambda_1^{'}(a,b)$ and $\lambda_2^{'}(a,b)$ of the beta-binomial p.d.f., given by (4) and (5). We can over-skew the beta p.d.f. $beta(a,b)$, by an additional parameter τ, with values $\tau>0$, by determining the beta p.d.f. $beta(a,b\tau)$. The beta p.d.f. $beta(a,b\tau)$ is negatively skewed, for $\tau<b^{-1}a$, and is positively skewed, for $\tau>b^{-1}a$. From the definition (7) of $\lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$, under the condition $a^2\tau+a\tau+b^2\tau-b^2\tau^2\le a^2+a$, it is seen that $\lambda_1^{'}(a,b\tau)-\lambda_2^{'}(a,b\tau)\le \lambda_1^{'}(a,b)-\lambda_2^{'}(a,b)$.

3). Rosenberg (1967) studies an application of the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$, given by (2), to the Monte Carlo evaluation of an integral. The same application can be organized for the multivariate Bernstein-type approximation in Pallini (2005) and for the multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), where $s>-1/2$, $\mathrm{m}=(m_1,\ldots,m_q)^T$, and $\mathrm{x}\in \mathrm{D}\subseteq R^q$. Most importantly, straightforward versions of the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), respectively, where $s>-1/2$, $x\in D$, $\mathrm{x}\in\mathrm{D}$, are both multivariate approximations for functions and approximate multivariate integrals of functions. Focussing on $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), where $s>-1/2$, $\mathrm{x}\in\mathrm{D}$, let us suppose that we are interested in the evaluation of an integral $\int_{\mathrm{D}} \mathrm{g(x)dx}$, where $\mathrm{D}\subseteq R^q$. In particular, we can start from an approximate integration rule of the form $C_{\mathrm{m}}^{(s)}(h_{\mathrm{m}}^{(s)};\mathrm{x},a,b)$, where $s>-1/2$, $h_{\mathrm{m}}^{(s)}:[0,1]^q\times\mathrm{D}\rightarrow \mathrm{R}^1$, and apply a procedure for a more efficient integration rule. See Wong (2001), and Hanselman and Littlefield (2005), chapter 24.

4). In the Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (8) and (9), respectively, where $s>-1/2$, $x\in D$ and $\mathrm{x}\in\mathrm{D}$, the linear kernels $m^{-s}(m^{-1}k-t)+x$, the linear kernels $m^{-s}(m^{-1}v-x)+x$ and $m_i^{-s}(m_i^{-1}v_i-x_i)+x_i$ can be substituted by nonlinear kernels, where $k=0,1,\ldots,m$, $k_i=0,1,\ldots,m_i$, $i=1,\ldots,q$, and $x\in D$, $\mathrm{x}=(x_1,\ldots,x_q)^{\mathrm{T}}\in\mathrm{D}$, respectively.

5). The Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), where $s>-1/2$, and $\mathrm{x}\in\mathrm{D}\subseteq R^q$, can be generalized by using a different approximation coefficient for each component. That is, we can use $\mathrm{s}=(s_1,\ldots,s_q)^T$ in the Bernstein-type approximation $C_{\mathrm{m}}^{(\mathrm{s})}(g;\mathrm{x},a,b)$, where $\mathrm{x}\in\mathrm{D}\subseteq R^q$. Another generalization of the Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), where $\mathrm{x}\in\mathrm{D}\subseteq R^q$, can be based on q different beta-binomial p.d.f.'s, $p_m(k;a_i,b_i)$, that can be defined from (3), for every $i=1,\ldots,q$.

6). Following DeVore and Lorentz (1993), chapter 1, it can be shown that the Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$, given by (8), is an integral operator with uniform convergence, as $m\rightarrow\infty$. If we suppose that $g(x)\neq 0$, for every $x\in D$, then we obtain $C_m^{(s)}(g;x,a,b)=\int_0^1 h_m(t,x)g(t)\,dt$, where $h_m(t,x)=\{g(x)\}^{-1}\{B(a,b)\}^{-1}\sum_{k=0}^{m} g(m^{-s}(m^{-1}k-t)+x)\, t^{a+k-1}(1-t)^{b+m-k-1}$ is the kernel of this integral operator, $s > -1/2$, $t \in [0,1]$, and $x \in D$. The definition of $C_m^{(s)}(g;x,a,b)$ by $h_m(t,x)$, $t \in [0,1]$, is equivalent to the problem of approximating $\{g(x)\}^2$ with $C_m^{(s)}(g;x,a,b)$, where $x \in D$. Similar results can be obtained for the multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\mathrm{x},a,b)$, given by (9), as $m_i \to \infty$, $i = 1,\ldots,q$.

