

BROWNIAN SHEET AND UNIFORMITY TESTS ON THE HYPERCUBE

Alejandra Cabaña¹

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Espanya

Enrique M. Cabaña

PEDECIBA, Montevideo, Uruguay

SUMMARY

A construction of p -parameter Brownian sheet on the hypercube $C = [0, 1]^p$ as a sum of 2^p independent Gaussian processes is obtained. The terms are closely related to Brownian pillows, and the probability laws of their $L^2(C)$ squared norms are computed. This allows us to propose consistent tests of uniformity for samples of i.i.d. random vectors on C . A comparison of powers of the new tests with those of several uniformity tests found in the statistical literature completes the article.

Keywords: Brownian sheet; Multivariate uniformity tests.

1. INTRODUCTION

Testing for uniformity in the hypercube $C := [0, 1]^p$ (where $p \geq 1$), in spite of the extreme particularity of such probabilistic model, is interesting for several reasons, specially in areas such as statistics, machine learning, and computational mathematics.

Many Monte Carlo methods generate samples that should ideally be uniformly distributed over $[0, 1]^p$. Testing uniformity ensures that these methods produce correct and unbiased samples, which is critical for accurate integration, optimization, or probabilistic modeling.

Uniformity in the hypercube may arise as a consequence of transforming a multivariate distribution. For example, probability integral transforms via cumulative distribution functions (see [Rosenblatt, 1952](#)) map arbitrary distributions into the uniform distribution on $[0, 1]^p$. Testing for uniformity can validate the correctness of these transformations and evaluate model fit. Even testing composite hypothesis as goodness-of-

¹ Corresponding Author. E-mail: AnaAlejandra.Cabana@uab.cat

fit to multivariate normality on $\mathbb{R}^p$ can be reduced via standardization and probability integral transformation to assess uniformity on $[0, 1]^p$.

Several authors base their uniformity tests on quite different statistics, namely, the distribution of distances between the data or the use of normal quantiles (Yang and Modarres, 2017), the depth of the elements of the sample (Hegazy and Green, 1975), their distance-to-boundary (Berrendero et al., 2006), random graphs over points of the sample (Ebner et al., 2020) or minimal covering trees of the sample graph (Petrie and Willemain, 2013). Some of these tests are included in the R (The R Project for Statistical Computing) package SHT.

It is striking that the use of sophisticated statistical methods is preferred over basing decisions solely on the distance between the empirical distribution of the sample and the uniform distribution. This preference is attributable to the fact that the distributions of the empirical processes under the null hypothesis of uniformity not only depend on the dimension but also on the sample size. Moreover, neither these distributions nor their asymptotic forms for large samples are well understood.

We base our tests in a decomposition of the Brownian sheet into a sum of independent processes, that generalizes the well known decomposition $w(t) = t \cdot w(1) + b(t)$ of a standard Wiener process $w(t)$ on $[0, 1]$ as the sum of a Brownian bridge $b(t)$ and the “ramp” $t \cdot w(1)$. The result is not surprising but we include a straightforward presentation in Section 2 for the sake of completeness. This decomposition will enable us to express the empirical process as a sum of asymptotically independent processes. This key insight allows us to propose tests similar in spirit to the Cramér–von Mises test but with fewer drawbacks than a direct generalization would entail.

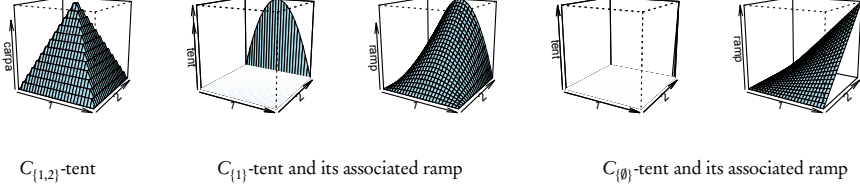
All the power comparisons in Section 4 are made via a Monte Carlo approximation of the distribution of the test statistics, since the asymptotic distribution is described by a series of random terms that is not suitable for calculation, and leads to conservative tests.

2. DECOMPOSITION OF p -BROWNIAN SHEET AS A SUM OF 2^p INDEPENDENT GAUSSIAN TERMS

The probabilistic result stated in Section 2.2 is based on a general property of functions with domain C that vanish on the set ∂^-C of points $\mathbf{t} = (t_1, t_2, \dots, t_p)^{\text{tr}} \in C$ with at least one component equal to zero. These functions have a unique decomposition as a sum of terms belonging to a special class that we call *ramps*.

2.1. Ramp components of a function $g : C \rightarrow \mathbb{R}$ that vanishes on ∂^-C .

Given a subset H of the set $J = \{1, 2, \dots, p\}$ and a point $\mathbf{t} = (t_1, t_2, \dots, t_p) \in C$, let us denote $\mathbf{t}_H$ the point with coordinates $(\mathbf{t}_H)_j = t_j$ if $j \in H$ and $(\mathbf{t}_H)_j = 1$ if $j \notin H$. Then, introduce the sets $C_H = \{\mathbf{t}_H : \mathbf{t} \in C\}$ and $\partial_H C_H = \{\mathbf{t}_H \in C_H : \text{for some } j \in H, t_j = 0 \text{ or } t_j = 1\}$. Each C_H is said to be a *face* of C of dimension equal to the cardinal $\#H$ of

Figure 1 – Tents and ramps in $[0, 1]^2$.

