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A STUDY ON A TWO UNIT PARALLEL SYSTEM WITH ERLANGIAN REPAIR TIME

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1. INTRODUCTION

Introduction of the redundancy, repair maintenance and the preventive maintenance are some of the well known methods by which the reliability of a system can be improved. Parallel systems have been extensively studied by several authors in the past. A detailed list of the work on two unit system is complied by (Osaki and Nakagawa 1976). It can be shown that any failure or repair time distribution can be approximated arbitrarily closely by a general erlang distribution (Cox 1970). In the present paper, an attempt has been made to study a two unit parallel system with erlangian distribution for the repair time. For convenience, an erlangian distribution with two stages has been considered. Most of the studies on several systems are confined in obtaining expressions for various measures of system performance but the associated inference problems are not considered. In 1994, P. Chandrasekhar and R. Natarajan studied a two unit parallel system and obtained exact confidence limits for the steady state availability of the system when the failure rate of an operable unit is a constant and the repair time of the failed unit is a two stage erlangian distribution. In general, the failure time and repair time are independent random variables and hence there is a need to study a model by relaxing this imposed condition. Hence an attempt is made in this paper to study a two unit parallel system , wherein the failure rate of a unit is a constant and the repair time distribution is a two stage erlangian distribution under the assumption that an operable unit will not fail, while the other unit is in the second stage of repair. Besides obtaining expressions for the system reliability, MTBF, availability and steady state availability, we obtain a CAN estimator and an asymptotic confidence interval for the steady state availability of the system and the MLE of the system reliability. The model and assumptions are given in the next section.

2. MODEL AND ASSUMPTIONS

The system under consideration is a two unit parallel system with a single repair facility. We have precisely the following assumptions:

(i) The units are similar and statistically independent. Each unit has a constant failure rate, say λ .
(ii) There is only one repair facility and the repair time distribution is a two stage erlangian distribution with probability density function (p.d.f.) given by

$$g(y)=(2\mu)^2 e^{-2\mu} y,\ 0< y<\infty\ ,\ \mu>0 \tag{1}$$

(iii) Eeach unit is new after repair.
(iv) Switch is perfect and the switchover is instantaneous.
(v) An operable unit will not fail, while the other unit is in the second stage of repair.

3. ANALYSIS OF THE SYSTEM

To analyse the behaviour of the system, we note that at any time t, the system will be found in any one of the following mutually exclusive and exhaustive states:

S_0 : Both the units are operating.

S_1 : One unit is operating and the other is in the first stage of repair.

S_2 : One unit is operating and the other is in the second stage of repair.

S_3 : One unit is in the first stage of repair and the other is waiting for repair.

S_4 : One unit is in the second stage of repair and the other is waiting for repair.

Since an erlangian distribution is the distribution of the sum of two independent and identically distributed exponential random variables, the stochastic process describing the behaviour of the system is a markov process. Let $p_i(t)$ $,i=0,1,2,3,4$ be the probability that the system is in state S_i at time t. Clearly the infinitesimal generator of the markov process is given by

$$Q=\begin{matrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{matrix}\begin{pmatrix} 2\lambda & 0 & -\mu & 0 & 0 \\ -2\lambda & (\lambda+\mu) & 0 & 0 & -\mu \\ 0 & -\mu & \mu & 0 & 0 \\ 0 & -\lambda & 0 & \mu & 0 \\ 0 & 0 & 0 & -\mu & \mu \end{pmatrix} \tag{1}$$

It may be noted that states S_0, S_1 and S_2 are system up-states, whereas S_3 and S_4 are system down states. We assume that initially, both the units are operable the state transition diagram is given in fig. 1.