7). Variants of the Bernstein polynomials that are discussed and studied in DeVore and Lorentz (1993), chapter 10, can also be regarded as extensions to the use of the binomial p.d.f. in Bernstein-like approximations. We recall that the most special cases of the beta p.d.f. are the arcsine distribution, the power distribution and the unform distribution. See Balakrishnan and Nevzorov (2003), chapter 16. Extensions to the beta-binomial p.d.f. $p_m(k;a,b)$, given by (3), where $k = 0,1,\ldots,m$, are discussed and studied in Wilcox (1981).

8. APPENDIX

8.1. *Basic properties of the Bernstein-type approximations (8) and (9)*

The Bernstein-type approximations $C_m^{(s)}(g;x,a,b)$ and $C_{\mathrm{m}}^{(\mathrm{s})}(g;\mathrm{x},a,b)$, given by (8) and (9), respectively, where $s > -1/2$, $x \in D$ and $\mathrm{x} \in \mathrm{D}$, respectively, are linear positive operators. Let γ_1 and γ_2 be finite constants. Let g, g_1, and g_2 be functions, $g(x)$, $g_1(x)$, and $g_2(x)$, $x \in D$. We have

$$C_m^{(s)}(\gamma_1 + \gamma_2 g;x,a,b) = \gamma_1 + \gamma_2 C_m^{(s)}(g;x,a,b)$$

$$C_m^{(s)}(g_1 + g_2;x,a,b) = C_m^{(s)}(g_1;x,a,b) + C_m^{(s)}(g_2;x,a,b),$$

where $s > -1/2$, $x \in D$. If $g_1(x) \le g_2(x)$, for all $x \in D$, we have

$$C_m^{(s)}(g_1;x,a,b) \le C_m^{(s)}(g_2;x,a,b),$$

$x \in D$. Multivariate versions of these properties hold for $C_{\mathrm{m}}^{(\mathrm{s})}(g;\mathrm{x},a,b)$, given by (9), where $s > -1/2$, $\mathrm{x} \in \mathrm{D}$.

8.2. *Uniform convergence of the Bernstein-type approximations (8) and (9)*

The uniform norm $\|g\|$ of the function $g(x)$, where $x \in D$, is defined as $\|g\| = \max_{x \in D} |g(x)|$. The Bernstein-type approximation $C_m^{(s)}(g;x,a,b)$, where $x \in D$, is given by (8). We want to show that, given a constant $\varepsilon > 0$, there exists a positive integer m_0, such that

$$\left| C_{m_0}^{(s)}(g;x,a,b) - g(x) \right| < \varepsilon , \tag{18}$$

for every $x \in D$.

For every $x \in D$, $C_m^{(s)}(1;x,a,b) = 1$, where $s > -1/2$. We define the functions $\mu_1(x) = x$ and $\mu_2(x) = x^2$. We have $C_m^{(s)}(\mu_1(x);x,a,b) = x$, and $C_m^{(s)}(\mu_2(x);x,a,b) = m^{-2s-1}\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\} + x^2$, where $s > -1/2$, and the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7).