H . The face $C_\emptyset$ has only the element $\mathbf{1}$ with all coordinates equal to one, and C_J is equal to C .

A function $g_H : C_H \rightarrow \mathbf{R}$ that vanishes on $\partial_H C_H$ shall be called a *tent*, or an H -tent if the domain is wanted to be mentioned explicitly.

To each H -tent g_H we associate an extension to the domain C , namely $R_{g_H}(t) = |t_{J \setminus H}| g_H(t_H)$, where $|(t_1, t_2, \dots, t_p)|$ denotes the product $\prod_{j=1}^p t_j$. Such extension is said to be an H -ramp. More precisely, the H -ramp associated to the H -tent g_H (Figure 1 shows some examples in $[0, 1]^2$).

The previous notations allow us to state the following theorem:

THEOREM 1.

(i) If $g : C \rightarrow \mathbf{R}$ vanishes on $\partial^- C$, then there exists a unique decomposition

$$g = \sum_{H \subset J} R_H \quad (1)$$

where each R_H is an H -ramp associated to an H -tent T_H .

(ii) The maps $g \mapsto \mathcal{R}_H g := R_H$ and $g \mapsto \mathcal{T}_H g := T_H$ are linear, preserve the continuity and satisfy $\sup_{t \in C} \mathcal{R}_H g(t) \leq K_p \sup_{t \in C} g(t)$, $\sup_{t \in C} \mathcal{T}_H g(t) \leq K_p \sup_{t \in C} g(t)$, where K_p depends on p but not on g .

PROOF. The proof of this Theorem, where the ramps R_H and tents T_H are obtained constructively, is deferred to the Appendix A. $\square$

2.2. Brownian tents, ramps, and a construction of Brownian sheet

The p -parameter Wiener process or p -Brownian sheet on C is the family of Gaussian centred variables $\{W(t) : t \in C\}$ with covariances $\mathbf{E}W(s)W(t) = |s \wedge t|$ (Chentsov, 1956; Walsh, 1986). Since W is a.s. 0 on the lower border $\partial^- C$ of C , Theorem 1 can be applied to conclude that there is a unique decomposition of W as a sum of ramps:

$$W(t) = \sum_{H \subset J} R_H(t), \quad R_H(t) = |t_{J \setminus H}| T_H(t_H). \quad (2)$$

An important property of this decomposition is stated below in the Corollary of Theorem 3.

DEFINITION 2.

- A Brownian H -tent is a centred Gaussian process on C_H with covariances

$$\mathbf{E}T_H(s)T_H(t) = \prod_{j \in H} (s_j \wedge t_j - s_j t_j). \quad (3)$$

- A Brownian H -ramp is a H -ramp associated to a Brownian H -tent.

Since $\mathbf{Var}T_H$ vanishes on $\partial_H C_H$, the Brownian H -tents are almost surely H -tents. These processes are already referred to as Brownian pillows in the mathematical literature (see, for instance, [Koning and Protasov, 2003](#); [Hashorva, 2008](#)). While an explicit definition of the Brownian pillow is provided in [Zhang \(2017\)](#), the term “pillow” is also used in the same article to describe a multivariate extension associated with a different decomposition of the Brownian sheet. To avoid potential misunderstandings, we prefer to retain the name “Brownian tent” here.

THEOREM 3. CONSTRUCTION OF A p -BROWNIAN SHEET AS A SUM OF INDEPENDENT BROWNIAN RAMPS.

Let $\{T_H^* : H \subset J\}$ be a family of independent Brownian H -tents, and R_H^* their corresponding Brownian ramps. Then the sum $W^* = \sum_{H \subset J} R_H^*$ is a p -Brownian sheet.

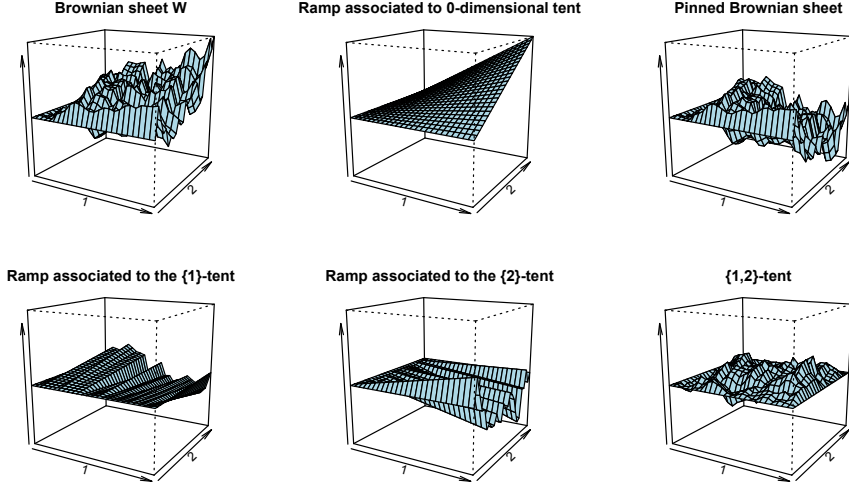
PROOF. Because of the independence of the ramps, the covariances of the sum $W^* = \sum_{H \subset J} R_H^*$ are

$$\begin{aligned} \mathbf{Cov}(W^*(s), W^*(t)) &= \sum_{H \subset J} \mathbf{Cov}(R_H^*(s), R_H^*(t)) \\ &= \sum_{H \subset J} |s_{J \setminus H}| |t_{J \setminus H}| \prod_{j \in H} (s_j \wedge t_j - s_j t_j) \\ &= |s| |t| \sum_{H \subset J} \prod_{j \in H} \alpha_j \quad \text{with } \alpha_j = \left(\frac{s_j \wedge t_j}{s_j t_j} - 1 \right). \end{aligned}$$