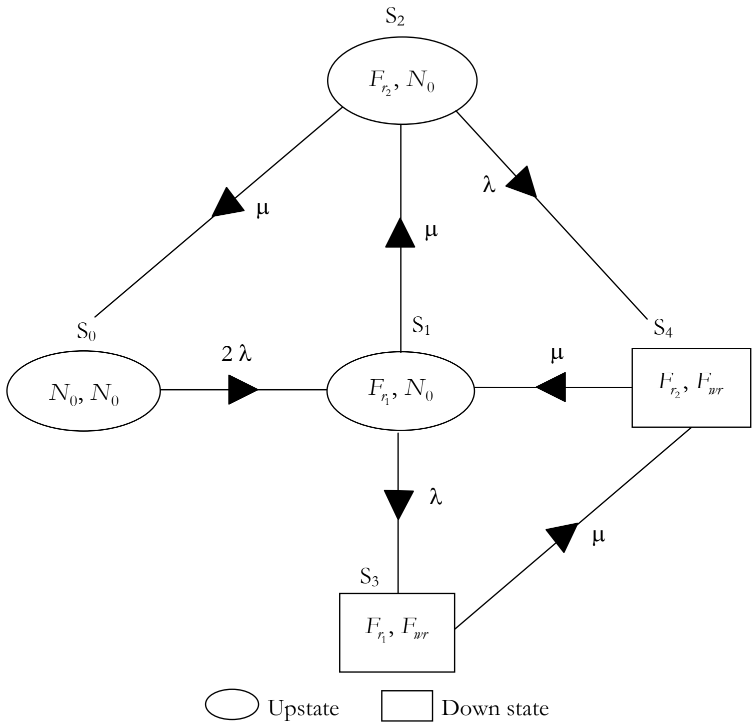

Figure 1 – State transition diagram.

SYMBOLS USED FOR THE STATES

N_0 : Unit is under operation

F_{r_i} : Failed unit is under phase $i^{th}\ (i=1,2)$ repair.

F_{wr} : Failed unit is waiting for repair.

We obtain the measures of system performance as follows:

3.1 *Reliability*

The system reliability $R(t)$ is the probability of failure free operation of the system in [0,t]. To derive an expression for the reliability of the system, we restrict the transitions of the Markov process to the system up-states namely S_0, S_1 and S_2. Using the infinitesimal generator given in (3.1), pertaining to these up-states and standard probabilistic arguments, we derive the following system of differential-difference equations:

$$p_0^{/}(t) = -2\lambda\, p_0(t) + \mu\, p_2(t) \tag{1}$$

$$p_1^{/}(t) = -(\lambda+\mu)\ p_1(t) + 2\lambda\, p_0(t) \tag{2}$$

$$p_2^{/}(t) = -\mu\, p_2(t) + \mu\, p_1(t) \tag{3}$$

Let $p_i^*(s)$ be the laplace transform of $p_i(t)$, i = 0, 1,2. Taking laplace transform on both the sides of diffrential-difference equations given above, solving for, $p_i^*(s)$, i = 0, 1, 2 and inverting, we get $p_i(t)$, i = 0, 1,2. Then the system reliability is given by

$$\text{R(t)} = \sum_{i=1}^{3} \frac{[(\alpha_i + 3\lambda + \mu)(\alpha_i + \mu) + 2\lambda\mu]}{\prod_{\substack{j=1 \\ j\neq i}}^{3} (\alpha_i - \alpha_j)}\, e^{\alpha_i t} \tag{4}$$

Where α_1, α_2 and α_3 are the roots of the cubic equation

$$s^3 + (3\lambda+\mu)s^2 + (2\lambda^2 + 5\lambda\mu + \mu^2)s + 2\lambda^2\mu$$

3.2 *Mean time before failure (MTBF)*

The system mean time before failure is given by

$$\text{MTBF} = p_1^*(0) + p_2^*(0) + p_3^*(0) = \frac{(5\lambda+\mu)}{2\lambda^2} \tag{1}$$

3.3 *System availability*

The system availability A(t) is the probability that the system operates within the tolerances at a given instant of time t and is obtained as follows:

Using the infinitesimal generator given in (3.1), we obtain the following system of differential-difference equations

$$p_0'(t) = -2\lambda\ p_0(t) + \mu\ p_2(t) \tag{1}$$

$$p_1'(t) = -(\lambda+\mu)\ p_1(t) + 2\lambda\ p_0(t) + \mu\ p_4(t) \tag{2}$$

$$p_2'(t) = -\mu\ p_2(t) + \mu\ p_1(t) \tag{3}$$

$$p_3'(t) = -\mu\ p_3(t) + \lambda\ p_1(t) \tag{4}$$

$$p_4'(t) = -\mu\ p_4(t) + \mu\ p_3(t) \tag{5}$$

Solving the above system of equations given in (3.3.1)-(3.3.5) using the fact that $\sum_{i=0}^{4} p_i(t) = 1$, we obtain

$$p_0(t) = \sum_{i=1}^{5} \frac{(\alpha_i+\mu)\ [(\alpha_i+\lambda+\mu)(\alpha_i+\mu)\ \mu-\lambda\mu^2]}{\prod_{\substack{j=1 \\ j\neq i}}^{3}(\alpha_i-\alpha_j)}\ e\alpha_i t \tag{6}$$

$$p_1(t) = 2\lambda \sum_{i=1}^{5} \frac{(\alpha_i+\mu)^3}{\prod_{\substack{j=1 \\ j\neq i}}^{5}(\alpha_i-\alpha_j)}\ e\alpha_i t \tag{7}$$

