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SUPERIORITY OF THE STOCHASTIC RESTRICTED LIU ESTIMATOR UNDER MISSPECIFICATION

M.H. Hubert, P. Wijekoon

1. INTRODUCTION

Misspecification of the regression model is a very serious problem in econometric theory. Generally researchers are concerned with two types of misspecification; excluding relevant variables and including irrelevant variables, where these two problems are treated separately. When the regression model is correctly specified it was shown that the Stochastic Restricted Liu estimator (SRLE) is better than the Liu estimator under certain conditions. In 1989 Trenkler and Wijekoon demonstrated that in the case of excluded variables, Mixed Regression Estimator (MRE) outperforms OLSE with respect to the mean squared error matrix sense. The purpose of this paper is to show that the SRLE based on the correct prior information is potentially better than the Liu estimator under the misspesified regression model due to the exclusion of relevant variables.

2. MODEL SPECIFICATION AND ESTIMATION

Starting from Kadiyala's (1986) approach we assume that the correctly specified multiple linear regression model is given by

$$\boldsymbol{Y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\delta} + \boldsymbol{\varepsilon} \tag{1}$$

with $\boldsymbol{\delta} = \boldsymbol{X}_2\boldsymbol{\beta}_2$, where, $\boldsymbol{Y}$ is an $(n \times 1)$ vector of observations on the response (or dependent) variable, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are $(n \times p_1)$ and $(n \times p_2)$ matrices of observations on the $p = p_1 + p_2$ regressors, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are $(p_1 \times 1)$ and $(p_2 \times 1)$ vectors of unknown parameters and $\boldsymbol{\varepsilon}$ is an $(n \times 1)$ vector of disturbances with $E(\boldsymbol{\varepsilon}) = \boldsymbol{0}$ and the variance covariance matrix $D(\boldsymbol{\varepsilon}) = \sigma^2\boldsymbol{I}$.

Suppose the model (1) is misspecified by excluding p_2 regressors as

$$\boldsymbol{Y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{u} \tag{2}$$

Furthermore, assume that some additional correctly specified prior information is available on $\boldsymbol{\beta}_1$ in the following form;

$$\boldsymbol{r} = \boldsymbol{R}\boldsymbol{\beta}_1 + \boldsymbol{v}\,, \tag{3}$$

where $\boldsymbol{r}$ is an $(j \times 1)$ stochastic known vector, $\boldsymbol{R}$ is a $(j \times p_1)$ matrix of full row rank j ($\leq p_1$) with known elements, $\boldsymbol{\beta}_1$ is the $(p_1 \times 1)$ coefficient vector of (2) and $\boldsymbol{v}$ is an $(j \times 1)$ random vector of disturbances satisfying

$$\mathrm{E}(\boldsymbol{v}) = 0, \qquad D(\boldsymbol{v}) = \boldsymbol{\Omega} > 0 \quad \text{and} \quad \mathrm{E}(\boldsymbol{v}\boldsymbol{u}') = 0\,. \tag{4}$$

Given $\boldsymbol{T}$ is a $(p_1 \times p_1)$ non singular matrix such that

$$\boldsymbol{X}_* = \boldsymbol{X}_1\boldsymbol{T}\,,\ \boldsymbol{R}_* = \boldsymbol{R}\boldsymbol{T}\,,\ \boldsymbol{T}'\boldsymbol{X}_1'\boldsymbol{X}_1\boldsymbol{T} = \boldsymbol{I}\,,\ \boldsymbol{R}_*'\boldsymbol{\Omega}^{-1}\boldsymbol{R}_* = \boldsymbol{\Lambda}\,, \tag{5}$$

where $\boldsymbol{\Lambda}$ is a $(p_1 \times p_1)$ diagonal matrix with elements

$$\lambda_i > 0 \ ; \text{if } i = 1, 2, \ldots, j$$

$$\lambda_i = 0 \ ; \text{if } i = j + 1, \ldots, p_1, \tag{6}$$

and j being the rank of $\boldsymbol{R}$ and $\boldsymbol{\gamma} = \boldsymbol{T}^{-1}\ \boldsymbol{\beta}_1$, we rewrite (1), (2) and (3) by using the simultaneous spectral decomposition of the two symmetric matrices $\boldsymbol{X}_1'\boldsymbol{X}_1$ and $\boldsymbol{R}'\boldsymbol{\Omega}^{-1}\boldsymbol{R}$ as