On the other hand, a simple manipulation shows that the covariances $|s \wedge t|$ of the p -Brownian sheet are the same as the covariances of W^* :

$$|s \wedge t| = \prod_{j=1}^p s_j \wedge t_j = |s| |t| \prod_{j=1}^p \frac{1}{s_j \vee t_j} = |s| |t| \prod_{j=1}^p (1 + \alpha_j) = |s| |t| \sum_{H \subset J} \prod_{j \in H} \alpha_j$$

thus proving our statement. $\square$

Figure 2 – Decomposition of Brownian Sheet ($p = 2$).

Since W is a copy of W^* , the ramps $\mathcal{R}_H W$ are copies of the Brownian H -ramps R_H^* and the tents $\mathcal{T}_H W$ are copies of the Brownian H -tents T_H^* , hence we have the following result.

COROLLARY 4. *The terms in the sum of Eq. (2) are independent Brownian ramps.*

REMARK 5. *In these arguments the uniqueness of the decomposition and continuity of the maps $\mathcal{R}_H$ both established by Theorem 1 play a decisive role.*

The particular case of decomposition for $p = 2$ is depicted in Figure 2.

2.3. Probability laws of Brownian tents and their squared norms

The probability distribution of the standard Brownian bridge is well known (see, for instance, [Durbin, 1973](#)) and may be found by obtaining the eigenfunctions $\{\psi_\nu := \sqrt{2} \sin(\nu\pi u)\}_{\nu \in N := \{1, 2, 3, \dots\}}$ of the covariance kernel $s \wedge t - st$, with eigenvalues $\lambda_\nu := \frac{1}{\nu^2 \pi^2}$. These eigenfunctions constitute an orthonormal basis of $L^2([0, 1])$ with the Lebesgue measure. Then the law of b is described by the Karhunen–Loève expansion $b(t) = \sum_{\nu \in N} \sqrt{\lambda_\nu} Z_\nu \Psi_\nu(t)$, where $\{Z_\nu : \nu \in N\}$ are i.i.d. standard Gaussian variables.

Since the covariance in Eq. (3) of a Brownian H -tent is a product of univariate kernels, the map $f(t) \mapsto \int_{C_H} \prod_{j \in H} (s_j \wedge t_j - s_j t_j) f(s) ds_H$ has eigenfunctions $\psi_\nu(t) = \prod_{j \in H} \lambda_{\nu_j} \phi_{\nu_j}(t_j)$, where $\nu \in \mathbb{N}^{\#H}$ is a multi-index with components $\nu_j \in \mathbb{N}$ for each

$j \in H$. These eigenfunctions are a complete orthonormal set on $L^2(C_H)$ with the Lebesgue measure, so that the Karhunen – Loève expansion

$$T_H(t_H) = \sum_{\nu \in \mathbb{N}^{\#H}} \sqrt{\lambda_\nu} Z_\nu \phi_\nu(t) = \sum_{\nu \in \mathbb{N}^{\#H}} \frac{Z_\nu}{|\nu| \tau^{\#H}} \prod_{j \in H} (\sqrt{2} \sin(\nu_j \pi t_j)) \quad (4)$$

holds with $\{Z_\nu : \nu \in \mathbb{N}^{\#H}\}$ i.i.d. standard Gaussian.

The probability law of the Brownian tent squared norm is obtained by integrating the square of Eq. (4) on C_H , and recalling that the eigenfunctions are an orthonormal set one gets

$$\|T_H\|^2 = \sum_{\nu \in \mathbb{N}^{\#H}} \frac{Z_\nu^2}{|\nu|^2 \tau^{2\#H}}. \quad (5)$$

Its c.d.f. shall be denoted $P_H(x) = \mathbf{P}\{\|T_H\|^2 \leq x\}$.

3. UNIFORMITY TESTS BASED ON THE H -TENTS OF THE EMPIRICAL PROCESS

3.1. Two consistent tests based on the asymptotic laws of the H -tents

Let $\mathcal{U}_n = \{U_1, U_2, \dots, U_n\}$ be a sample of i.i.d. C -valued random variables $U_i = (U_{i,1}, U_{i,2}, \dots, U_{i,p})^{\text{tr}}$, $i = 1, 2, \dots, n$, with continuous distribution F , $F_{\mathcal{U}_n}(t) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{U_i \leq t\}}$ its empirical distribution function and $W_{\mathcal{U}_n}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\mathbf{1}_{\{U_i \leq t\}} - \mathbf{1}_t)$ the empirical process with respect to the uniform distribution.