$$p_2(t) = 2\ \lambda\ \mu \sum_{i=1}^{5} \frac{(\alpha_i+\mu)^2}{\prod_{\substack{j=1 \\ j\neq i}}^{5}(\alpha_i-\alpha_j)}\ e\alpha_i t \tag{8}$$

$$p_3(t) = \lambda^2 \sum_{i=1}^{5} \frac{(\alpha_i+\mu)^3}{\prod_{\substack{j=1 \\ j\neq i}}^{5}(\alpha_i-\alpha_j)}\ e\alpha_i t \tag{9}$$

$$p_4(t) = \lambda^2 \sum_{i=1}^{5} \frac{(\alpha_i+\mu)^3}{\prod_{\substack{j=1 \\ j\neq i}}^{5}(\alpha_i-\alpha_j)}\ e\alpha_i t \tag{10}$$

Where α_1, α_2 , α_3 , α_4 and α_5 are the roots of the equation

$$\begin{aligned} & s^5 + (3\lambda + 4\mu)\ s^4 + [4\lambda\mu + 3\mu^2 + (2\lambda+\mu)(\lambda+3\mu)]s^3 \\ & + [(\lambda+\mu)\mu^2 + (2\lambda+\mu)\mu(3\mu+2\lambda) + 2\lambda\mu(\lambda+3\mu) - 2\lambda^2]s^2 \\ & + \mu[(2\lambda+\mu)(\lambda+\mu)\mu + 2\lambda\mu(3\mu+2\lambda) - 4\lambda^2]s + 2\lambda\mu^2[\mu(\lambda+\mu) - \lambda] \end{aligned}$$

Since s_0, s_1 and s_2 correspond to system up-states, the availability of the system is given by

$$A(t) = p_0(t) + p_1(t) + p_2(t) \tag{11}$$

3.4 *Steady state availability*

The system steady state availability is given by

$$A_\infty = \lim_{t\to\infty} A(t) \quad = \lim_{s\to 0} sA^*(s) \tag{1}$$

$$= \frac{(\mu^3 + 4\lambda)}{(\lambda+\mu)(2\lambda+\mu)+6\lambda\mu^2}$$

In the following sections, we obtain a CAN estimator, a 100 (1-α)% asymptotic confidence interval for steady state availability of the system and MLE of the system reliability.

4. CONFIDENCE INTERVAL FOR STEADY STATE AVAILABILITY OF THE SYSTEM

Let $X_1, X_2, X_3, \ldots X_n$ be a random sample of times to failure of the unit with p.d.f. given by

$$f(x) = \lambda\, e^{-\lambda x} \,,\, 0 < x < \infty \,,\, \lambda > 0 \tag{1}$$

Let $Y_1, Y_2, Y_3, \ldots Y_n$ be a random sample times to repair with the p.d.f. as in (2.1). It is clear that $E(\bar{X}) = \frac{1}{\lambda}$ and $E\left(\frac{\bar{Y}}{2}\right) = \frac{1}{\mu}$ where $\bar{X}$ and $\bar{Y}$ are the sample means of time to failure and time to repair of the unit respectively. It can be shown that $\bar{X}$ and $\frac{\bar{Y}}{2}$ are the maximum likelihood estimators (MLEs) of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively.let $\theta_1 = \frac{1}{\lambda}$ and $\theta_2 = \frac{1}{\mu}$. Clearly , the steady state availability given in (3.4.1) reduces to

$$A_\infty = \frac{\theta_1\theta_2(\theta_1 + 4\theta_2^3)}{(\theta_1+\theta_2)(2\theta_2+\theta_1)+6\theta_1} \tag{2}$$

Hence the mle of A_∞ is given by

$$\hat{A}_\infty = \frac{\bar{X}\ \bar{Y}[2\bar{X} + \bar{Y}^3]}{2[(2\bar{X}+\bar{Y})(\bar{X}+\bar{Y})+12\bar{X}]} \tag{3}$$

It may be noted that $\hat{A}_\infty$ is a real valued function in $\bar{X}$ and $\bar{Y}$ which is also differentiable. Now consider the following application of multivariance central limit theorem. See (Rao 1974).