$$\boldsymbol{Y} = \boldsymbol{X}_*\boldsymbol{\gamma} + \boldsymbol{\delta} + \boldsymbol{\varepsilon} \tag{7}$$

$$\boldsymbol{Y} = \boldsymbol{X}_*\boldsymbol{\gamma} + \boldsymbol{u} \tag{8}$$

$$\boldsymbol{r} = \boldsymbol{R}_*\boldsymbol{\gamma} + \boldsymbol{v} \tag{9}$$

The Ordinary Least Squares Estimator (OLSE) $\hat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ in (8) is

$$\hat{\boldsymbol{\gamma}} = (\boldsymbol{X}_*'\boldsymbol{X}_*)^{-1}\boldsymbol{X}_*'\boldsymbol{Y} = \boldsymbol{X}_*'\boldsymbol{Y} \tag{10}$$

with bias

$$B(\hat{\boldsymbol{\gamma}}) = \boldsymbol{X}_*'\boldsymbol{\delta} \tag{11}$$

and dispersion matrix

$$D(\hat{\boldsymbol{\gamma}}) = \sigma^2\boldsymbol{I}_{p_1}\,. \tag{12}$$

Kejian Liu (1993) introduced a new biased estimator called the *Liu estimator* (LE), and showed that this estimator is superior to the OLSE both in the scalar mean squared error and mean squared error matrix sense.

Now the Liu estimator for the model in (8) is given by

$$\hat{\gamma}_{d} = F_{d}\hat{\gamma} = F_{d}X_{*}'Y \tag{13}$$

where,

$$F_{d} = (X_{*}'X_{*} + I)^{-1}(X_{*}'X_{*} + dI) \text{ for } 0 < d < 1 \tag{14}$$

But in this case

$$F_{d} = \left(\frac{1+d}{2}\right)I = k_1 I \text{, with } k_1 = \left(\frac{1+d}{2}\right). \tag{15}$$

Thus the LE under misspecification is,

$$\hat{\gamma}_{d} = k_1 X_{*}'Y \tag{16}$$

with bias

$$\mathrm{B}(\hat{\gamma}_{d}) = (F_{d} - I)\gamma + F_{d}X_{*}'\delta = k_1 X_{*}'\delta - k_2\gamma, \tag{17}$$

where, $k_2 = \left(\frac{1-d}{2}\right)$, and dispersion matrix

$$D(\hat{\gamma}_{d}) = F_{d}D(\hat{\gamma})F_{d}' = k_1^2\sigma^2 I. \tag{18}$$

Hence the Mean Squared Error matrix (MSE) of $\hat{\gamma}_{d}$ is

$$\mathrm{MSE}(\hat{\gamma}_{d}) = k_1^2\sigma^2 I + (k_1 X_{*}'\delta - k_2\gamma)(k_1 X_{*}'\delta - k_2\gamma)'. \tag{19}$$

The Mixed Regression Estimator (MRE) due to Thiel and Goldberger (1961), for the transformed model turns out to be

$$\hat{\gamma}_{*} = (I + \sigma^2\Lambda)^{-1}(X_{*}'Y + \sigma^2 R_{*}'\Omega^{-1}r). \tag{20}$$

The bias vector and the dispersion matrix of $\hat{\gamma}_{*}$ are given by

$$\mathrm{B}(\hat{\gamma}_{*}) = (I + \sigma^2\Lambda)^{-1} X_{*}'\delta \tag{21}$$

and

$$\mathrm{D}(\hat{\gamma}_{*}) = \sigma^2(I + \sigma^2\Lambda)^{-1} \tag{22}$$

respectively. Based on $\hat{\boldsymbol{\gamma}}_*$, now the Stochastic Restricted Liu Estimator (SRLE) under misspecification can be defined as

$$\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}} = \boldsymbol{F_d}\hat{\boldsymbol{\gamma}}_* = k_1\hat{\boldsymbol{\gamma}}_* \tag{23}$$