Then it is well known that

- If F is uniform in C , $W_{\mathcal{U}_n}$ converges in law to the pinned Brownian sheet $W_0(t) = W(t) - \mathbf{1}_t W(1)$ as n goes to infinity (Pyke, 1975), and because of the continuity of the map $\mathcal{T}$ defined in Theorem 1-(ii) the H -tents $T_{n,H} := \mathcal{T}_H W_{\mathcal{U}_n}$ of $W_{\mathcal{U}_n}$ converge jointly in law to the H -tents of W_0 , which are the H -tents $T_H := \mathcal{T}_H W$ for $H \neq \emptyset$. This implies that $\|T_{n,H}\|^2 \xrightarrow{\mathcal{D}} \|T_H\|^2$ jointly for $H \subset J, H \neq \emptyset$, and hence

- the vector p_n of components

$$p_{n,H} = 1 - P_H(\|T_{n,H}\|^2) \quad (6)$$

is asymptotically uniform on $[0, 1]^{2^p - 1}$;

- consequently the random variables $Q(1 - p_{n,H})$, where Q is the quantile function of the squared standard normal are asymptotically i.i.d. $\sim \chi^2$ with one degree of freedom, so that $\mathcal{S} := \sum_{H \subset J, H \neq \emptyset} Q(1 - p_{n,H})$ converges in law to a χ_f^2 distribution with $f := 2^p - 1$ degrees of freedom.

- If F is not uniform in C , $\lim_{n \rightarrow \infty} \|W_{\mathcal{U}_n}\|^2 = \infty$ a.s., so that the triangle inequality applied to Eq. (2) implies that at least for one non-empty H , $\lim_{n \rightarrow \infty} \|T_{n,H}\|^2 = 3^{\#J \setminus H} \lim_{n \rightarrow \infty} \|R_{n,H}\|^2 = \infty$ and hence at least one component of $\mathbf{p}_n$ tends to zero and S tends to infinity a.s.

The aforementioned dichotomy supports the rejection of the null hypothesis that F follows a uniform distribution when $\mathbf{p}_{\min} := \min\{p_{n,H} : H \subset J, H \neq \emptyset\}$ is smaller than a given constant, and likewise when S exceeds a specific constant. Both procedures result in consistent tests. We refer to the test with the rejection region $\mathbf{p}_{\min} < c$ as the minimum asymptotic test, abbreviated as m-as-test, and the test with the rejection region $S > k$ as the sum asymptotic test, or s-as-test. For large values of n , the asymptotic significance level of the m-as-test is $\alpha = 1 - (1 - c)^{2^p - 1}$. Meanwhile, selecting $k = Q_{2^p - 1}(1 - \alpha)$ —the $1 - \alpha$ quantile of the $\chi_{2^p - 1}^2$ distribution—yields an s-as-test with an asymptotic significance level of α .

3.2. Finite samples tests

The practical implementation of the tests in Section 3.1 requires the computation of the probabilities P_H and the empirical tents $T_{n,H}$. These latter statistics are easily calculated as we show in Section 3.4. However, the same is not true for the probability P_H because the nice, compact formula in Eq. (5) does not allow for a simple calculation.

For this reason, we will replace the asymptotic tests with tests for each n , using the distribution $P_{n,H}$ of the statistics $\|T_{n,H}\|^2$ instead of its asymptotic distribution. Likewise, the exact calculation of probabilities needed to compute the new statistics will be replaced by estimates based on Monte Carlo simulations.

In summary, the decision procedures we propose are the following:

3.2.1. The minimum test (abbreviated m-test) for samples of size n

1. Generate the list $\mathcal{H}$ of nonempty subsets $H \subset J$,
2. for each $H \in \mathcal{H}$ compute the statistic $\|T_{n,H}\|^2$,
3. introduce the p -values $\tilde{p}_{n,H} = 1 - P_{n,H}(\|T_{n,H}\|^2)$,
4. generate a large number $\{\mathcal{U}_n^r : r = 1, 2, \dots, R\}$ of independent samples of size n of uniform random values on $C = [0, 1]^p$,
5. for each $\mathcal{U}_n^r$ compute the squared norms $\|T_H^r\|^2$ and estimate $\tilde{p}_{n,H}$ by means of the statistic $\hat{p}_{n,H} := \frac{\sum_{r=1}^R \mathbf{1}_{\{\|T_H^r\|^2 > \|T_{n,H}\|^2\}} + 1}{R + 1}$,
6. Reject the null hypothesis of uniformity if $\min_{\{H \in \mathcal{H}\}} \hat{p}_{n,H}$ is smaller than $1 - (1 - \alpha)^{1/\#\mathcal{H}}$ where α is the desired significance level of the test.

3.2.2. The sum test (abbreviated s-test) for samples of size n

1.-5. Repeat steps 1.-5. of Section 3.2.1,

6. compute the statistic $\hat{S} = \sum_{H \in \mathcal{H}} Q_1(1 - \hat{p}_H)$ where Q_1 denotes the quantile function of the χ^2 distribution with one degree of freedom,
7. reject the null hypothesis of uniformity if $\hat{S}$ is greater than the quantile $1 - \alpha$ of the χ^2 distribution with $\#\mathcal{H}$ degrees of freedom.