Suppose that $\| g \| = M$. We take $x_0 \in D$. We have

$$-2M \le g(x_0) - g(x) \le 2M , \tag{19}$$

where $x_0, x \in D$. The function g is continuous; given $\varepsilon_1 > 0$, there exists a constant $\delta > 0$, such that

$$-\varepsilon_1 < g(x_0) - g(x) < \varepsilon_1 , \tag{20}$$

for $| x_0 - x | < \delta$, and $x_0, x \in D$. From (19) and (20), it follows that $-\varepsilon_1 - 2M \le g(x_0) - g(x) \le \varepsilon_1 + 2M$, and then

$$-\varepsilon_1 - 2M\delta^{-2}(x_0 - x)^2 \le g(x_0) - g(x) \le \varepsilon_1 + 2M\delta^{-2}(x_0 - x)^2 , \tag{21}$$

for $x_0, x \in D$. In fact, if $| x_0 - x | < \delta$, (20) implies (21), $x_0, x \in D$. If $| x_0 - x | \ge \delta$, then $\delta^{-2}(x_0 - x)^2 \ge 1$ and (19) implies (21), $x_0, x \in D$. Following Appendix 8.1, (21) becomes

$$-\varepsilon_1 - 2M\delta^{-2}C_m^{(s)}((x_0 - x)^2;x,a,b) \le C_m^{(s)}(g;x,a,b) - g(x)$$

$$\le \varepsilon_1 + 2M\delta^{-2}C_m^{(s)}((x_0 - x)^2;x,a,b) , \tag{22}$$

for $x_0, x \in D$. We observe that $(x_0 - x)^2 = x_0^2 - 2x_0x + x^2$, $x_0, x \in D$. It follows that

$$C_m^{(s)}((x_0 - x)^2;x,a,b) = m^{-2s-1}\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\} , \tag{23}$$

for $x \in D$, where the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7). We have $C_m^{(s)}((x_0 - x)^2;x,a,b) = O(m^{-2s-1})$, as $m \to \infty$, $x \in D$. Finally, we have

$$\left| C_m^{(s)}(g;x,a,b) - g(x) \right| \le \varepsilon_1 + 2M\delta^{-2}m^{-2s-1}\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\} ,$$

$x \in D$, where the quantity $\lambda_1'(a,b) - \lambda_2'(a,b)$ is given by (7). Setting $\varepsilon_1 = \varepsilon/2$, for any $m_0 > [4M\delta^{-2}\varepsilon^{-1}\{\lambda_1'(a,b) - \lambda_2'(a,b)\}]^{1/(2s+1)}$, where $s > -1/2$, and the quantity $\lambda_1'(a,b) - \lambda_2'(a,b)$ is given by (7), the uniform convergence (18) is proved.

The condition $s > -1/2$ is required for the uniform convergence. The convergence $C_m^{(s)}(g; x, a, b) \to g(x)$, for $s > -1/2$, is uniform, at any point of continuity $x \in D$, as $m \to \infty$, in the sense that the upper bound (23) for the uniform norm does not depend on x, $x \in D$.

The multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g; \mathrm{x}, a, b)$, where $s > -1/2$, and $\mathrm{x} \in \mathrm{D}$, is given by (9). We observe that q is fixed and does not depend on m. Considering the uniform norm $\|g\|$ of the function $g(\mathrm{x})$, $\mathrm{x} \in \mathrm{D}$, defined as $\|g\| = \max_{\mathrm{x} \in \mathrm{D}} |g(\mathrm{x})|$, we want to show that, given a constant $\varepsilon > 0$, there exist positive integers $\mathrm{m}_0 = (m_{01}, \ldots, m_{0q})^T$, such that

$$\left| C_{\mathrm{m}_0}^{(s)}(g; \mathrm{x}, a, b) - g(\mathrm{x}) \right| < \varepsilon, \tag{24}$$

for every $\mathrm{x} \in \mathrm{D}$.

For every $\mathrm{x} \in \mathrm{D}$, $C_{\mathrm{m}}^{(s)}(1; \mathrm{x}, a, b) = 1$, where $s > -1/2$. We define $\mu_1(\mathrm{x}) = \sum_{i=1}^{q} x_i$ and $\mu_2(\mathrm{x}) = \sum_{i=1}^{q} x_i^2$. We have $C_{\mathrm{m}}^{(s)}(\mu_1(\mathrm{x}); \mathrm{x}, a, b) = \sum_{i=1}^{q} x_i$, and $C_{\mathrm{m}}^{(s)}(\mu_2(\mathrm{x}); \mathrm{x}, a, b) = \left(\sum_{i=1}^{q} m_i^{-2s+1} \right) \{\lambda_1'(a,b) - \lambda_2'(a,b)\} + \sum_{i=1}^{q} x_i^2$, where $s > -1/2$, and the quantity $\lambda_1'(a,b) - \lambda_2'(a,b)$ is given by (7).