Furthermore the bias vector, the dispersion matrix and the corresponding MSE matrix of $\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}}$ are given by

$$\mathrm{B}(\hat{\boldsymbol{\gamma}}_{srd}) = k_1(\boldsymbol{I} + \sigma^2\boldsymbol{\Lambda})^{-1}\boldsymbol{X}_*'\delta - k_2\boldsymbol{\gamma}, \tag{24}$$

$$\mathrm{D}(\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}}) = \boldsymbol{F_d}D(\hat{\boldsymbol{\gamma}}_*)\boldsymbol{F}_d' = k_1^2\sigma^2(\boldsymbol{I} + \sigma^2\boldsymbol{\Lambda})^{-1} \tag{25}$$

and

$$\begin{aligned}\mathrm{MSE}(\hat{\boldsymbol{\gamma}}_{srd}) = {} & k_1^2\sigma^2(\boldsymbol{I} + \sigma^2\boldsymbol{\Lambda})^{-1} + \\ & (k_1(\boldsymbol{I} + \sigma^2\boldsymbol{\Lambda})^{-1}\boldsymbol{X}_*'\boldsymbol{\delta} - k_2\boldsymbol{\gamma})(k_1(\boldsymbol{I} + \sigma^2\boldsymbol{\Lambda})^{-1}\boldsymbol{X}_*'\boldsymbol{\delta} - k_2\boldsymbol{\gamma})'\end{aligned} \tag{26}$$

respectively.

3. MSE MATRIX COMPARISONS OF SRLE AND LE

Now we investigate under which conditions $\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}}$ is superior to $\hat{\boldsymbol{\gamma}}_{\boldsymbol{d}}$ in the MSE matrix criterion. That is we need to find conditions, which ensure the nonnegative definiteness (n.n.d.) of $MSE(\hat{\boldsymbol{\gamma}}_d) - MSE(\hat{\boldsymbol{\gamma}}_{srd})$.

Let us use the following notations:

$$\boldsymbol{\Sigma} = \boldsymbol{I} + \sigma^2\boldsymbol{\Lambda} \quad \text{and} \quad \boldsymbol{\eta} = \boldsymbol{X}_*'\boldsymbol{\delta}. \tag{27}$$

After some straightforward calculations, we obtain

$$\begin{aligned}\mathrm{MSE}(\hat{\boldsymbol{\gamma}}_d) - \mathrm{MSE}(\hat{\boldsymbol{\gamma}}_{srd}) = {} & k_1^2\sigma^4\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1} + (k_1\boldsymbol{\eta} - k_2\boldsymbol{\gamma})(k_1\boldsymbol{\eta} - k_2\boldsymbol{\gamma})' \\ & -(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} - k_2\boldsymbol{\gamma})(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} - k_2\boldsymbol{\gamma})'\end{aligned} \tag{28}$$

Trenkler and Wijekoon (1989) discussed the dominance of $\hat{\boldsymbol{\gamma}}_*$ over $\hat{\boldsymbol{\gamma}}$ using the following lemma derived by Baksalary and Trenkler (1988). We use the same lemma to obtain a necessary and sufficient condition for the superiority of the SRLE over the LE.

Lemma 1. (Baksalary and Trenkler 1988)
Let $\boldsymbol{A} \geq 0$ of type $n \times n$, let $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ be vectors of type $n \times 1$ and let α_1 and α_2 be positive scalars. Then

$$\boldsymbol{A}+\alpha_1^{-1}\boldsymbol{a}_1\boldsymbol{a}_1'-\alpha_2^{-1}\boldsymbol{a}_2\boldsymbol{a}_2' \tag{29}$$

is nonnegative definite if and only if the following conditions hold:
either

$$\boldsymbol{a}_i \in \Re(\boldsymbol{A}), \quad i=1,2 \tag{30}$$

and

$$(\boldsymbol{a}_1'\boldsymbol{A}^-\boldsymbol{a}_1+\alpha_1)(\boldsymbol{a}_2'\boldsymbol{A}^-\boldsymbol{a}_2-\alpha_2)\le(\boldsymbol{a}_1'\boldsymbol{A}^-\boldsymbol{a}_2)^2 \tag{31}$$