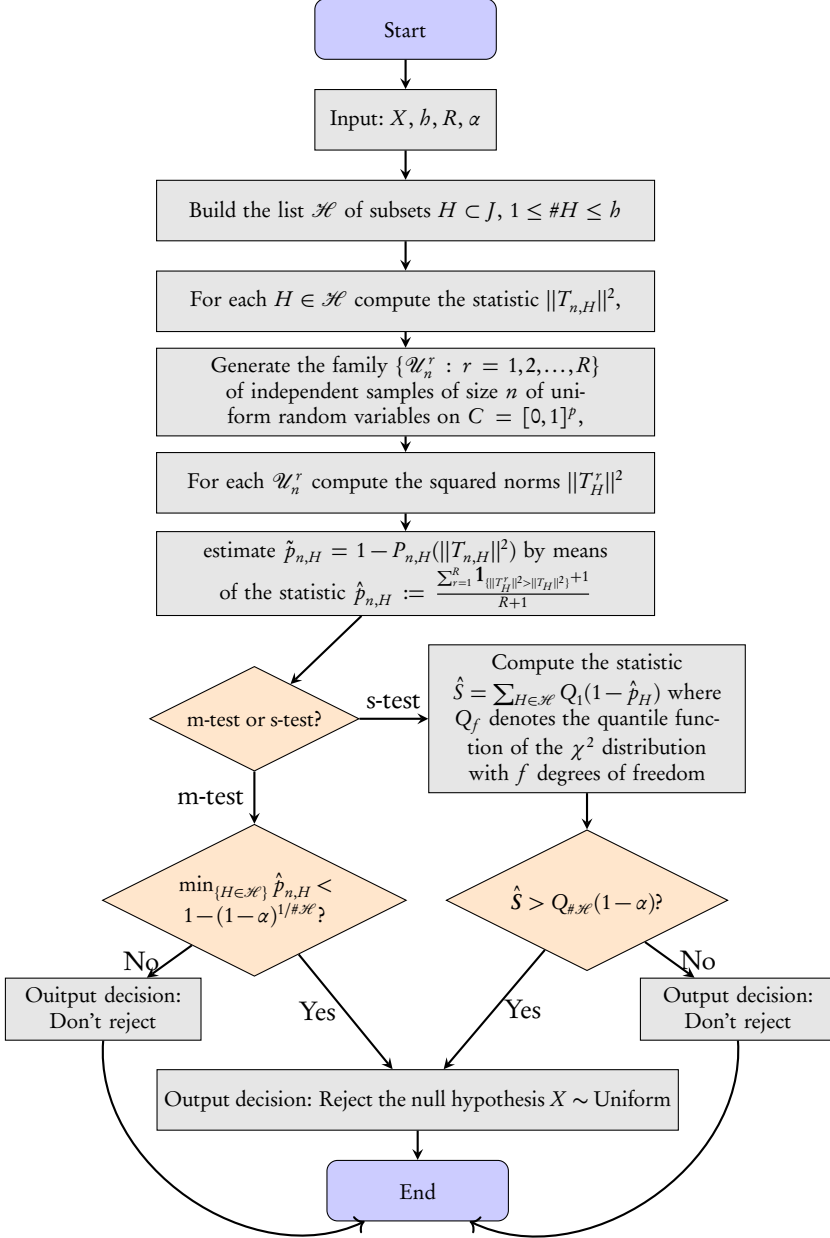
3.3. Partial tests

If p is very large, consistency can be sacrificed to reduce the number of empirical tests to be computed, by substituting a partial family of subsets of J , such as $\{H : H \subset J, 0 < \#H \leq h\}$ with $h < p$ for the whole family $\mathcal{H}$ of nonempty subsets in the steps of Sections 3.2.1 and 3.2.2.

The computations of empirical powers in Section 4 include examples of partial tests.

Figure 3 contains a schematic description of both m- and s-tests, including the partial versions.

Figure 3 – A schematic summary of the proposed procedures to test uniformity of a multivariate sample X with significance level α



3.4. Computing the H -tent of $W_{\mathcal{U}_n}$

The linearity of the map $\mathcal{T}$ allows to express the H -tents of samples of size n in terms of H -tents of samples of size one.

The empirical distribution function of the sample of size one $\mathcal{U}_1 = \{U\}$, $U = (U_1, U_2, \dots, U_p)^{\text{tr}}$, is $F_{\mathcal{U}_1}(t) = \mathbf{1}_{\{U \leq t\}} = \prod_{j=1}^p \mathbf{1}_{\{U_j \leq t_j\}}$ and the empirical process is $W_{\mathcal{U}_1}(t) = \mathbf{1}_{\{U \leq t\}} - |t| = \prod_{j=1}^p \mathbf{1}_{\{U_j \leq t_j\}} - \prod_{j=1}^p t_j$.

