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ESTIMATING THE PARAMETERS OF THE NORMAL, EXPONENTIAL AND GAMMA DISTRIBUTIONS USING MEDIAN AND EXTREME RANKED SET SAMPLES

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1. INTRODUCTION

McIntyre (1952) intuitively introduced and applied the method of ranked set sampling (RSS) in estimating mean pasture yields as a more efficient and cost-effective method than the method of simple random sampling (SRS). This method is useful in situations where units under consideration are easier and cheaper to order than to measure with respect to a characteristic of interest. The method draws from a population of interest, n random samples each of size n. The members of each random sample are ordered with respect to the characteristic under study. From the first ordered set, the smallest unit is chosen for measurement as is the second smallest from the second ordered sample. This continues until the largest element from the last random sample is measured. The set of measured elements then constitutes the ranked set sample of size n. This process may be repeated m times (m cycles) to yield a sample of size nm. For classified and extensive review of work done in RSS, see (Patil et al, 1994). However, we will briefly review some of the literature relevant to this study.

Takahasi and Wakimoto (1968) independently described the method of RSS and presented a sound mathematical argument, which supports McIntyre's intuitive assertion. (Dell and Clutter, 1972) showed that errors in ranking reduce the precision of the RSS mean relative to the SRS mean. However, the RSS mean remains dominant over the SRS mean until ranking is so poor as to yield a random sample when it performs just as well as the SRS mean.

Stokes (1980) proposed a RSS estimator of population variance analogous to the SRS unbiased estimator and showed it to be asymptotically unbiased and demonstrated its dominance over the SRS unbiased estimator. (Sinha et al, 1996) proposed some best linear unbiased estimators (BLUEs) of the parameters of the normal and exponential distributions under RSS and some modifications of it. (Stokes, 1995) studied the maximum likelihood estimators (mle's) under RSS of the parameters of the location-scale family having cumulative distribution function (cdf) of the form $F\left(\dfrac{x-\mu}{\sigma}\right)$ with F known. Assuming the usual regularity conditions, Stokes (1995) considered several examples and demonstrated the dominance of the mle's under RSS over other estimators. BLUE's of the location and scale parameters were proposed in the same study and shown to do nearly as well as their maximum likelihood counterparts in most cases.

Muttlak (1997) proposed median ranked set sampling (MRSS), a modification of ranked set sampling, which upon picking and ordering the n random samples selects the median element of each ordered set if n is odd. However, if n is even, it selects the (n / 2)th smallest observation from each of the first n / 2 ordered sets and the ((n + 2) / 2)th smallest observation from the second n / 2 sets. This selection procedure yield a median ranked set sample of size n, and may be repeated m times to give a MRSS of size mn. Muttlak (1997) showed the MRSS mean to be an unbiased estimator when the underlying distribution is symmetric and biased otherwise. In both cases, his estimator has been shown under various distributions, to dominate the RSS sample mean.

The method of ERSS as studied my Samawi *et al.* (1996) draws n times, a random sample of size n from a population under consideration. For even set size n, the largest and smallest units are alternately taken from the first to the nth random sample. The resulting sample of n /2 each of first and nth order statistics forms the extreme ranked set sample. On the other hand, if n is odd, the largest and smallest units are alternately selected from the first random sample to the (n – 1)th random sample. From the nth random sample, either the mean of the largest and smallest unit is chosen or the median of the whole set. In this study, we will consider taking the median from the nth sample.

In this paper, we show that the mle's of the normal mean (i.e. the location parameter of the normal distribution) and the scale parameters of the exponential and gamma distributions under MRSS dominate all other estimators. We further show that the mle of the normal standard deviation (i.e. the scale parameter of the normal distribution) under ERSS is the most efficient. We exhibit a similar trend in the linear unbiased estimators. In Sections 2, we present the maximum likelihood estimators and discuss the results in Section 3. A brief discussion of the two-parameter family is presented in Section 4. Section 5 considers the linear unbiased estimators whose results are discusses in Section 6. We conclude the work in Section 7.

Throughout this work, we shall denote by $X_{i(r)j}$, the r^{th} smallest unit from the i^{th} sample of the j^{th} cycle.

2. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we consider the maximum likelihood estimation of the location and scale parameters under MRSS and ERSS paying attention to the odd and even set sizes.

2.1. *Maximum likelihood estimation under MRSS*

Case 1. Odd set sizes

Suppose $\{X_{i(p)j},\ i=1,\ 2,\ ...,n;\ j=1,\ 2,\ ...,m\}$ is a MRSS from a population with cdf and pdf of the form $F\left(\frac{x-\mu}{\sigma}\right)$ and $\frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right)$ respectively, where p = (n + 1) / 2. Further, let $Z_{i(p)j}=\frac{X_{i(p)j}-\mu}{\sigma}$. Then the $Z_{i(p)j}$'s are independent and identically distributed with pdf see David (1981)

$$f_p(z)=\frac{1}{B(p,\ p)}F^{p-1}(z)[1-F(z)]^{p-1}\frac{1}{\sigma}f(z) \tag{1}$$

Thus, the log likelihood function of the MRSS of odd set size is

$$\begin{aligned} L_{MRSS1} &= K_1 - mn\ln\sigma + \left(\frac{n-1}{2}\right)\sum_{j=1}^{m}\sum_{i=1}^{n}\{\ln F(Z_{i(p)j}) + \ln[1-F(Z_{i(p)j})]\} \\ &+ \sum_{j=1}^{m}\sum_{i=1}^{n}\ln f(Z_{i(p)j}) \end{aligned} \tag{2}$$

where K_1 is a constant.

If σ is known, then the mle of μ is the solution of the equation

$$\frac{n-1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\{\frac{f(Z_{i(p)j})}{1-F(Z_{i(p)j})}-\frac{f(Z_{i(p)j})}{F(Z_{i(p)j})}\right\}-\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{f'(Z_{i(p)j})}{f(Z_{i(p)j})}=0\,. \tag{3}$$

Also, the Fisher information of μ from the sample is

$$\begin{aligned} I_{mn1}(\mu) &= \frac{mn}{\sigma^2}E\left\{\left[(\frac{f'(z)}{f(z)})^2-\frac{f''(z)}{f(z)}\right]g(z\right\} \\ &+ \frac{mn(n-1)}{2\sigma^2}E\left\{\left[\frac{[2F(z)-1]f'(z)}{F(z)[1-F(z)]}+\frac{[F^2(z)+(1-F(z))^2]f^2(z)}{[1-F(z)]^2F^2(z)}\right]g(z)\right\} \end{aligned} \tag{4}$$

where

$$g(z)=\frac{1}{B(p,p)}F^{p-1}(z)[1-F(z)]^{p-1}. \tag{5}$$

To compare the resulting mle with that from a SRS of the same size, we define following (Stokes, 1995), the asymptotic relative precision

$$\lim_{m\to\infty} RP(\hat{\mu}_{mle1}, \hat{\mu}_{ML}) = \frac{I_{mn1}(\mu)}{I_{mn}(\mu)}, \tag{6}$$

where $I_{mn}(\mu)$ is the Fisher information of μ from the SRS.