or $\boldsymbol{a}_1 \notin \Re(\boldsymbol{A})$, $i = 1, 2$ and $Q_A\boldsymbol{a}_2=\lambda Q_A\boldsymbol{a}_1$ for some λ (32)
where $\boldsymbol{Q}_A=\boldsymbol{I}-\boldsymbol{A}\boldsymbol{A}^+$, and

$$(\boldsymbol{a}_1-\lambda\boldsymbol{a}_1)'\boldsymbol{A}^-(\boldsymbol{a}_2-\lambda\boldsymbol{a}_1)\le\alpha_2-\lambda^2\alpha_1 \tag{33}$$

with the inequalities (31) and (33) being independent of the choice of a generalized inverse $\boldsymbol{A}^-$ of $\boldsymbol{A}$, where $\Re(\boldsymbol{A})$ stands for the column space of the matrix $\boldsymbol{A}$, and $\boldsymbol{A}^+$ is the Moore-Penrose inverse of $\boldsymbol{A}$.

Note that $k_1^2\sigma^4\boldsymbol{\Lambda}\Sigma^{-1}$ is non negative definite. To apply this lemma, let

$$\boldsymbol{A}=k_1^2\sigma^4\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1} \tag{34}$$

$$\boldsymbol{a}_1=k_1\boldsymbol{\eta}-k_2\boldsymbol{\gamma} \tag{35}$$

$$\boldsymbol{a}_2=k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta}-k_2\boldsymbol{\gamma} \tag{36}$$

$$\alpha_i=1 \text{ for } i=1,2 \tag{37}$$

<u>*Case I*</u>

If $\boldsymbol{a}_1\in\Re(\boldsymbol{A})$ then $\boldsymbol{a}_2\in\Re(\boldsymbol{A})$, since $\boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{\Lambda}$ commute. After some straightforward calculations, we obtain

$$\boldsymbol{a}_1'\boldsymbol{A}^-\boldsymbol{a}_1=\frac{1}{k_1^2\sigma^4}(k_1\boldsymbol{\eta}'-k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^-\boldsymbol{\Sigma}(k_1\boldsymbol{\eta}-k_2\boldsymbol{\gamma}), \tag{38}$$

$$\boldsymbol{a}_2'\boldsymbol{A}^-\boldsymbol{a}_2=\frac{1}{k_1^2\sigma^4}(k_1\boldsymbol{\eta}'\boldsymbol{\Sigma}^{-1}-k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^-\boldsymbol{\Sigma}(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta}-k_2\boldsymbol{\gamma}) \tag{39}$$

and

$$\boldsymbol{a}_1'\boldsymbol{A}^-\boldsymbol{a}_2=\frac{1}{k_1^2\sigma^4}(k_1\boldsymbol{\eta}'-k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^-\boldsymbol{\Sigma}(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta}-k_2\boldsymbol{\gamma}). \tag{40}$$

Then by Lemma 1,

$\text{MSE}(\hat{\boldsymbol{\gamma}}_{\boldsymbol{d}}) - \text{MSE}(\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}}) \geq 0$, if and only if

$$\begin{aligned}&\{(k_1\boldsymbol{\eta}' - k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^{-}\boldsymbol{\Sigma}(k_1\boldsymbol{\eta} - k_2\boldsymbol{\gamma}) + k_1^2\sigma^4\}\\ &\qquad \times\{(k_1\boldsymbol{\eta}'\boldsymbol{\Sigma}^{-1} - k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^{-}\boldsymbol{\Sigma}(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} - k_2\boldsymbol{\gamma}) - k_1^2\sigma^4\}\\ &\qquad \leq \{(k_1\boldsymbol{\eta}' - k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^{-}\boldsymbol{\Sigma}(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} - k_2\boldsymbol{\gamma})\}^2\end{aligned} \tag{41}$$

Note that when $d = 1$, $k_1 = 1$ and $k_2 = 0$. Then the LE $\hat{\boldsymbol{\gamma}}_{\boldsymbol{d}}$ = OLSE $\hat{\boldsymbol{\gamma}}$ and the SRLE $\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}}$ = MRE $\hat{\boldsymbol{\gamma}}_{*}$, and hence equation (41) is equal to the condition obtained by Trenkler and Wijekoon (1989).