Suppose that $\|g\| = M$. We take $\mathrm{x}_0 = (x_{01}, \ldots, x_{0q})^T$, where $\mathrm{x}_0 \in \mathrm{D}$. We observe that $(|\mathrm{x}_0 - \mathrm{x}|)^2 = \sum_{i=1}^{q} (x_{0i}^2 + x_i^2 - 2x_{0i}x_i)$, $\mathrm{x}_0, \mathrm{x} \in \mathrm{D}$. The uniform convergence (24) follows from $C_{\mathrm{m}}^{(s)}((|\mathrm{x}_0 - \mathrm{x}|)^2; \mathrm{x}, a, b) = \sum_{i=1}^{q} O(m_i^{-2s-1})$, as $m_i \to \infty$, for $s > -1/2$, where $i = 1, \ldots, q$, $\mathrm{x}_0, \mathrm{x} \in \mathrm{D}$. Under the condition $s > -1/2$, the convergence $C_{\mathrm{m}}^{(s)}(\mathrm{g}; \mathrm{x}, a, b) \to g(\mathrm{x})$ is uniform at any point of continuity $\mathrm{x} \in \mathrm{D}$, as $m_i \to \infty$, where $i = 1, \ldots, q$.

8.3. *Degrees of approximation (10) and (11)*

For every $\delta > 0$, we denote by $\xi(x_0, x; \delta)$ the maximum integer less than or equal to $\delta^{-1}\left|x_0 - x\right|$, where $x_0, x \in D$. We recall the definition of the modulus of continuity $\omega(\delta)$, where $\delta > 0$. We have

$$\left|g(x_0) - g(x)\right| \leq \omega(\delta)\{1 + \xi(x_0, x;\delta)\}, \tag{25}$$

$x_0, x \in D$.

The Bernstein-type approximation $C_m^{(s)}(g; x, a, b)$ is given by (8), where $s > -1/2$, and $x \in D$. Then, we have that

$$\begin{aligned}
&\left|C_m^{(s)}(g; x, a, b) - g(x)\right| \\
&\leq \{B(a,b)\}^{-1}\int_0^1 \sum_{k=0}^{m}\left|g(m^{-s}(m^{-1}v - t) + x) - g(x)\right|\binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1}\,dt \\
&\leq \omega(\delta)\{B(a,b)\}^{-1}\int_0^1 \sum_{k=0}^{m}\{1 + \xi(x_0, x;\delta)\}\binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1}\,dt \\
&\leq \omega(\delta)\{B(a,b)\}^{-1}\int_0^1 \sum_{k=0}^{m}\{1 + \delta^{-1}\left|m^{-s}(m^{-1}k - t)\right|\}\binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1}\,dt \\
&\leq \omega(\delta)\{B(a,b)\}^{-1}\int_0^1 \sum_{k=0}^{m}\{1 + \delta^{-2}m^{-2s-2}(k - mt)^2\}\binom{m}{k} t^{a+k-1}(1-t)^{b+m-k-1}\,dt,
\end{aligned}$$

$x \in D$. It follows that

$$\left|C_m^{(s)}(g; x, a, b) - g(x)\right| \leq \omega(\delta)\,[1 + \delta^{-2}m^{-2s-1}\{\lambda_1'(a,b) - \lambda_2'(a,b)\}],$$

$x \in D$. Setting $\delta = m^{-1/2}$, we finally have the degree of approximation (10).

For every $\delta > 0$, we denote by $\xi(\mathrm{x}_0, \mathrm{x};\delta)$ the maximum integer less than or equal to $\delta^{-1}\left|\mathrm{x}_0 - \mathrm{x}\right|$, where $\delta^{-1}\left|\mathrm{x}_0 - \mathrm{x}\right| = \delta^{-1}\left(\sum_{i=1}^{q}(x_{0i} - x_i)^2\right)^{1/2}$, and $\mathrm{x}_0, \mathrm{x} \in \mathrm{D}$. We have $\left|g(\mathrm{x}_0) - g(\mathrm{x})\right| \leq \omega(\delta)\{1 + \xi(\mathrm{x}_0, \mathrm{x};\delta)\}$, where $\omega(\delta)$ is the modulus of continuity, $\delta > 0$, and $\mathrm{x}_0, \mathrm{x} \in \mathrm{D}$.