Similarly, if μ is known, then the mle of σ is the solution of the equation

$$\begin{aligned} &-mn + \frac{n-1}{2}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\{-\frac{Z_{i(p)j} f(Z_{i(p)j})}{F(Z_{i(p)j})} + \frac{Z_{i(p)j} f(Z_{i(p)j})}{1-F(Z_{i(p)j})}\right\} \\ &-\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{Z_{i(p)j} f'(Z_{i(p)j})}{f(Z_{i(p)j})} = 0 \end{aligned} \tag{7}$$

and the Fisher information of σ from the MRSS is

$$\begin{aligned} I_{mn1}(\sigma) = &\frac{mn}{\sigma^2}\mathrm{E}\left\{\left[\left(\frac{Z_r f'(Z_r)}{f(Z_r)}\right)^2 - \frac{Z_r^2 f''(Z_r) + 2Z_r f'(Z_r)}{f(Z_r)}\right] g(z) - 1\right\} \\ &+ \frac{mn(n-1)}{2\sigma^2}\mathrm{E}\left\{\left[\frac{[2F(Z_r)-1][Z_r^2 f'(Z_r) + 2Z_r f(Z_r)]}{[1-F(Z_r)]F(Z_r)} + \right.\right. \\ &\left.\left. + \frac{[F^2(Z_r) + (1-F(Z_r))^2] Z_{i(p)j}^2 f^2(Z_r)}{F^2(Z_r)[1-F(Z_r)]^2}\right] g(z)\right\}. \end{aligned} \tag{8}$$

The comparison between the mle of σ under MRSS and the SRS maximum likelihood estimator is done using the asymptotic relative precision

$$\lim_{m\to\infty} RP(\hat{\sigma}_{mle1}, \hat{\sigma}_{ML}) = \frac{I_{mn1}(\sigma)}{I_{mn}(\sigma)}. \tag{9}$$

Case 2. Even set sizes

Let $\{\{X_{i(q)j}\}_{i=1}^{q} \bigcup \{X_{i(q+1)j}\}_{i=q+1}^{n};\ j = 1,\ 2,\ ...,\ m$, where $q = n/2\}$ be a MRSS from the type of distribution considered in case 1, where *n* and *m* are the set size and the cycle size respectively. Let $Z_{i(q)j}$ and $Z_{i(q+1)j}$ be the corresponding standardized order statistics. Then their respective pdf's are

$$f_q(z) = \frac{1}{\sigma\ B(q, q+1)} F^{(n-2)/2}(z)[1-F(z)]^{n/2} f(z) \tag{10}$$

and

$$f_{q+1}(z) = \frac{1}{\sigma \; B(q+1,q)} F^{n/2}(z)[1-F(z)]^{(n-2)/2} f(z). \tag{11}$$

The log likelihood function is

$$L_{MRSS2} = K_2 - mn\ln\sigma + \sum_{j=1}^{m}\left\{\sum_{i=1}^{q}\left[\frac{n-2}{2}\ln F(Z_{i(q)j}) + \frac{n}{2}\ln[1-F(Z_{i(q)j})] + \ln f(Z_{i(q)j})\right]\right.$$
$$\left. + \sum_{i=q+1}^{n}\left[\frac{n}{2}\ln F(Z_{i(q+1)j}) + \frac{n-2}{2}\ln[1-F(Z_{i(q+1)j})] + \ln f(Z_{i(q+1)j})\right]\right\} \tag{12}$$

where K_2 is a constant. The MRSS mle of μ when σ is known is the solution of the equation

$$\sum_{j=1}^{m}\left\{\sum_{i=1}^{q}\left[-\frac{n-2}{2}\frac{f(Z_{i(q)j})}{F(Z_{i(q)j})} + \frac{n}{2}\frac{f(Z_{i(q)j})}{[1-F(Z_{i(q)j})]} - \frac{f'(Z_{i(q)j})}{f(Z_{i(q)j})}\right]\right.$$
$$\left. + \sum_{i=q+1}^{n}\left[-\frac{n}{2}\frac{f(Z_{i(q+1)j})}{F(Z_{i(q+1)j})} + \frac{n-2}{2}\frac{f(Z_{i(q+1)j})}{[1-F(Z_{i(q+1)j})]} - \frac{f'(Z_{i(q+1)j})}{f(Z_{i(q+1)j})}\right]\right\} = 0 \tag{13}$$

The corresponding Fisher information is

$$I_{mn2}(\mu) = \frac{mn}{2\sigma^2}\mathrm{E}\left\{(g_1(z)+g_2(z))\left[\left(\frac{f'(z)}{f(z)}\right)^2 - \frac{f''(z)}{f(z)}\right]\right\}$$
$$+\frac{mn}{4\sigma^2}\mathrm{E}\left\{n\left[\left(\frac{f'(z)}{1-F(z)} + \left(\frac{f(z)}{1-F(z)}\right)^2\right)g_1(z) - \left(\frac{f'(z)}{F(z)} - \left(\frac{f(z)}{F(Z_r)}\right)^2\right)g_2(z)\right]\right.$$
$$\left.+(n-2)\left[\left(\frac{f'(z)}{1-F(z)} + \left(\frac{f(z)}{1-F(z)}\right)^2\right)g_2(z) - \left(\frac{f'(z)}{F(z)} - \left(\frac{f(z)}{F(z)}\right)^2\right)g_1(z)\right]\right\}, \tag{14}$$

where $g_1(z) = \frac{1}{B(q,q+1)} F^{n/2}(z)[1-F(z)]^{(n-2)/2}$ and $g_2(z) = \frac{1}{B(q,q+1)} F^{(n-2)/2}(z)[1-F(z)]^{n/2}$. Thus, the corresponding asymptotic relative precision with respect to the SRS mle is

$$\lim_{m\to\infty} RP(\hat{\mu}_{mle2}, \hat{\mu}_{ML}) = \frac{I_{mn2}(\mu)}{I_{mn}(\mu)}. \tag{15}$$

On the other hand, if μ is known and σ is of interest, we can obtain the MRSS maximum likelihood estimator of σ as the solution of the equation

$$
\begin{aligned}
mn + \sum_{j=1}^{m}\Bigg\{\sum_{i=1}^{q}\Bigg[(n-2)\frac{Z_{i(q)j}f(Z_{i(q+1)j}}{F(Z_{i(q)j})} - (n)\frac{Z_{i(q)j}f(Z_{i(q+1)j}}{1-F(Z_{i(q)j})} + \frac{Z_{i(q)j}f'(Z_{i(q+1)j}}{f(Z_{i(q)j})}\Bigg] \\
+ \sum_{i=q+1}^{n}\Bigg[(n)\frac{Z_{i(q)j}f(Z_{i(q+1)j}}{F(Z_{i(q)j})} - (n-2)\frac{Z_{i(q)j}f(Z_{i(q+1)j}}{1-F(Z_{i(q)j})} + \frac{Z_{i(q)j}f'(Z_{i(q+1)j}}{f(Z_{i(q)j})}\Bigg]\Bigg\} = 0.
\end{aligned} \tag{16}
$$

The corresponding Fisher information and the asymptotic relative precision with respect to the corresponding SRS mle of σ are respectively given by

$$
\begin{aligned}
I_{mn2}(\sigma) = \frac{mn}{2\sigma^2}\mathrm{E}\Bigg\{\Bigg[\left(\frac{zf'(z)}{f(z)}\right)^2 - \frac{z^2 f''(z) + 2zf'(z)}{f(z)}\Bigg][g_1(z) + g_2(z)] - 2\Bigg\} \\
+\frac{\mathrm{mn}}{4\sigma^2}E\Bigg\{(n-2)\Bigg[\left(\frac{Z_r^2 f'(z) + 2zf(z)}{1-F(z)} + \left(\frac{zf(z)}{1-F(z)}\right)^2\right\} g_2(z) \\
-\left(\frac{z^2 f'(z) + 2zf(z)}{F(z)} - \left(\frac{zf(z)}{F(z)}\right)^2\right) g_1(z)\Bigg] \\
+ n\Bigg[\left(\frac{z^2 f'(z) + 2zf(z)}{1-F(z)} + \left(\frac{Z_r f(z)}{1-F(z)}\right)^2 g_1(z)\right. \\
-\left(\frac{z^2 f'(z) + 2zf(z)}{F(z)} - \left(\frac{Z_r f(z)}{F(z)}\right)^2 g_2(z)\Bigg]\Bigg\}
\end{aligned} \tag{17}
$$

and

$$
\lim_{m\to\infty} RP(\hat{\sigma}_{mle1}, \hat{\sigma}_{ML}) = \frac{I_{mn2}(\sigma)}{I_{mn}(\sigma)}. \tag{18}
$$

2.2. *Maximum likelihood estimation under ERSS*

We now consider the maximum likelihood estimation of the location and scale parameters using ERSS in the spirit of the previous section.