The ramps and tents of $W_{\mathcal{U}_1}$ can be obtained inductively by applying the operations described in the proof of Theorem 1, but a simpler alternative way is to develop

$$\begin{aligned} \mathbf{1}_{\{U \leq t\}} &= \prod_{j=1}^p \mathbf{1}_{\{U_j \leq t_j\}} = |t| \prod_{j=1}^p \left(1 + \left(\frac{\mathbf{1}_{\{U_j \leq t_j\}}}{t_j} - 1 \right) \right) \\ &= |t| \sum_{H \subset J} \prod_{j \in H} \left(\frac{\mathbf{1}_{\{U_j \leq t_j\}}}{t_j} - 1 \right) = \sum_{H \subset J} |t_{J \setminus H}| \prod_{j \in H} (\mathbf{1}_{\{U_j \leq t_j\}} - t_j) \\ &= |t| + \sum_{H \subset J, H \neq \emptyset} |t_{J \setminus H}| \prod_{j \in H} (\mathbf{1}_{\{U_j \leq t_j\}} - t_j), \end{aligned}$$

so that

$$W_{\mathcal{U}_1}(t) = \sum_{H \subset J, H \neq \emptyset} |t_{J \setminus H}| \prod_{j \in H} (\mathbf{1}_{\{U_j \leq t_j\}} - t_j). \quad (7)$$

The products $\prod_{j \in H} (\mathbf{1}_{\{U_j \leq t_j\}} - t_j)$ are H -tents because they vanish if $t_H \in \partial_H C_H$, that is, if at least one of the t_j , for $j \in H$, is 0 or 1. Therefore, the right-hand term of Eq. (7) is the decomposition of $W_{\mathcal{U}_1}$ as a sum of H -ramps, thus proving that

$$\mathcal{R}_H(W_{\mathcal{U}_1}) = |t_{J \setminus H}| \prod_{j \in H} (\mathbf{1}_{\{U_j \leq t_j\}} - t_j) \quad \text{and} \quad \mathcal{T}_H(W_{\mathcal{U}_1}) = \prod_{j \in H} (\mathbf{1}_{\{U_j \leq t_j\}} - t_j).$$

The first part of next statement follows by applying the linear operator $\mathcal{T}_H$ to the equality $W_{\mathcal{U}_n}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n W_{\{U_i\}}(t)$ and part (ii) follows by noticing that the integral in $\frac{1}{n} \int_{C_H} \sum_{b,i=1}^n \prod_{j \in H} (\mathbf{1}_{\{U_{b,j} \leq t_j\}} - t_j) (\mathbf{1}_{\{U_{i,j} \leq t_j\}} - t_j) dt$ commutes not only with the sum but also with the product because of the factorization of the integrand:

THEOREM 6.

(i) The H -tent of the empirical process is

$$T_{n,H}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \prod_{j \in H} (\mathbf{1}_{\{U_{i,j} \leq t_j\}} - t_j),$$

and

(ii) its squared norm is

$$\|T_{n,H}\|^2 = \frac{1}{n} \sum_{b,i=1}^n \prod_{j \in H} \left(\frac{U_{b,j}^2 + U_{i,j}^2}{2} - U_{b,j} \vee U_{i,j} + \frac{1}{3} \right). \quad (8)$$

4. A BRIEF EMPIRICAL DESCRIPTION OF POWERS

4.1. Empirical comparison of the powers of several uniformity tests

Yang and Modarres (2017) compare the powers of their uniformity tests Q_1, Q_2, Q_3 based on the distances between observations and C_N based on the norm of the transformation of the sample obtained by applying elementwise the inverse of the standard Normal c.d.f. to the sample points, with the powers of tests M^2 based on three types of depth of the sample points (Hegazy and Green, 1975), BCV based on their distances-to-boundary (Berrendero et al., 2006) and MST based on a minimum covering tree of the sample graph (Petrie and Willemain, 2013).

For that purpose they construct two tables, both for samples in $[0, 1]^2$, one for alternatives of dependence and the other for alternatives of shape.

Their first table contains the power of the different uniformity tests against the following copula alternatives:

- AMH (Ali-Mikhail-Haq): $C_\theta(u, v) = \frac{uv}{1 - \theta(1-u)(1-v)}$, $-1 \leq \theta < 1$,
- FGM (Farlie-Gumbe-Morgenstern): $C_\theta(u, v) = uv + \theta uv(1-u)(1-v)$, $\theta \in [-1, 1]$,
- Clayton: $C_\theta(u, v) = \max[u^{-\theta} + v^{-\theta} - 1, 0]^{-1/\theta}$, $\theta \in [-1, \infty)$, $\theta \neq 0$ and
- Plackett: $C_\theta(u, v) = uv$ if $\theta = 0$, and for $\theta > 0$, $C_\theta(u, v) = \frac{1}{2}(\theta - 1)A - \sqrt{A^2 - 4uv(\theta - 1)}$, with $A = 1 + (u + v)(\theta - 1)$.

Observe that AMH and Clayton are Archimedean copulas, while FGM and Plackett are non Archimedean.

In their second table the alternatives are vectors with i.i.d. components distributed $\text{Beta}(\alpha, \beta)$ for several values of the parameters.

Our Tables 1 and 2 reproduce those of Yang and Modarres and add at the end of each line, the estimated powers of our m- and s-tests based on 1000 replications of samples for each size and alternative.

Our tests show a good performance in detecting copula alternatives, outperforming in some cases all the competitors. As for the alternatives with i.i.d. components, they exhibit a power similar to that of some of the others, occasionally surpassing some of them.