Case II

If $\boldsymbol{a}_1 \notin \Re(\boldsymbol{A})$ then

$$Q_A = \boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{+} = \boldsymbol{I} - \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{+} = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{I}_{j\times p_1} \end{pmatrix} = (\boldsymbol{I} - \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{+})\boldsymbol{\Sigma}^{-1} \tag{42}$$

and, consequently

$$Q_A\boldsymbol{a}_2 = \lambda Q_A\boldsymbol{a}_1 \text{ with } \lambda = 1 . \tag{43}$$

Further it can be also shown that

$$(\boldsymbol{a}_2 - \boldsymbol{a}_1)'\boldsymbol{A}^{-}(\boldsymbol{a}_2 - \boldsymbol{a}_1) = \boldsymbol{\eta}'\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} . \tag{44}$$

Hence to satisfy condition (33) in lemma 1 we must have

$$\boldsymbol{\eta}'\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} \leq 0 . \tag{45}$$

Note that this is the same condition obtained by Trenkler and Wijekoon (1989) when comparing the OLSE with MRE under misspecification.

Theorem 1

The SRLE $\hat{\boldsymbol{\gamma}}_{srd}$ dominates the LE $\hat{\boldsymbol{\gamma}}_{d}$ under misspecification with respect to the MSE-matrix criterion if and only if,
(i) either

$$\begin{aligned}&\{(k_1\boldsymbol{\eta}' - k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^{-}\boldsymbol{\Sigma}(k_1\boldsymbol{\eta} - k_2\boldsymbol{\gamma}) + k_1^2\sigma^4\}\\ &\qquad \times\{(k_1\boldsymbol{\eta}'\boldsymbol{\Sigma}^{-1} - k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^{-}\boldsymbol{\Sigma}(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} - k_2\boldsymbol{\gamma}) - k_1^2\sigma^4\}\\ &\qquad \leq \{(k_1\boldsymbol{\eta}' - k_2\boldsymbol{\gamma}')\boldsymbol{\Lambda}^{-}\boldsymbol{\Sigma}(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{\eta} - k_2\boldsymbol{\gamma})\}^2\end{aligned}$$

when $\boldsymbol{a}_i \in \Re(\boldsymbol{A})$, for $i = 1,2$
(ii) or
$\boldsymbol{\eta}'\boldsymbol{\Lambda\Sigma}^{-1}\boldsymbol{\eta} \le 0$, which is equivalent to $\eta_i = 0,\ i = 1,\cdots,j$, where η_i denote the i^th component of the vector $\boldsymbol{\eta}$ when $\boldsymbol{a}_1 \notin \Re(\boldsymbol{A})$.

4. MSE MATRIX COMPARISONS OF PREDICTORS

We assume that (1) holds for further realizations of the dependent variable. In other words if $\boldsymbol{Y}_0$ and $\boldsymbol{\varepsilon}_0$ are $(m \times 1)$ vectors and $\boldsymbol{X}_{10}$ and $\boldsymbol{X}_{20}$ are $(m \times p_1)$ and $(m \times p_2)$ matrices then

$$\boldsymbol{Y}_0 = \boldsymbol{X}_{10}\boldsymbol{\beta}_1 + \boldsymbol{\delta} + \boldsymbol{\varepsilon}_0, \quad \text{with } \boldsymbol{\delta} = \boldsymbol{X}_{20}\boldsymbol{\beta}_2 \tag{46}$$

The error vector $\boldsymbol{\varepsilon}_0$ satisfies the following conditions:

$$\mathrm{E}(\boldsymbol{\varepsilon}_0) = \mathbf{0}, \quad \text{and} \quad \mathrm{E}(\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_0') = \sigma^2 \boldsymbol{I}_m \tag{47}$$