Case 1. Even set sizes

Suppose that we have an even set size. Then the m-cycle ERSS in set notation is

$$
\{X_{i(1)j}; i = 1, 2, ..., n; j = 1, 2, ..., m\} \bigcup \{X_{i(n)j}; i = 1, 2, ..., n; j = 1, 2, ..., m\}.
$$

It us no difficult to obtain the log likelihood function as

$$L_{E_1} = K - mn\ln\sigma + \sum_{j=1}^{m}\left\{\sum_{i=1}^{n/2}[(n-1)\ln[1-F(Z_{i(1)j})] + \ln f(Z_{i(1)j})]\right.$$
$$\left. + \sum_{i=(n/2)+1}^{n}[(n-1)\ln F(Z_{i(n)j}) + \ln f(Z_{i(n)j})]\right\}. \tag{19}$$

Suppose that σ is known. Then the mle of μ, $\hat{\mu}_{ML_{E_1}}$ is the solution of the equation

$$\sum_{j=1}^{m}\left\{\sum_{i=1}^{n/2}\left[(n-1)\frac{f(Z_{i(1)j})}{1-F(Z_{i(1)j})} - \frac{f'(Z_{i(1)j})}{f(Z_{i(1)j})}\right] - \right.$$
$$\left.\sum_{i=(n/2)+1}^{n}\left[(n-1)\frac{f(Z_{i(n)j})}{F(Z_{i(n)j})} + \frac{f'(Z_{i(n)j})}{f(Z_{i(n)j})}\right]\right\} = 0\,. \tag{20}$$

We obtain the Fisher information for μ for the ERSS as

$$I_{mn}^{E_1}(\mu) = \frac{mn}{2\sigma^2}E\left\{\left[\left(\frac{f'(z)}{f(z)}\right)^2 - \frac{f''(z)}{f(z)}\right][h_1(z)+h_2(z)]\right\}$$
$$+\frac{mn(n-1)}{2\sigma^2}E\left\{\left[\frac{f'(z)}{1-F(z)} + \left(\frac{f(z)}{1-F(z)}\right)^2\right]h_1(z) - \left[\frac{f'(z)}{F(z)} - \left(\frac{f(z)}{F(z)}\right)^2\right]h_2(z)\right\} \tag{21}$$

where $h_1(z) = n[1-F(z)]^{n-1}$ and $h_2(z) = nF^{(n-1)}(z)$. Thus we can find the asymptotic relative precision of the ERSS estimator relative to that of the SRS, $\hat{\mu}_{ML}$, by

$$\lim_{m\to\infty} RP(\hat{\mu}_{ML_{E_1}}, \hat{\mu}_{ML}) = \frac{I_{mn}^{E_1}(\mu)}{I_{mn}(\mu)}\,. \tag{22}$$

Similarly, if μ is known, then the ERSS mle of σ, $\hat{\sigma}_{ML_{E_2}}$ is the solution of the equation

$$mn - \sum_{j=1}^{m}\left\{\sum_{i=1}^{n/2}\left[(n-1)\frac{Z_{i(1)j}f(Z_{i(1)j})}{1-F(Z_{i(1)j})} - \frac{Z_{i(1)j}f'(Z_{i(1)j})}{f(Z_{i(1)j})}\right] - \right.$$
$$\left.\sum_{i=(n/2)+1}^{n}\left[(n-1)\frac{Z_{i(1)j}f(Z_{i(n)j})}{F(Z_{i(n)j})} + \frac{Z_{i(1)j}f'(Z_{i(n)j})}{f(Z_{i(n)j})}\right]\right\} = 0 \tag{23}$$

and the Fisher information about σ from the ERSS is

$$I_{mn}^{E_1}(\sigma) = \frac{mn}{2\sigma^2} E\left\{\left[\left(\frac{zf'(z)}{f(z)}\right)^2 - \frac{z^2 f''(z) + 2zf'(z)}{f(z)}\right][h_1(z) + h_2(z)] - 2\right\}$$
$$+ \frac{mn}{2\sigma^2} E\left\{(n-1)\left[\frac{z^2 f'(z) + 2zf(z)}{1-F(z)} + \left(\frac{zf(z)}{1-F(z)}\right)^2\right] \right. \tag{24}$$
$$\left. h_1(z) \; -(n-1)\left[\frac{z^2 f'(z) + 2zf(z)}{F(z)} + \left(\frac{zf(z)}{F(z)}\right)^2\right] h_2(z)\right\},$$

where h_1 and h_2 are as previously defined. Hence, we find the asymptotic relative precision of the ERSS estimator to that of the SRS by

$$\lim_{m\to\infty} RP(\hat{\sigma}_{ML_{E_1}}, \hat{\sigma}_{ML}) = \frac{I_{mn}^{E_1}(\sigma)}{I_{mn}(\sigma)}. \tag{25}$$

Case 2. Odd set sizes

Suppose now that we have an odd set sized ERSS with m cycles from the location-scale family of distributions. Then the ERSS is the set

$$\left\{X_{i(1)j};\ \mathrm{i} = 1, 2, ..., \frac{\mathrm{n\text{-}1}}{2}; \mathrm{j} = 1, 2, ..., \mathrm{m}\right\} \bigcup \left\{X_{i(n)j};\ \mathrm{i} = \frac{\mathrm{n\text{-}1}}{2} + 1, 2, ..., \mathrm{n\text{-}1}; \mathrm{j} = 1, 2, ..., \mathrm{m}\right\}$$
$$\bigcup \left\{X_{n\left(\frac{n+1}{2}\right)j};\mathrm{j} = 1, 2, ..., \mathrm{m}\right\}.$$

The log likelihood function of the ERSS is

$$L_{E_2} = K' - mn\ln\sigma + \sum_{j=1}^{m}\left\{\sum_{i=1}^{n'}[(n-1)\ln[1-F(Z_{i(1)j})] + \ln f(Z_{i(1)j})]\right.$$
$$\left. + \sum_{i=n'+1}^{n-1}[(n-1)\ln F(Z_{i(n)j}) + \ln f(Z_{i(n)j})]\right\} \tag{26}$$
$$+ \sum_{j=1}^{m}\left\{\left(\frac{n-1}{2}\right)[[\ln[1-F(Z_{n(p)j})] + \ln F(Z_{n(p)j})] + \ln f(Z_{n(p)j})\right\},$$

where $n' = (n-1)/2$ and $p = (n+1)/2$.

Therefore, the mle of μ when σ is known, is the solution of the equation

$$\sum_{j=1}^{m}\left\{\sum_{i=1}^{n'}\left[(n-1)\frac{f(Z_{i(1)j})}{1-F(Z_{i(1)j})}-\frac{f'(Z_{i(1)j})}{f(Z_{i(1)j})}\right]-\sum_{i=n'+1}^{n-1}\left[(n-1)\frac{f(Z_{i(n)j})}{F(Z_{i(n)j})}-\frac{f'(Z_{i(n)j})}{f(Z_{i(n)j})}\right]\right\}$$

$$\sum_{j=1}^{m}\left\{\left(\frac{n-1}{2}\right)\left[\frac{f(Z_{ni(1)j})}{1-F(Z_{n(1)j})}-\frac{f(Z_{n(1)j})}{F(Z_{n(1)j})}\right]-\frac{f'(Z_{n(1)j})}{f(Z_{n(1)j})}\right\}=0. \tag{27}$$

The Fisher information for μ from the ERSS when σ is known is

$$I_{mn}^{E_2}(\mu)=\frac{mn}{\sigma^2}\mathrm{E}\left\{\frac{1}{n}\left[\frac{n-1}{2}[h_1(z)+h_2(z)]+g(z)\right]\left[\left(\frac{f'(z)}{f(z)}\right)^2-\frac{f''(z)}{f(z)}\right]\right\}$$

$$+\frac{mn(n-1)}{2\sigma^2}\mathrm{E}\left\{\frac{1}{n}((n-1)h_1(z)+g(z))\left[\frac{f'(z)}{1-F(z)}+\left(\frac{f(z)}{1-F(z)}\right)^2\right]\right. \tag{28}$$

$$\left.-\frac{1}{n}((n-1)h_2(z)+g(z))\left[\frac{f'(z)}{F(z)}+\left(\frac{f(z)}{F(z)}\right)^2\right]\right\},$$

where $h_1(z)$ and $h_2(z)$ are as previously defined and

$$g(z)=\frac{1}{B(p,n-p+1)}F^{p-1}(z)[1-F(z)]^{n-p}. \tag{29}$$

Therefore, the asymptotic relative precision of the ERSS estimator with respect to that of the SRS is

$$\lim_{m\to\infty}RP(\hat{\mu}_{ML_{E_2}},\hat{\mu}_{ML})=\frac{I_{mn}^{E_2}(\mu)}{I_{mn}(\mu)} \tag{30}$$