The multivariate Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g; \mathrm{x}, a, b)$ is given by (9), where $s > -1/2$, and $\mathrm{x} \in \mathrm{D}$. We have

$$\left|C_{\mathrm{m}}^{(s)}(g; \mathrm{x}, a, b) - g(\mathrm{x})\right| \leq \omega(\delta)\left[1 + \delta^{-2}\sum_{i=1}^{q} m_i^{-2s-1}\{\lambda_1'(a,b) - \lambda_2'(a,b)\}\right],$$

$x \in D$. Setting $\delta = m^{-1/2}$, where $m = \sum_{i=1}^{q} m_i$, we finally have the degree of approximation (11).

8.4. *Orders of error in probability (14) and (15)*

The Bernstein-type approximation $C_m^{(s)}(g;\bar{x},a,b)$ is given by (12), where $s > -1/2$. Let $g'(x) = (dx)^{-1} dg(x)$ and $g''(x) = (dx)^{-2} d^2 g(x)$ be the first two derivatives of the function $g(x)$, where $x \in D$. We recall that the quantity $\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)$ is given by (7). By Taylor expanding the function $g(m^{-s}(m^{-1}k - t) + \bar{x})$ around μ, for every $k = 0,1,\ldots,m$, we have

$$
\begin{aligned}
C_m^{(s)}(g;\bar{x},a,b) = g(\mu) & \\
& + g'(\mu)(\bar{x} - \mu) \\
& + \frac{1}{2}[g''(\mu) m^{-2s-1}\{\lambda_1^{'}(a,b) - \lambda_2^{'}(a,b)\} + g''(\mu)(\bar{x} - \mu)^2] \\
& + \cdots \\
& = g(\mu) + O_p(n^{-1/2}) + O(m^{-2s-1}),
\end{aligned}
$$

where $s > -1/2$, as $m \to \infty$, and $n \to \infty$. Order $O(m^{-2s-1}) + O_p(n^{-1/2})$ of error in probability in (14), as $m \to \infty$, and $n \to \infty$, is thus proved.

The Bernstein-type approximation $C_{\mathrm{m}}^{(s)}(g;\bar{\mathrm{x}},a,b)$ is given by (13), where $s > -1/2$. By Taylor expanding the function

$$
g\begin{pmatrix} m_1^{-s}(m_1^{-1}k_1 - t_1) + \bar{x}_1 \\ \vdots \\ m_q^{-s}(m_q^{-1}k_q - t_q) + \bar{x}_q \end{pmatrix}
$$

around $\mu = (\mu_1,\ldots,\mu_q)^T$, for every $k_i = 1,\ldots,m_i$, $i = 1,\ldots,q$, we can prove the order $\sum_{i=1}^{q} O(m_i^{-2s-1}) + O_p(n^{-1/2})$ of error in probability in (15), $s > -1/2$, as $m_i \to \infty$, where $i = 1,\ldots,q$, and $n \to \infty$.

8.5. *Asymptotic normality (16) and (17)*

Following (14), we have that $n^{1/2}\{C_m^{(s)}(g;\overline{x},a,b)-g(\mu)\}$, where $s>-1/2$, is asymptotically equivalent to $n^{1/2}\{g(\overline{x})-g(\mu)\}$, as $m\to\infty$ and $n\to\infty$. An application of the Central Limit Theorem then shows the asymptotic normality in (16), as $m\to\infty$, and $n\to\infty$.

Following (15), we have $n^{1/2}\{C_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}},a,b)-g(\mu)\}$, where $s>-1/2$, $\mathrm{m}=(m_1,\ldots,m_q)^T$, is asymptotically equivalent to $n^{1/2}\{g(\overline{\mathrm{x}})-g(\mu)\}$, as $m_i\to\infty$, where $i=1,\ldots,q$, and $n\to\infty$. An application of the Central Limit Theorem then shows the asymptotic normality in (17), as $m_i\to\infty$, where $i=1,\ldots,q$, and $n\to\infty$.