Also $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}$ are uncorrelated such that $E(\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}') = \mathbf{0}$. We further assume that $\boldsymbol{X}_{10}$ is fixed, and $\boldsymbol{X}_{10}$, $\boldsymbol{X}_{20}$ are known. Then again rewriting model (46) using the transformation, we obtain

$$\boldsymbol{Y}_0 = \boldsymbol{X}_*\boldsymbol{\gamma} + \boldsymbol{\delta} \tag{48}$$

Now the Liu Predictor (LP) and the Stochastic Restricted Liu Predictor (SRLP) is defined as

$$\hat{\boldsymbol{Y}}_d = \boldsymbol{X}_*\hat{\boldsymbol{\gamma}}_d \tag{49}$$

and

$$\hat{\boldsymbol{Y}}_{srd} = \boldsymbol{X}_*\hat{\boldsymbol{\gamma}}_{srd} \tag{50}$$

respectively.
The MSE matrix of the Liu predictor is

$$\begin{aligned}\mathrm{MSE}(\hat{\boldsymbol{Y}}_d) = \boldsymbol{X}_*\mathrm{MSE}(\hat{\boldsymbol{\gamma}}_d)\boldsymbol{X}_*' - \boldsymbol{\delta}(k_1\boldsymbol{X}_*'\boldsymbol{\delta} - k_2\boldsymbol{\gamma})'\boldsymbol{X}_*' \\ - \boldsymbol{X}_*(k_1\boldsymbol{X}_*'\boldsymbol{\delta} - k_2\boldsymbol{\gamma})\boldsymbol{\delta}' + \boldsymbol{\delta\delta}'\end{aligned}. \tag{51}$$

Similarly the MSE matrix of SRLP turns out to be

$$\begin{aligned}\text{MSE}(\hat{\boldsymbol{Y}}_{srd}) = \boldsymbol{X}_*\text{MSE}(\hat{\boldsymbol{\gamma}}_{srd})\boldsymbol{X}'_* - \boldsymbol{\delta}(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'_*\boldsymbol{\delta} - k_2\boldsymbol{\gamma})'\boldsymbol{X}'_* \\ - \boldsymbol{X}_*(k_1\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'_*\boldsymbol{\delta} - k_2\boldsymbol{\gamma})\boldsymbol{\delta}' + \boldsymbol{\delta}\boldsymbol{\delta}'\end{aligned}. \tag{52}$$

After some straightforward calculations we can obtain

$$\text{MSE}(\hat{\boldsymbol{Y}}_{\boldsymbol{d}}) - \text{MSE}(\hat{\boldsymbol{Y}}_{\boldsymbol{srd}}) = \boldsymbol{D} - \boldsymbol{\theta}\boldsymbol{\theta}' \tag{53}$$

where

$$\boldsymbol{D} = \boldsymbol{X}_*(\text{MSE}(\hat{\boldsymbol{\gamma}}_{\boldsymbol{d}}) - \text{MSE}(\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}}))\boldsymbol{X}'_* + \boldsymbol{\delta}\boldsymbol{\delta}' + k_1^2\sigma^4\boldsymbol{X}_*\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'_*\boldsymbol{\delta}\boldsymbol{\delta}'\boldsymbol{X}_*\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'_*$$

and $\theta = \boldsymbol{\delta} + k_1\sigma^2\boldsymbol{X}_*\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}'_*\boldsymbol{\delta}$ respectively. To study the MSE-matrix superiority of SRLP over LP we can use the following lemma.

Lemma 2 (Baksalary and Kala 1983)

Let $\boldsymbol{B} \geq 0$ of type $n \times n$ matrix, $\boldsymbol{b}$ is an $n \times 1$ vector and λ is a positive real number. Then the following conditions are equivalent.

(i) $\lambda\boldsymbol{B} - \boldsymbol{b}\boldsymbol{b}' \geq 0$

(ii) $\boldsymbol{B} \geq 0$, $\boldsymbol{b} \in \Re(\boldsymbol{B})$ and $\boldsymbol{b}'\boldsymbol{B}^-\boldsymbol{b} \leq \lambda$

where $\Re(\boldsymbol{B})$ stands for the column space of $\boldsymbol{B}$, and the latter inequality is independent of the choice of a g-inverse $\boldsymbol{B}^-$ of $\boldsymbol{B}$

According to this lemma now we can state the following theorem.