Similarly, the ERSS mle of σ, $\hat{\sigma}_{ML_{E_2}}$, is the solution of the equation

$$-mn+\sum_{j=1}^{m}\left\{\sum_{i=1}^{n'}\left[(n-1)\frac{Z_{i(1)j}f(Z_{i(1)j})}{1-F(Z_{i(1)j})}-\frac{Z_{i(1)j}f'(Z_{i(1)j})}{f(Z_{i(1)j})}\right]\right.$$

$$\left.-\sum_{i=n'+1}^{n-1}\left[(n-1)\frac{Z_{i(n)j}f(Z_{i(n)j})}{F(Z_{i(n)j})}-\frac{Z_{i(1)j}f'(Z_{i(n)j})}{f(Z_{i(n)j})}\right]\right\} \tag{31}$$

$$+\sum_{j=1}^{m}\left\{\left(\frac{n-1}{2}\right)\left[\frac{Z_{n(p)j}f(Z_{n(p)j})}{1-F(Z_{n(p)j})}-\frac{Z_{n(p)j}f(Z_{n(p)j})}{F(Z_{n(p)j})}\right]-\frac{Z_{n(p)j}f'(Z_{n(p)j})}{f(Z_{n(p)j})}\right\}=0$$

and the ERSS Fisher information for σ is

$$I_{mn}^{E_2}(\sigma)=\frac{mn}{\sigma^2}E\left\{\frac{1}{n}\left[\left(\frac{zf'(z)}{f(z)}\right)^2-\frac{z^2 f''(z)+2zf'(z)}{f(z)}\right]\left(\left(\frac{n-1}{2}\right)[h_1(z)+h_2(z)]+g(z)\right)-1\right\}$$
$$+\frac{mn(n-1)}{2\sigma^2}E\left\{\frac{1}{n}\left[\frac{z^2 f'(z)+2zf(z)}{1-F(z)}+\left(\frac{zf(z)}{1-F(z)}\right)^2\right][[(n-1)h_1(z)+g(z)]\right.$$
$$\left.-\frac{1}{n}\left[\frac{z^2 f'(z)+2zf(z)}{F(z)}+\left(\frac{zf(z)}{F(z)}\right)^2\right][[(n-1)h_2(z)+g(z)]\right\}. \tag{32}$$

Hence the asymptotic relative precision is

$$\lim_{m\to\infty} RP(\hat{\sigma}_{ML_{E_2}},\hat{\sigma}_{ML})=\frac{I_{mn}^{E_2}(\sigma)}{I_{mn}(\sigma)}. \tag{33}$$

3. RESULTS I

In this section, we present the asymptotic relative precision of the mle's for MRSS and ERSS in comparison with the asymptotic relative precision derived by Stokes (1995) and other relative precision values under the normal, exponential and gamma distributions. We will present one illustration of how the values are computed for the various distributions.

Let $X_1, X_2, \ldots, X_{mn}$ be a random sample of size mn from a normal population with unknown mean, μ and unit variance. We know that the mle of μ from this random sample is the sample mean (i.e. $\hat{\mu}_{ML}=\bar{X}$) with Fisher information, $I_{mn}(\mu)=mn$. For the MRSS with odd set size n, we can find the mle of μ using equation (3) as the solution of the equation

$$\sum_{j=1}^{m}\sum_{i=1}^{n} Z_{i(p)j}+\left(\frac{n-1}{2}\right)\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{[2\Phi(Z_{i(p)j})-1]\phi(Z_{i(p)j})}{[1-\Phi(Z_{i(p)j})]\Phi(Z_{i(p)j})}=0,$$

where $\Phi(.)$ and $\phi(.)$ respectively denote the cumulative distribution function and the probability density function of the standard normal variable. Using equations (4) and (5), the Fisher information is

$$I_{mn1}(\mu)=\frac{mn}{\sigma^2}E[g(z)]$$
$$+\frac{mn(n-1)}{2}E\left\{\left(\frac{[\Phi^2(z)+[1-\Phi(z)^2]\phi^2(z)}{[1-\Phi(z)]^2\Phi^2(z)}-\frac{z[2\Phi(z)-1]\phi(z)}{\Phi(z)[1-\Phi(z)]}\right)g(z)\right\},$$

where $g(z) = \dfrac{1}{B(p, n-p+1)} \Phi^{p-1}(z)[1-\Phi(z)]^{n-p}$ and p = (n + 1) / 2.

From equation (9), the asymptotic relative precision of the MRSS with respect to the SRS is

$$\lim_{m\to\infty} RP(\hat{\mu}_{ML_1}, \hat{\mu}_{ML})$$
$$= E[g(z)] + \frac{(n-1)}{2} E\left\{\left(\frac{[\Phi^2(z) + [1-\Phi(z)^2]\phi^2(z)}{[1-\Phi(z)]^2\Phi^2(z)} - \frac{z[2\Phi(z)-1]\phi(z)}{\Phi(z)[1-\Phi(z)]}\right) g(z)\right\}.$$

We compute the above expectations and all other expectations that follow by numerical integration using Mathematica 2.2.

For the MRSS with even set size, *n*, the mle, using equation (13), is the solution of the equation

$$\sum_{j=1}^{m}\left[\sum_{i=1}^{q} Z_{i(q)j} + \sum_{i=q+1}^{n} Z_{i(q)j}\right] + \frac{1}{2}\sum_{j=1}^{m}\left\{\sum_{i=1}^{q}\frac{[(2n-2)\Phi(Z_{i(q)j}) - n + 2]\phi(Z_{i(q)j})}{[1-\Phi(Z_{i(q)j})]\Phi(Z_{i(q)j})}\right.$$
$$\left. + \sum_{i=q+1}^{n}\frac{[(2n-2)\Phi(Z_{i(q+1)j}) - n]\phi(Z_{i(q+1)j})}{[1-\Phi(Z_{i(q+1)j})]\Phi(Z_{i(q+1)j})}\right\} = 0,$$

and from equation (14), the Fisher information is given by

$$I_{mn2}(\mu) = mnE\{g_1(Z_r) + g_2(Z_r)\}$$
$$+ \frac{mn}{4} E\left\{ n\left[\left(\frac{-Z_r\phi(Z_r)}{1-\Phi(Z_r)} + \left(\frac{\phi(Z_r)}{1-\Phi(Z_r)}\right)^2\right) g_1(Z_r) - \left(\left(\frac{-Z_r\phi(Z_r)}{\Phi(Z_r)} + \left(\frac{\phi(Z_r)}{\Phi(Z_r)}\right)^2\right) g_2(Z_r)\right)\right]\right.$$
$$\left. +(n-2)\left[\left(\frac{-Z_r\phi(Z_r)}{1-\Phi(Z_r)} + \left(\frac{\phi(Z_r)}{1-\Phi(Z_r)}\right)^2\right) g_2(Z_r) - \left(\left(\frac{-Z_r\phi(Z_r)}{\Phi(Z_r)} + \left(\frac{\phi(Z_r)}{\Phi(Z_r)}\right)^2\right) g_1(Z_r)\right)\right]\right\},$$

where $g_1(z) = \dfrac{1}{B(q, q+1)} \Phi^{q-1}(z)[1-\Phi(z)]^{q}$ and $g_2(z) = \dfrac{1}{B(q+1, q)} \Phi^{q}(z)[1-\Phi(z)]^{q-1}$.