TABLE 1
Empirical power of uniformity tests against copula alternatives

alternative	n	M_S^2	M_L^2	M_T^2	C_N	BCV	MST	Q1	Q2	Q3	m-test	s-test
AMH $\theta = 0.9$	10	0.376	0.038	0.062	0.056	0.056	0.066	0.065	0.121	0.127	0.137	0.144
	25	0.328	0.118	0.054	0.068	0.062	0.112	0.066	0.170	0.164	0.359	0.339
	50	0.504	0.166	0.060	0.078	0.072	0.154	0.063	0.233	0.204	0.695	0.648
FGM $\theta = 1$	10	0.672	0.046	0.096	0.060	0.044	0.044	0.055	0.090	0.094	0.093	0.086
	25	0.590	0.076	0.060	0.072	0.040	0.052	0.052	0.104	0.115	0.238	0.250
	50	0.390	0.072	0.050	0.062	0.040	0.094	0.049	0.126	0.127	0.459	0.431
Clayton $\theta = 2$	10	0.384	0.016	0.078	0.088	0.078	0.164	0.097	0.257	0.237	0.372	0.319
	25	0.638	0.472	0.076	0.074	0.136	0.592	0.101	0.427	0.370	0.888	0.849
	50	0.984	0.850	0.060	0.090	0.194	0.894	0.098	0.640	0.566	0.998	0.998
Plackett $\theta = 5$	10	0.572	0.026	0.064	0.078	0.051	0.082	0.078	0.162	0.153	0.185	0.171
	25	0.414	0.170	0.046	0.072	0.078	0.152	0.076	0.234	0.210	0.536	0.513
	50	0.632	0.356	0.038	0.086	0.082	0.270	0.071	0.349	0.295	0.860	0.839

The numbers in boldface point out the cases in which our m-test outperforms the others.

TABLE 2
Empirical powers of uniformity tests against bivariate i.i.d. Beta alternatives

α	β	M_S^2	M_L^2	M_T^2	C_N	BCV	MST	Q1	Q2	Q3	m-test	s-test
.5	.5	<i>0.140</i>	<i>0.356</i>	<i>0.472</i>	<i>0.268</i>	0.998	<i>0.106</i>	0.997	0.999	0.999	0.444	0.683
	1	0.330	<i>0.242</i>	<i>0.182</i>	1.000	0.976	<i>0.254</i>	<i>0.184</i>	<i>0.415</i>	0.386	0.998	1.000
	2	<i>0.950</i>	<i>0.698</i>	<i>0.090</i>	1.000	1.000	0.998	<i>0.998</i>	<i>0.951</i>	<i>0.991</i>	1.000	1.000
	3	0.996	<i>0.776</i>	<i>0.086</i>	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	1	0.056	<i>0.054</i>	<i>0.044</i>	0.056	0.056	<i>0.030</i>	0.048	<i>0.042</i>	0.074	0.048	0.048
	2	<i>0.124</i>	<i>0.254</i>	<i>0.018</i>	1.000	<i>0.066</i>	<i>0.856</i>	<i>0.971</i>	<i>0.495</i>	<i>0.965</i>	1.000	1.000
	3	<i>0.374</i>	<i>0.456</i>	<i>0.070</i>	1.000	<i>0.426</i>	1.000	1.000	<i>0.221</i>	1.000	1.000	1.000
2	2	0.262	0.222	<i>0.070</i>	<i>0.030</i>	0.992	0.880	1.000	0.949	1.000	0.108	0.207
	3	<i>0.172</i>	<i>0.314</i>	<i>0.096</i>	<i>0.806</i>	0.998	0.998	1.000	1.000	1.000	0.977	0.994
3	3	<i>0.166</i>	<i>0.426</i>	<i>0.150</i>	<i>0.030</i>	1.000	1.000	1.000	<i>0.544</i>	1.000	0.720	0.935

Numbers in italics indicate the cases with power smaller than the power of our s-test.

4.2. Performance of our partial tests against six-dimensional copulas

Table 3 shows the empirical powers of the partial tests proposed in Section 3.3, applied to samples of 50 normal copulas distributed as $\Phi_R(\Phi^{-1}(U_1), \Phi^{-1}(U_2), \dots, \Phi^{-1}(U_p))$, where $U_1, U_2, \dots, U_p$ are i.i.d. Uniform on $[0, 1]$, Φ is the standard Normal c.d.f. and Φ_R is the c.d.f. of the centred Normal vector in $\mathbf{R}^p$ with variance

$$R = \begin{pmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \rho & \rho & 1 & \dots & \rho \\ \dots & \dots & \dots & \dots & \dots \\ \rho & \rho & \rho & \dots & 1 \end{pmatrix}$$

TABLE 3
Empirical powers of the YM tests C_N , Q1, Q2, Q3 and our partial m- and s-tests.

alternative	Yang & Modarres tests				partial tests		b
	C_N	Q1	Q2	Q3	m-test	s-test	
normal copula $\rho = 0.05$	0.052	0.043	0.055	0.077	0.041	0.050	1
					0.065	0.098	2
					0.023	0.147	3
					0.000	0.193	4
					0.000	0.220	5
					0.000	0.228	6
normal copula $\rho = 0.10$	0.051	0.050	0.107	0.113	0.037	0.053	1
					0.113	0.236	2
					0.042	0.238	3
					0.000	0.286	4
					0.000	0.306	5
					0.000	0.308	6
normal copula $\rho = 0.15$	0.056	0.060	0.220	0.191	0.034	0.056	1
					0.193	0.477	2
					0.079	0.417	3
					0.000	0.420	4
					0.000	0.426	5
					0.000	0.429	6
normal copula $\rho = 0.20$	0.058	0.072	0.366	0.314	0.036	0.057	1
					0.327	0.706	2
					0.150	0.603	3
					0.000	0.601	4
					0.000	0.590	5
					0.000	0.588	6
normal copula $\rho = 0.30$	0.060	0.091	0.720	0.683	0.034	0.056	1
					0.684	0.965	2
					0.414	0.906	3
					0.000	0.881	4
					0.000	0.861	5
					0.000	0.857	6
normal copula $\rho = 0.40$	0.065	0.116	0.929	0.954	0.030	0.063	1
					0.911	1.000	2
					0.744	0.988	3
					0.000	0.974	4
					0.000	0.968	5
					0.000	0.968	6

The numbers in boldface point out the cases in which our s-test outperforms the others.