Dipartimento di Statistica e Matematica
Applicata all'Economia
Università di Pisa

ANDREA PALLINI

REFERENCES

N. BALAKRISHNAN, V.B. NEVZOROV (2003), *A primer on statistical distributions*, Wiley and Sons, New York.

O.E. BARNDORFF-NIELSEN, D.R. COX (1989), *Asymptotic techniques for use in Statistics*, Chapman & Hall, London.

R.A. BECKER, J.M. CHAMBERS, A.R. WILKS (1988), *The new S Language*, Wadsworth and Brooks/ Cole, Pacific Grove, California.

E.W. CHENEY (1982), *Introduction to approximation theory*, 2nd Edition, AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island.

E.K.P. CHONG, S.H. ŻAK (1996), *An introduction to optimization*, Wiley and Sons, New York.

P.J. DAVIS (1963), *Interpolation and approximation*, Blaisdell Publishing Company, Waltham, Massachusetts.

R.A. DEVORE, G.G. LORENTZ (1993), *Constructive approximation*, Grundlehren der mathematischen Wissenschaften 303, Springer-Verlag, Berlin.

W. FELLER (1968), *An introduction to probability theory and its applications*, Volume I, 3rd Edition, Wiley and Sons, New York.

W. FELLER (1971), *An introduction to probability theory and its applications*, Volume II, 2nd Edition, Wiley and Sons, New York.

P. HALL (1992), *The bootstrap and edgeworth expansion*, Springer-Verlag, New York.

D. HANSELMAN, B. LITTLEFIELD (2005), *Mastering MATLAB® 7*, Pearson Prentice Hall, Upper Saddle River, New Jersey.

N.L. JOHNSON, A.W. KEMP, S. KOTZ (2005), *Univariate discrete distributions*, 3rd Edition, Wiley and Sons, New York.

N.L. JOHNSON, S. KOTZ, S. BALAKRISHNAN (1997), *Discrete multivariate distributions*, Wiley and Sons, New York.

P.P. KOROVKIN (1960), *Linear operators and approximation theory*, Hindustan Publishing Corporation, Delhi.

G.G. LORENTZ (1986), *Bernstein polynomials*, Chelsea Publishing Company, New York.

S. MONTIEL, A. ROS (2005), *Curves and surfaces*, AMS Graduate Studies in Mathematics 69, American Mathematical Society, Real Sociedad Mathemàtica Española, Providence, Rhode Island.

F.W.J. OLVER (1997), *Asymptotics and special functions*, A. K. Peters, Natick, Massachusetts.

A. PALLINI (2005), *Bernstein-type approximation of smooth functions*, "Statistica", 65, 169-191.

G.M. PHILLIPS (2003), *Interpolation and approximation by polynomials*, Springer-Verlag, New York.

R.J. RIVLIN (1981), *An introduction to the approximation of functions*, Reprint of the 1969 Edition, Dover Publications, Mineola, New York.

L. ROSENBERG (1967), *Bernstein polynomials and Monte Carlo integration*, "SIAM Journal of Numerical Analysis", 4, 566-574.

P.K. SEN, J.M. SINGER (1993), *Large Sample Methods in Statistics: An introduction with applications,* Chapman & Hall, New York.

R.R. WILCOX (1981), *A review of the Beta-Binomial model and its extensions*, "Journal of Educational Statistics", 6, 3-32.

R. WONG (2001), *Asymptotic approximations of integrals*, Reprint of the 1989 Edition, SIAM Classics in Applied Mathematics 34, Society for Industrial and Applied Mathematics, Philadelphia.

SUMMARY

Bernstein-type approximation using the beta-binomial distribution

The Bernstein-type approximation using the beta-binomial distribution is proposed and studied. Both univariate and multivariate Bernstein-type approximations are studied. The uniform convergence and the degree of approximation are studied. The Bernstein-type estimators of smooth functions of population means are also proposed and studied.