Thus, the asymptotic relative precision

$$\lim_{m\to\infty} RP(\hat{\mu}_{mle2}, \hat{\mu}_{ML}) = \frac{I_{mn2}(\mu)}{I_{mn}(\mu)}$$
$$= \mathrm{E}\{g_1(z) + g_2(z)\} + \frac{1}{4}\mathrm{E}\left\{ n\left[\left(\frac{-z\phi(z)}{1-\Phi(z)} + \left(\frac{\phi(z)}{1-\Phi(z)}\right)^2\right) g_1(Z_r) - \left(\left(\frac{-z\phi(z)}{\Phi(z)} + \left(\frac{\phi(z)}{\Phi(z)}\right)^2\right) g_2(z)\right)\right]\right.$$
$$\left. +(n-2)\left[\left(\frac{-z\phi(z)}{1-\Phi(z)} + \left(\frac{\phi(z)}{1-\Phi(z)}\right)^2\right) g_2(z) - \left(\left(\frac{-z\phi(z)}{\Phi(z)} + \left(\frac{\phi(z)}{\Phi(z)}\right)^2\right) g_1(z)\right)\right]\right\}.$$

Using the derived results of Section 2.2, this problem can similarly be considered under ERSS. In a similar fashion, we use the derived results in obtaining the appropriate asymptotic relative precision in the estimation of σ from the normal, exponential and gamma distributions using MRSS and ERSS.

Table 1 compares the asymptotic relative precision values for mle's of μ from N (μ, 1) under each of RSS Stokes (1995), MRSS and ERSS. It also displays the relative precision values for the corresponding non-parametric estimators of μ under each of RSS Dell and Clutter (1972), MRSS Muttlak (1997) and ERSS computed following the methods of Samawi *et al.* (1996). We observe that the MRSS mle of μ dominates all the other estimators beside the non-parametric estimator proposed by Muttlak (1997).

TABLE 1

The asymptotic relative precision and relative precision values for estimators of μ from N (μ, 1)

Set size	Asymptotic relative precision (Maximum likelihood)			Relative precision (Non-parametric methods)		
	RSS	MRSS	ERSS	RSS	MRSS	ERSS
2	1.48	1.48	1.48	1.47	1.47	1.47
3	1.96	2.23	1.96	1.91	2.23	1.91
4	2.44	2.78	2.10	2.35	2.77	2.03
5	2.92	3.49	2.56	2.77	3.49	2.41
6	3.40	4.07	2.53	3.19	4.06	2.40
7	3.88	4.75	3.00	3.59	4.75	2.73
8	4.36	5.34	2.87	4.00	5.34	2.68

In Table 2, we show the relative precision values for the mle's under RSS Stokes (1995), MRSS and ERSS. We also show that for the RSS non-parametric estimator Stokes (1980). Clearly, the ERSS mle dominates the other methods in estimating σ from the normal distribution.

TABLE 2

The asymptotic relative precision and relative precision values for estimators of σ from N (0, σ^2)

Set size	Asymptotic relative precision (Maximum likelihood)			Relative precision (Non-parametric methods)
	RSS	MRSS	ERSS	RSS
2	1.14	1.14	1.14	1.00
3	1.27	0.98	1.27	1.08
4	1.41	1.08	1.74	1.18
5	1.54	0.98	1.85	1.27
6	1.68	1.05	2.40	1.38
7	1.81	0.99	2.49	1.48
8	1.95	1.04	3.06	1.57

Table 3 displays the asymptotic relative precision for mle's of σ from the exponential distribution ($F(x) = 1 - \exp[-x/\sigma]$) under RSS Stokes (1995), MRSS and ERSS. It also shows the relative precision values for the non-parametric estimators under RSS Dell and Clutter (1972), MRSS Muttlak (1997) and ERSS computed following Samawi *et al.* (1996). Again, we see that the MRSS mle's are the dominant estimators.

TABLE 3

The asymptotic relative precision and relative precision values for estimator of the scale parameter of the exponential distribution

Set size	Asymptotic relative precision (Maximum likelihood)			Relative precision (Non-parametric methods)		
	RSS	MRSS	ERSS	RSS	MRSS	ERSS
2	1.40	1.40	1.40	1.33	1.33	1.33
3	1.81	1.92	1.81	1.64	2.25	1.64
4	2.21	2.37	2.06	1.92	2.44	1.17
5	2.62	2.88	2.44	2.19	2.23	1.32
6	3.02	3.33	2.58	2.45	2.14	0.75
7	3.42	3.83	2.96	2.70	1.80	0.81
8	3.83	4.29	3.03	2.94	1.67	0.46

In Tables 4 and 5, we show the results for the estimators of σ from Gamma (2) and Gamma (3) distributions (i.e. from $F(x) = \frac{1}{\Gamma(\alpha)} \int_0^{x/\sigma} t^{\alpha-1} e^{-t}$, $\alpha > 0$ with α = 2 and 3 respectively).

The values for the RSS maximum likelihood estimators were computed following Stokes (1995). The values for the non-parametric RSS and MRSS estimators were obtained from Dell and Clutter (1972) and Muttlak (1997) respectively. We computed the values for the ERSS non-parametric estimators following Samawi *et al.* (1996). We observe here that the MRSS maximum likelihood estimators do better than all the other estimators except for set size of 3 when the MRSS non-parametric estimators do better.

TABLE 4

The asymptotic relative precision and relative precision values for estimators of the scale parameter Gamma (2.0)

Set size	Asymptotic relative precision (Maximum likelihood)			Relative precision (Non-parametric methods)		
	RSS	MRSS	ERSS	RSS	MRSS	ERSS
2	1.44	1.44	1.44	1.39	1.39	1.39
3	1.88	2.07	1.88	1.75	2.23	1.75
4	2.32	2.56	2.08	2.10	2.56	1.45
5	2.76	3.16	2.50	2.42	2.64	1.66
6	3.20	3.67	2.56	2.74	2.70	1.09
7	3.64	4.26	2.98	3.05	2.48	1.18
8	4.07	4.78	2.95	3.35	2.40	0.72

TABLE 5

The asymptotic relative precision and relative precision values for estimators of the scale parameter from Gamma (3.0)

Set size	Asymptotic relative precision (Maximum likelihood)			Relative precision (Non-parametric methods)		
	RSS	MRSS	ERSS	RSS	MRSS	ERSS
2	1.45	1.45	1.45	1.41	1.41	1.41
3	1.90	2.12	1.90	1.80	2.24	1.80
4	2.36	2.63	2.08	2.16	2.62	1.60
5	2.81	3.26	2.52	2.52	2.85	1.84
6	3.26	3.80	2.55	2.87	3.01	1.32
7	3.71	4.42	2.98	3.20	2.91	1.44
8	4.17	4.96	2.92	3.54	2.90	0.94

4. THE TWO-PARAMETER FAMILY

From all the work above, we assumed one of the parameters known and estimated the other. That is to say we were dealing with the one-parameter family of distributions. The obvious question now is what if both the location and scale parameters are unknown. This gives rise to the two-parameter problem. In this case, the usual principle of maximum likelihood requires us to obtain the first derivative of the appropriate log likelihood function with respect to each parameter, set each result equal to zero, and simultaneously solve the resulting equations for the parameters. For instance, if μ and σ are not known and we have a MRSS of odd set size, we obtain their maximum likelihood estimates by simultaneously solving equations (3) and (7) for μ and σ. Similarly, if we have an even set size, we set equation (2) equal to zero and solve simultaneously with equation (16) as in the case of the odd set size.

We will now investigate the performance of these estimators for the case of odd set sizes against the performance of the corresponding SRS estimators. We will do this using the Fisher information matrix, which has $I_{mn1}(\mu)$ and $I_{mn1}(\sigma)$ (equations (4) and (8) respectively) as its diagonal elements. The off diagonal elements are given by

$$-\mathrm{E}\left\{\frac{\partial^2 L_{MRSS1}}{\partial\mu\partial\sigma}\right\}=\frac{mn}{\sigma^2}\mathrm{E}\left\{\left[z\left(\frac{f'(z)}{f(z)}\right)^2-\frac{zf''(z)+f'(z)}{f(z)}\right]g(z)\right\}$$
$$+\frac{mn(n-1)}{2\sigma^2}\mathrm{E}\left\{\left[\frac{[2F(z)-1][zf'(z)+f(z)]}{F(z)[1-F(z)]}+\frac{z[F^2(z)+(1-F(z))^2]f^2(z)}{F^2(z)[1-F(z)]^2}\right]g(z)\right\}.$$

where $\frac{1}{B(p,p)}F^{p-1}(z)[1-F(z)]^{n-p}$ and $p=\frac{n+1}{2}$. Just as in the case of RSS Stokes (1995), it has been verified that the off-diagonal elements are zero for symmetric distributions. Note that

$$\frac{\partial^2 L_{MRSS1}}{\partial\mu\,\partial\sigma}=\frac{\partial^2 L_{MRSS1}}{\partial\sigma\,\partial\mu}.$$

In the case of SRS, the diagonal elements are given by $I_{mn}(\mu)$ and $I_{mn}(\sigma)$, the Fisher information for μ and σ respectively from a SRS. The off-diagonal elements are given by

$$-\mathrm{E}\left\{\frac{\partial^2 L}{\partial\mu\partial\sigma}\right\}=\mathrm{E}\left\{\frac{zf''(z)+f'(z)}{f(z)}-z\left(\frac{f'(z)}{f(z)}\right)^2\right\}.$$

This is also zero for symmetric distributions.