Suppose $T_1', T_2', \ldots$ are independent and identically distributed k-dimensional random variables such that

$$T_n' = (T_{1n}, T_{2n}, ..., T_{kn}), n = 1,2,3,....$$

Having the first and second order moments $E(T_n) = \mu$ and $D(T_n) = \Sigma$. Define the sequence of random variables

$$\overline{T}_n = (\overline{T}_{1n}, \overline{T}_{2n}, ..., \overline{T}_{kn}), n = 1,2,3,....$$

where

$$\overline{T}_n = \frac{1}{n}\sum_{j=1}^{n} T_{ij}, \quad \begin{matrix} i=1,2,.......k \\ j=1,2,.....,n \end{matrix}$$

then, $\sqrt{n}(\overline{T}_n - \mu) \xrightarrow{d} N(0,\Sigma) \quad as \quad n \to \infty$

by applying the above application of multivariate central limit theorem, it readily follows that

$$\sqrt{n}\left[\left(\bar{X}, \frac{\bar{Y}}{2}\right) - (\theta_1, \theta_2)\right] \xrightarrow{d} N(0,\Sigma) \quad as \quad n \to \infty$$

Where the dispersion matrix $\Sigma = ((\sigma_{ij}))_{2x2.}$.
again, from (RAO, 1974), we have

$$\sqrt{n}(\hat{A}_\infty - A_\infty) \xrightarrow{d} N(0, \sigma^2(\theta)) \quad as \quad n \to \infty$$

where

$$\theta = (\theta_1, \theta_2) \text{ and}$$

$$\sigma^2(\theta) = \sum_{i=1}^{2}\left(\frac{\partial A_\infty}{\partial \theta_i}\right)^2 \sigma_{ii}$$

$$= \frac{\begin{matrix} 2[2\theta_2\{(\theta_1+\theta_2)(\theta_1+2\theta_2+6\theta_1)\}(\theta_1+2\theta_2^3)^2 - (2\theta_1+3\theta_2+6)]^2\theta_1^2 \\ +[(\theta_1+\theta_2)(\theta_1+2\theta_2+6\theta_1)\theta_1(\theta_1+16\theta_2^3) - \{\theta_1\theta_2(\theta_1+4\theta_2^3)(3\theta_1+4\theta_2)]^2\theta_2^2 \end{matrix}}{[(\theta_1+\theta_2)(\theta_1+2\theta_2+6\theta_1)]^2}$$

hence $\hat{A}_\infty$ is a CAN estimator of A_∞.

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator $\hat{\theta}$ namely $\hat{\theta} = \left(\bar{X}, \frac{\bar{Y}}{2}\right)$.

Let $\hat{\theta}^2 = \sigma^2(\hat{\theta})$ since $\sigma^2(\theta)$ is a continious function of θ, $\hat{\sigma}^2$ is a consistent estimator of $\sigma^2(\theta)$ i.e., $\hat{\sigma}^2 \xrightarrow{p} \sigma^2 \quad as \quad n \to \infty$

by slutskey theorem, we have

$$\frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} \xrightarrow{d} N(0,1)$$

i.e. $$P\left[-k_{\frac{\alpha}{2}} < \frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} < k_{\frac{\alpha}{2}}\right] = (1-\alpha),$$

$k_{\frac{\alpha}{2}}$ is obtained from normal tables hence, a 100(1-α)% asymptotic confidence interval for A_∞ is givn by $\hat{A}_\infty \pm k_{\frac{\alpha}{2}} . \frac{\sigma}{\sqrt{n}}$

5. MLE OF SYSTEM RELIABILITY

Since $\overline{X}$ and $\frac{\overline{Y}}{2}$ are the MLEs of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively, by applying the method given in (ZACKS, 1992) , we obtain the MLE of system reliability as

$$\hat{\mathrm{R}}(t) = \sum_{i=1}^{3} \frac{[\hat{\alpha}_i \overline{X}\overline{Y} + 3\overline{Y} + 2\overline{X})(\hat{\alpha}_i \overline{Y} + 2) + 4\overline{X}]}{\overline{X}\overline{Y}^2 \prod_{\substack{j=1 \\ j \neq i}}^{3} (\alpha_i - \alpha_j)} \; e^{\hat{\alpha}_i t} \tag{1}$$

where $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\alpha}_3$ are the roots of the cubic equation

$$(\overline{X}\,\overline{Y})^2 s^3 + \overline{X}\,\overline{Y}(4\overline{X} + 3\overline{Y})s^2 + (4\overline{X}^2 + 10\overline{X}\,\overline{Y} + 2\overline{Y}^2)s + 4\overline{Y} = 0$$

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SUMMARY

A study on a two unit parallel system with erlangian repair time

A two unit parallel system where in the failure rate of a unit is a constant and the repair time distribution is a two stage erlangian distribution is considered. Measures of system performance such as reliability, MTBF ,system availability and steady state availability are derived. Also, a consistent asymptotically normal (CAN) estimator, a 100(1-α) % asymptotic confidence interval for the steady state availability of the system and the maximum likelihood estimator (MLE) of the system reliability are obtained.