Theorem 2

The following two statements are equivalent.

(i) $\text{MSE}(\hat{\boldsymbol{Y}}_{\boldsymbol{d}}) - \text{MSE}(\hat{\boldsymbol{Y}}_{\boldsymbol{srd}}) \geq 0$

(ii) $\boldsymbol{D} \geq 0$, $\boldsymbol{\theta} \in \text{R}(\boldsymbol{D})$ and $\boldsymbol{\theta}'\boldsymbol{D}^-\boldsymbol{\theta} \leq 1$.

Note that in section 3 conditions were derived under which $\text{MSE}(\hat{\boldsymbol{\gamma}}_{\boldsymbol{d}}) - \text{MSE}(\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}})$ is n.n.d.. Obviously these conditions are sufficient for $\boldsymbol{D} \geq 0$. Therefore we can conclude that there are situations where $\hat{\boldsymbol{Y}}_{\boldsymbol{srd}}$ outperforms $\hat{\boldsymbol{Y}}_{\boldsymbol{d}}$ with respect to the mean squared error matrix criterion.

Also when $d = 1$, then $k_1 = 1$, $\hat{\boldsymbol{\gamma}}_{\boldsymbol{srd}} = \hat{\boldsymbol{\gamma}}_*$, and $\hat{\boldsymbol{\gamma}}_{\boldsymbol{d}} = \hat{\boldsymbol{\gamma}}$. In this case the matrix $\boldsymbol{D}$ and vector $\boldsymbol{\theta}$ correspond to the predictors of $\hat{\boldsymbol{\gamma}}_*$ and $\hat{\boldsymbol{\gamma}}$ obtained by Trenkler and Wijekoon (1989).

5. CONCLUSION

Theorem 1 gives the conditions, in which the SRLE $\hat{\gamma}_{srd}$ dominates the LE $\hat{\gamma}_d$ under misspecification with respect to the MSE-matrix criterion. To compare the corresponding predictors the results given in theorem 2 can be used. We also note that in both cases when $d = 1$, then $k_1 = 1$, $\hat{\gamma}_{srd} = \hat{\gamma}_*$, and $\hat{\gamma}_d = \hat{\gamma}$, and hence the results obtained by Trenkler and Wijekoon (1989) can nicely be presented.

Department of Economics and Management, M.H. HUBERT
Vavuniya Campus, Vavuniya, Sri Lanka

Department of Statistics and Computer Science, P. WIJEKOON
University of Peradeniya, Sri Lanka

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RIASSUNTO

Superiorità dell'Estimatore Stocastico Ristretto di Liu sotto l'ipotesi di non-specificazione

Questo sommario tratta l'uso delle corrette antecedenti informazioni nella stima dei coefficienti di regressione quando il modello di regressione è non-specificato a causa dell'esclusione di qualche rilevante regressore variabile. In particolare l'attenzione è focalizzata sull'Estimatore Stocastico Ristretto di Liu, introdotto da Hubert e Wijekoon (2004), che supera lo stimatore Liu rispetto al criterio dell'errore quadratico medio della matrice. Inoltre la superiorità del calcolatore "Estimator Stocastico Liu" sul calcolatore Liu è stata anche esaminata e si è concluso che vi sono situazioni dove l'Estimatore Stocastico Ristretto di Liu è superiore al calcolatore Liu rispetto al criterio dell'errore quadratico medio della matrice anche se il modello è non-specificato.

SUMMARY

Superiority of the Stochastic Restricted Liu Estimator under misspecification

This paper deals with the use of correct prior infromation in the estimation of regression coefficients when the regression model is misspecified due to the exclusion of some relevant regressor variables. In particular, the attention is focused on the Stochastic Restricted Liu estimator introduced by Hubert and Wijekoon (2004), which outperforms Liu estimator with respect to the matrix mean squared error matrix criterion. Further the superiority of the Stochastic Restricted Liu predictor over the Liu predictor is also examined, and concluded that there are situations where the Stochastic Restricted Liu predictor outperforms the Liu predictor with respect to the mean squared error matrix criterion even the model is misspecified.