Thus, to compare the MRSS estimators with those of the SRS, we compare the determinants of the information matrices

$$\begin{bmatrix} I_{mn1}(\mu) & -E\left\{\dfrac{\partial^2 L_{MRSS1}}{\partial\mu\,\partial\sigma}\right\} \\ -E\left\{\dfrac{\partial^2 L_{MRSS1}}{\partial\mu\,\partial\sigma}\right\} & I_{mn1}(\sigma) \end{bmatrix} \text{ and } \begin{bmatrix} I_{mn}(\mu) & -E\left\{\dfrac{\partial^2 L}{\partial\mu\,\partial\sigma}\right\} \\ -E\left\{\dfrac{\partial^2 L}{\partial\mu\,\partial\sigma}\right\} & I_{mn}(\sigma) \end{bmatrix},$$

which are the information matrices for the MRSS and RSS estimators respectively. This clearly shows that the trend observed in the relative precision values in the case of the one-parameter family also holds here even though the actual estimates may differ. This observation agrees with Stokes (1995) in the case of RSS. The analysis for even set sizes follow similarly.

5. LINEAR UNBIASED ESTIMATORS

The results of the maximum likelihood estimators apart from being asymptotic may be difficult to find as they involve numerically solving complicated equations. In this section, we propose some unbiased estimators of the parameters considered in the previous sections in terms of the MRSS and ERSS, which are relatively very easy to compute.

Following Lloyd (1952) as in Stokes (1995), we define $\alpha_{(.:n)} = \mathrm{E}[Z_{i(.)}]$ and $v_{(.:n)} = \mathrm{Var}[Z_{i(.)}]$, where $Z_{i(.)} = \dfrac{X_{i(.)} - \mu}{\sigma}$ and $X_{i(.)}$ is the $(.)^{th}$ order statistic in the i^{th} set. Thus, it follows that $\mathrm{E}[X_{i(.)}] = \mu + \sigma\alpha_{(.:n)}$ and $\mathrm{Var}[X_{i(..:n)}] = \sigma^2 v_{(.:n)}$.

5.1. *Linear Unbiased Estimation under MRSS*

We maintain as in the previous sections that $p = (n + 1) / 2$ and $q = n / 2$.

Case 1. Odd set sizes

Suppose that we have an odd set size, *n*, and that σ is known. Then the proposed estimator of μ from a MRSS with *m* cycles is

$$\hat{\mu}_{\mathrm{UB}_1} = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n} X_{i(p)j} - \sigma\alpha_{(p:n)} . \tag{34}$$

It is easy to verify that this estimator is unbiased for μ and has variance

$$\text{Var}[\hat{\mu}_{UB_1}] = \frac{\sigma^2}{mn} v_{(p:n)} . \quad (35)$$

Suppose now that μ is known and we wish to estimate σ using MRSS. Then for odd set sizes, the proposed unbiased estimator of σ is

$$\hat{\sigma}_{UB_1} = \frac{1}{\alpha_{(p:n)}} \left\{ \frac{1}{mn} \sum_{j=1}^{m} \sum_{i=1}^{n} X_{i(p)j} - \mu \right\}, \quad (36)$$

This estimator has variance

$$\text{Var}[\hat{\sigma}_{UB_1}] = \frac{\sigma^2 v_{(p:n)}}{mn\alpha^2_{(p:n)}} . \quad (37)$$

It is clear that the results of equation (36) and (37) cannot be used if the underlying distribution is symmetric, as in this case, $\alpha_{(p:n)} = 0$ for all odd set sizes.

Case 2. Even set sizes

For even set sizes with known σ, the unbiased estimator of μ is give by

$$\hat{\mu}_{UB_2} = \frac{1}{mn} \sum_{j=1}^{m} \left[\sum_{i=1}^{q} X_{i(q)j} + \sum_{i=q+1}^{n} X_{i(q+1)j} \right] - \frac{\sigma}{2} (\alpha_{(q:n)} + \alpha_{(q+1:n)}) , \quad (38)$$

with variance

$$\text{Var}[\hat{\mu}_{UB_2}] = \frac{\sigma^2}{2mn} (v_{(q:n)} + v_{(q+1:n)}) . \quad (39)$$

On the other hand, the unbiased estimator of σ when μ is known is

$$\hat{\sigma}_{UB_2} = \frac{1}{mn} \sum_{j=1}^{m} \left[\frac{1}{\alpha_{(q:n)}} \sum_{i=1}^{q} X_{i(q)j} + \frac{1}{\alpha_{(q+1:n)}} \sum_{i=q+1}^{n} X_{i(q+1)j} \right] - \frac{\mu}{2} \left[\frac{1}{\alpha_{(q:n)}} + \frac{1}{\alpha_{(q+1:n)}} \right], \quad (40)$$

which has variance

$$\text{Var}[\hat{\sigma}_{UB_2}] = \frac{\sigma^2}{2mn} \left[\frac{v_{(q:n)}}{\alpha^2_{(q:n)}} + \frac{v_{(q+1:n)}}{\alpha^2_{(q+1:n)}} \right] . \quad (41)$$

5.2. *Linear unbiased estimation under ERSS*

Case 1. Odd set sizes

For an odd set size n, the ERSS of m cycles in set notation is

$$\{X_{i(1)j}; i=1,2,...,(n-1)/2; j=1,2,...m\} \bigcup \{X_{i(n)j}; i=((n-1)/2)+1,...,n-1; j=1,2,...m\} \bigcup \{X_{n(p)j}; j=1,2,...m\}$$

where $p = (n+1)/2$.

Suppose σ is known and n is odd, then the ERSS unbiased estimator $\hat{\mu}_1$ of μ is

$$\hat{\mu}_1 = \frac{1}{mn}\sum_{j=1}^{m}\left\{\sum_{i=1}^{p'} X_{i(1)j} + \sum_{i=p'+1}^{n-1} X_{i(n)j} + X_{n(p)j}\right\} - \frac{(n-1)(\alpha_{(1:n)} + \alpha_{(n:n)})\sigma + 2\alpha_{(p:n)}}{2n}, \quad (42)$$

where $p' = (n-1)/2$ and $p = (n+1)/2$. This estimator has variance

$$\text{Var}[\hat{\mu}_1] = \frac{[(n-1)(v_{(1:n)} + v_{(n:n)}) + 2v_{(p:n)}]\sigma^2}{2mn^2}. \quad (43)$$

If σ is unknown and the underlying distribution is symmetric, then

$$\hat{\mu}_1 = \frac{1}{mn}\sum_{j=1}^{m}\left\{\sum_{i=1}^{p'} X_{i(1)j} + \sum_{i=p'+1}^{n-1} X_{i(n)j} + X_{n(p)j}\right\}, \quad (44)$$

which coincides with the estimator proposed by Samawi *et al.* (1996) and

$$\text{Var}[\hat{\mu}_1] = \frac{[(n-1)v_{(1:n)} + v_{(p:n)}]\sigma^2}{mn^2}. \quad (45)$$

From equation (44), it is clear that knowledge of the nuisance parameter is not required in the estimation of μ when σ is known and the underlying distribution is symmetric.