The powers of Yang and Modarres (YM) tests computed by using the R package SHT are added for the sake of comparison.

5. FINAL COMMENTS

Both m- and s-tests are competitive with other tests proposed in the statistical literature, and show a good performance in detecting copula alternatives. It should be noted that the simplicity of the formula in Eq. (8) for obtaining the $\|T_H\|^2$ allows for a simple calculation of the test statistics.

Neither of the two new tests is more powerful than the other. Which has better results depends on the alternatives.

ACKNOWLEDGEMENTS

The first author was supported by grant PID2021-123733NB-I00, Ministerio de Ciencia, Innovación y Universidades, Spain.

APPENDIX

A. PROOF OF THEOREM 1

In order to prove the Theorem we apply the following Lemma:

LEMMA 7. *Each H -ramp vanishes on all faces $C_{H'}$ with $\#H' = \#H$, $H' \neq H$.*

PROOF (OF LEMMA 7). Assume that R_H is the ramp associated to the H -tent T_H . There exist $j \in H \setminus H'$, so that if $t \in C_{H'}$, then $t_j = 1$ and consequently the j -th component of t_H is one, that is, t_H belongs to ∂C_H , this implies that $T_H(t_H) = 0$, and hence the H -ramp $R_H(t) = |t_{J \setminus H}| T_H(t_H)$ vanishes. $\square$

PROOF (OF THEOREM 1). The map $T_\emptyset : 1 \mapsto g(1)$ is a $\emptyset$ -tent, with $\emptyset$ -ramp $R_\emptyset(t) = |t|g(1)$. We introduce now the function $g_0 = g - R_\emptyset$ that vanishes on $C_\emptyset$.

For each H with cardinal one, the restriction of g_0 to C_H is a tent, because $t \in \partial_H C_H$ implies $t \in C_\emptyset$ or $t \in \partial^- C$. Its ramp R_H vanishes on the other faces of dimension $\#H$, as stated in Lemma 7, so that the function $g_1 = g_0 - \sum_{\#H=1} R_H = g - \sum_{\#H \leq 1} R_H$ vanishes on $\{C_H : \#H \leq 1\}$.

Now set $b = 2, 3, \dots, p$ succesively and

- recall that g_{b-1} vanishes on $\{C_H : \#H < b\}$, and notice that for $\#H = b$, the restriction of g_{b-1} to C_H is a tent, because $t \in \partial_H C_H$ implies that either t belongs to a face of dimension $b - 1$ or it belongs to $\partial^- C$,

- let R_H denote the ramp associated to $g_{b-1}(t_H)$, that vanishes on the remaining faces of dimension b as implied by Lemma 7,
- introduce the function $g_b = g_{b-1} - \sum_{\#H=b} R_H = g - \sum_{\#H \leq b} R_H$ that vanishes on C_H for all H with cardinal smaller or equal than b .

In particular, for $b = p$, $g_p = g - \sum_{\#H \leq p} R_H = 0$ is the same as Eq. (1), thus proving the statement (i) of the Theorem.

The ramps R_H are obtained by applying stepwise to g two kind of operations:

1. restrictions of the domain and multiplication by continuous factors smaller or equal than one, which are linear, preserve the continuity and do not increase the supremum of the absolute value, and
2. subtracting from g $\binom{p}{b}$ ramps obtained by the first kind of operations, which preserves the continuity and the linearity.

Let us denote $\rho_b = \sup_{t \in C} |g_{b-1}|$, so that for $\#H = b$, $\sup_{t \in C} |R_H| \leq \rho_b$ and therefore $\rho_{b+1} \leq (1 + \binom{p}{b})\rho_b$. If $\rho = \sup_{t \in C} |g(t)|$ then $\rho_1 \leq (1 + \binom{p}{0})\rho = 2\rho$, $\rho_2 \leq (1 + \binom{p}{1})\rho_1 = 2(p+1)\rho$ and, in general $\rho_b \leq \prod_{j=0}^{b-1} (1 + \binom{p}{j})\rho$. Therefore, the statement (ii) holds with $K_p = \prod_{j=0}^{p-1} (1 + \binom{p}{j})$. $\square$

B. ADDED IN PROOF

An R package for the computation of the tests is provided by the authors at <https://github.com/emcabana/MuniCandS>.

The computation of the minimum of a large collection $(2^p - 1)$ of estimated p-values with a discrete range (step 5 in Section 3.2.1) produces the unwanted collapse of the vanishing empirical powers, as seen in Table 3 for $p \geq 4$. The package MuniCandS improves the estimation of the p-values by means of a continuous interpolation, thus fixing that drawback.

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