Now if μ is known, then the unbiased estimator of σ is

$$\hat{\sigma}_1 = \frac{2n}{(n-1)(\alpha_{(1:n)} + \alpha_{(n:n)}) + 2\alpha_{(p:n)}}\left\{\frac{1}{mn}\sum_{j=1}^{m}\left[\sum_{i=1}^{p'} X_{i(1)j} + \sum_{i=p'+1}^{n-1} X_{i(n)j} + X_{n(p)j}\right] - \mu\right\}, \quad (46)$$

with variance,

$$\text{Var}[\hat{\sigma}_1] = \frac{2[(n-1)(v_{(1:n)} + v_{(n:n)}) + 2v_{(p:n)}]\sigma^2}{m[(n-1)(\alpha_{(1:n)} + \alpha_{(n:n)}) + 2\alpha_{(p:n)}]^2}. \quad (47)$$

Equations (46) and (47) are undefined where the underlying distribution is symmetric, as the denominators vanish in that case. The use of the partial ERSS or PERSS in Section 5.3 remedies this situation.

Case 2. Even set sizes

For an even set *n*, the *m*-cycle ERSS in set notation is

$$\{X_{i(1)j}; i=1,2,...,n/2; j=1,2,...,m\}\bigcup\{X_{i(n)j}; i=(n/2)+1,...,n; j=1,2,...,m\}$$

Suppose that we have an even set size, *n*, with σ known. Then the unbiased estimator of μ is

$$\hat{\mu}_2 = \frac{1}{mn}\sum_{j=1}^{m}\left\{\sum_{i=1}^{q} X_{i(1)j} + \sum_{i=q+1}^{n} X_{i(n)j}\right\} - \frac{\sigma}{2}(\alpha_{(1:n)} + \alpha_{(n:n)}), \tag{48}$$

where $q = n/2$. This estimator has variance

$$\text{Var}[\hat{\mu}_2] = \frac{\sigma^2}{2mn}(v_{(1:n)} + v_{(n:n)}) . \tag{49}$$

For symmetric distributions, equation (48) reduces to

$$\hat{\mu}_2 = \frac{1}{mn}\sum_{j=1}^{m}\left\{\sum_{i=1}^{q} X_{i(1)j} + \sum_{i=q+1}^{n} X_{i(n)j}\right\}, \tag{50}$$

which again coincides with the estimator in (Samawi et al, 1996) and

$$\text{Var}[\hat{\mu}_2] = \frac{\sigma^2 v_{(1:n)}}{mn} = \frac{\sigma^2 v_{(n:n)}}{mn} . \tag{51}$$

Conversely, if μ is known, then the unbiased estimator of σ and its variance are respectively given by

$$\hat{\sigma}_2 = \frac{1}{mn}\sum_{j=1}^{m}\left\{\frac{1}{\alpha_{(1:n)}}\sum_{i=1}^{q} X_{i(1)j} + \frac{1}{\alpha_{(n:n)}}\sum_{i=q+1}^{n} X_{i(n)j}\right\} - \frac{\mu}{2}\left[\frac{1}{\alpha_{(1:n)}} + \frac{1}{\alpha_{(n:n)}}\right] \tag{52}$$

and

$$\text{Var}[\hat{\sigma}_2] = \frac{\sigma^2}{2mn}\left[\frac{v_{(1:n)}}{\alpha^2_{(1:n)}} + \frac{v_{(n:n)}}{\alpha^2_{(n:n)}}\right]. \tag{53}$$

If the underlying distribution is symmetric, then equation (52) simplifies to

$$\hat{\sigma}_2 = \frac{1}{mn\alpha_{(1:n)}} \sum_{j=1}^{m} \left\{ \sum_{i=1}^{q} X_{i(1)j} - \sum_{i=q+1}^{n} X_{i(n)j} \right\}, \tag{54}$$

and equation (53) to

$$\text{Var}[\hat{\sigma}_2] = \frac{\sigma^2 \nu_{(1:n)}}{mn\alpha_{(1:n)}^2} = \frac{\sigma^2 \nu_{(n:n)}}{mn\alpha_{(n:n)}^2}. \tag{55}$$

5.3. *Estimation under partial extreme ranked set sampling (PERSS)*

We will now propose partial extreme ranked set sampling (PERSS), a modification of ERSS for odd set sizes. In this method, if n is odd, (n –1) random samples are taken from the population of interest. From the first (n -1)/2 sets, the smallest observation is selected for measurement and the largest observation from each of the remaining (n – 1)/2 sets. This yields a sample of size (n – 1) units. The cycle may be repeated m times to give a sample of size m (n –1). This modification is necessary for two reasons; first, the proposed ERSS linear unbiased estimator of σ when μ is known and n is odd is inapplicable when the underlying distribution is symmetric. Second, the performance of ERSS for odd set sizes involves the selection of the median from one of the sets, which calls for proper ranking of the observations in that set, whereas the partial ERSS does not. Thus, for odd set sizes, the m-cycle PERSS as described is the set

$$\{X_{i(1)j}; i = 1, 2, ...(n-1)/2; j = 1, 2, ..., m\}$$
$$\bigcup \{X_{i(n)j}; i = (n-1)/2+1, ..., n-1; j = 1, 2, ..., m\}.$$

We note from the algorithm that even though PERSS is applied to odd set sizes, it yields samples of even sizes. For example, PERSS performed on set sizes of 3, 5, 7 and 9 will respectively yields samples of size 2, 4, 6 and 8 per cycle.

Suppose we have a PERSS as described above. Then the linear unbiased estimator of μ, assuming that σ is known, is given by

$$\hat{\mu}_1' = \frac{1}{m(n-1)} \sum_{j=1}^{m} \left\{ \sum_{i=1}^{p'} X_{i(1)j} + \sum_{i=p'+1}^{n-1} X_{i(n)j} \right\} - \frac{\sigma}{2}(\alpha_{(1:n)} + \alpha_{(n:n)}) \tag{56}$$

with variance,

$$\text{Var}[\hat{\mu}_1'] = \frac{\sigma^2}{2m(n-1)}(\nu_{(1:n)} + \nu_{(n:n)}). \tag{57}$$

On the other hand, if μ is known, then the linear unbiased estimator of σ is given by

$$\hat{\sigma}_1{}' = \frac{1}{m(n-1)}\sum_{j=1}^{m}\left\{\frac{1}{\alpha_{(1:n)}}\sum_{i=1}^{p'} X_{i(1)j} + \frac{1}{\alpha_{(n:n)}}\sum_{i=p'+1}^{n-1} X_{i(n)j}\right\} - \frac{\mu}{2}\left[\frac{1}{\alpha_{(1:n)}} + \frac{1}{\alpha_{(n:n)}}\right] \quad (58)$$

and

$$\text{Var}[\hat{\sigma}_1{}'] = \frac{\sigma^2}{2m(n-1)}\left[\frac{\nu_{(1:n)}}{\alpha^2_{(1:n)}} + \frac{\nu_{(n:n)}}{\alpha^2_{(n:n)}}\right]. \quad (59)$$

It is clear from equations (56) and (58) that if the underlying distribution is symmetric, then the nuisance parameter is not required in the estimation of the parameter of interest.

6. RESULTS II

In this section, we compare the proposed linear unbiased estimators in Section 5 with the best SRS estimators. 'Best' here implies estimators that have achieved the lower bound variance, which is the reciprocal of the Fisher information. Results are presented for the estimation of μ from $N(\mu, 1)$ and σ from $N(0, \sigma^2)$, Exp(σ), gamma(2, σ) and gamma(3,σ).

Let $X_1, X_2, \ldots, X_{mn}$ be a random sample of size mn from a distribution with cdf $F\left(\frac{x-\mu}{\sigma}\right)$. It is easy to show that the Fisher information of μ and σ from this sample under the usual regularity conditions see Stokes (1995) are respectively given by

$$I_{mn}(\mu) = \frac{mn}{\sigma^2} E\left\{\frac{f'(Z_r)}{f(Z_r)}\right\}^2 \quad (60)$$

and

$$I_{mn}(\sigma) = \frac{mn}{\sigma^2} E\left\{\left[\frac{Z_r f'(Z_r)}{f(Z_r)}\right]^2 - 1\right\}, \quad (61)$$

where $Z_r = (X_r - \mu)/\sigma$.

Thus, the respective lower bound variance of the SRS unbiased estimators of μ and σ are given by the reciprocals of the last two equations. Therefore, the proposed ERSS estimators are compared with the SRS best estimators using the relative precision

$$RP(\hat{\mu}_\kappa, \hat{\mu}_{ML}) = \frac{1/I_{mn}(\mu)}{Var[\hat{\mu}_\kappa]} \text{ and } RP(\hat{\sigma}_\kappa, \hat{\sigma}_{ML}) = \frac{1/I_{mn}(\sigma)}{Var[\hat{\sigma}_\kappa]},$$

where κ = 1, 2 and $\hat{\mu}_{ML}$ and $\hat{\sigma}_{ML}$ denote the best SRS estimators of μ and σ respectively. In a similar fashion, the MRSS and the PERSS estimators are compared with the 'best' SRS estimators.

Table 6 shows the results for odd set sizes. Clearly, all the estimators under RSS, MRSS and ERSS estimators of μ from $N(\mu,1)$, and of σ from Exp(σ), gamma(2, σ) and gamma(3, σ) are more efficient than the corresponding best SRS estimators. The gain in using the proposed ERSS estimators is moderate as compared to that using MRSS, but the proposed estimators will serve as the best alternatives where complete and accurate ranking of observations is difficult as compared to picking the smallest and the largest units.

TABLE 6

The relative precision of the RSS, MRSS and ERSS linear unbiased estimators for odd set sizes

Distribution	Sampling Scheme	Sample Size per Cycle				
		3	5	7	9	11
N (μ, 1)	RSS	1.93	2.94	3.82	4.77	5.75
	MRSS	2.23	3.49	4.75	6.02	7.29
	ERSS	1.91	2.41	2.73	2.98	3.17
Exp (σ)	RSS	1.80	2.59	3.39	4.19	4.99
	MRSS	1.92	2.87	3.83	4.79	5.75
	ERSS	1.63	2.05	2.42	2.75	3.04
Gamma (2, σ)	RSS	1.86	2.73	3.59	4.46	4.93
	MRSS	2.07	3.16	4.26	5.36	6.46
	ERSS	1.75	2.17	2.48	2.74	2.96
Gamma (3, σ)	RSS	1.89	2.77	3.67	4.56	5.46
	MRSS	2.12	3.26	4.42	5.57	6.73
	ERSS	1.80	2.23	2.53	2.77	2.97

In Table 7, results are presented for the RSS, MRSS ERSS and PERSS estimators of μ from $N(\mu, 1)$ and σ from $N(0, \sigma^2)$, Exp(σ), gamma(2, σ) and gamma(3,σ) for even set sizes per cycle. Note that in the case of RSS, MRSS and ERSS, the sample size per cycle is the same as the set size, but for the PERSS, the sample size per cycle is the set size minus one as explained earlier.

It is clear from Table 7 that the PERSS estimator of σ from $N(0, \sigma^2)$ is the most efficient for when the set size per cycle is greater than two. In estimating the other parameters, the trend is similar to that observed in Table 6 except that the PERSS estimators dominate all the rest when the sample size per cycle is two. We finally note that PERSS has improved the efficiency of ERSS.

TABLE 7

The relative precision of the RSS, MRSS and ERSS linear unbiased estimators and that for the corresponding PERSS for even set sizes per cycle

Distribution	Sampling Scheme	Sample Size per Cycle				
		2	4	6	8	10
N (μ, 1)	RSS	1.47	2.40	3.35	4.30	5.25
	MRSS	1.47	2.77	4.06	5.34	6.62
	ERSS	1.47	2.03	2.40	2.68	2.90
	PERSS	1.79	2.23	2.55	2.80	3.00
N (0, σ^2)	RSS	0.23	0.60	0.92	1.21	1.50
	MRSS	0.23	0.12	0.08	0.06	0.05
	ERSS	0.23	1.08	1.93	2.72	3.44
	PERSS	0.64	1.51	2.33	3.09	3.78
Exp (σ)	RSS	1.40	2.19	2.99	3.79	4.59
	MRSS	1.29	2.30	3.28	4.25	5.22
	ERSS	1.29	1.51	1.60	1.66	1.69
	PERSS	1.42	1.56	1.63	1.68	1.71
Gamma (2, σ)	RSS	1.43	2.29	3.16	4.03	4.90
	MRSS	1.37	2.52	3.64	4.76	5.87
	ERSS	1.37	1.75	1.96	2.10	2.20
	PERSS	1.59	1.86	2.03	2.15	2.24
Gamma (3, σ)	RSS	1.44	2.33	3.22	4.11	5.01
	MRSS	1.40	2.60	3.78	4.94	6.11
	ERSS	1.40	1.84	2.09	2.27	2.41
	PERSS	1.65	1.98	2.19	2.34	2.46

7. CONCLUSIONS

In this paper, we have proposed maximum likelihood estimators (mle's) of the parameters of the normal, exponential and gamma distributions in the light of the location-scale family of distributions, using median ranked set sampling (MRSS) and extreme ranked set sampling (ERSS). Under these sampling schemes, we have also proposed some linear unbiased estimators (lue's) of the same parameters, which are a lot more easily computable than the mle's.

The mle's of the normal mean, μ, and the scale parameters of the exponential and gamma distributions under MRSS are found to dominate all other estimators. Similarly, the lue's of these same parameters under MRSS are the most dominant among all the lue's considered.

The mle's of the normal standard deviation under ERSS are the most efficient among the mle's while the PERSS lue of the same parameter dominates all the lue's. The PERSS scheme is generally seen to be an improvement of the ERSS scheme.

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RIASSUNTO

Stime dei parametri della distribuzioni normale, esponenziale e gamma basate sulla mediana e sugli estremi

In questo lavoro vengono presi in considerazioni gli stimatori di massima verosimiglianza (mle's) come stimatori lineari corretti dei parametri delle distribuzioni normale, esponenziale e gamma, considerate nel contesto delle distribuzioni di locazione e scala, cioè di distribuzioni con funzione di ripartizione del tipo $F((x-\mu)/\sigma)$, utilizzando un metodo di campionamento caratterizzato da un ordinamento basato sulla mediana (MRSS) e sugli estremi (ERSS). I metodi MRSS e ERSS rappresentano una variazione del campionamento ordinale RRS e risultano di più facile applicazione e meno esposti ai problemi di errata classificazione. Gli mle's della media della normale e dei parametri di scala della distribuzione esponenziale e gamma, determinati secondo il metodo MRSS, mostrano comportamenti migliori degli altri stimatori, mentri quelli della deviazione standard basati sul metodo ERSS sono i migliori. Un comportamento analogo si osserva per i lue. Si propone inoltre una modifica degli ERSS relativamente al campionamento basato sull'ordinamento parziale degli estremi per campioni di numerosità dispari per generane di numerosità pari. È mostrato come il lue per la deviazione standard della normale sia il più efficiente di tutti i lue per gli stessi parametri. Inoltre i lue PERSS sono i più efficienti quando la dimensione campionaria per ciclo è pari a due.

SUMMARY

Estimating the parameters of the normal, exponential and gamma distributions using median and extreme ranked set samples

In this paper, we propose maximum likelihood estimators (mle's) as well as linear unbiased estimators (lue's) of the parameters of the normal, exponential and gamma distributions in the light of the location-scale family of distributions - i.e. distributions with cumulative distribution functions of the form $F((x-\mu)/\sigma)$, using median ranked set sampling (MRSS) and extreme ranked set sampling (ERSS). MRSS and ERSS are modifications of ranked set sampling (RSS), which are more practicable and less prone to problems resulting from erroneous ranking. The mle's of the normal mean and the scale parameters of the exponential and gamma distributions under MRSS are shown to dominate all other estimators, while the mle of the normal standard deviation under ERSS is the most efficient. A similar trend is observed in the lue's. A modification of ERSS namely partial extreme ranked set sampling (PERSS) is proposed for odd set sizes to generate even-sized samples. The lue of the normal standard deviation under this modification is shown to be the most efficient of all the lue's of the same parameter. Among the lue's considered, the PERSS lue's are the most efficient when the sample size per cycle is two.