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FAILURE EXTROPY FROM A QUANTILE PERSPECTIVE

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SUMMARY

The present paper explores the domain of failure extropy through the lens of quantile-based concepts, introducing novel results to enhance our understanding of extreme events. Failure extropy, the measure of disorder and uncertainty associated with system breakdowns, is a critical aspect of risk assessment and reliability analysis. This study presents a comprehensive investigation into quantile-based methodologies, focusing on their applicability. We introduce a robust nonparametric estimator for failure extropy and lifetime data analysis.

Keywords: Bias; Failure extropy; Mean square error (MSE); Quantile function.

1. INTRODUCTION

This is an era of technological advancements and complex interconnected systems; understanding extreme events and failure modes is inevitable. Failure extropy, a measure capturing the chaos and unpredictability inherent in system breakdowns, is a pivotal concept in risk analysis and reliability engineering. Traditional statistical methods often fail to address the tails of distributions where extreme events lie. The present paper endeavours to contribute to this evolving field by focusing on the quantile-based concepts of failure extropy to enhance the precision of risk assessments.

Recently, Lad *et al.* (2015) showed that the Shannon entropy has a complementary dual function known as extropy. For a non-negative, absolutely continuous random variable X with survival function $\bar{F} = 1 - F$, cumulative distribution function (cdf) F

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and probability density function (pdf) f, the extropy is defined by

$$\eta(X) = -\frac{1}{2}\int_0^\infty f^2(x)dx. \tag{1}$$

Kayal (2021) proposed the definition of extropy as analogous to the definition of the cumulative entropy by Di Crescenzo and Longobardi (2009), which is known as failure extropy. The failure extropy of random variable X with support S, denoted by $\xi_f(X)$ is defined as

$$\xi_f(X) = -\frac{1}{2}\int_S F^2(x)dx. \tag{2}$$

Like extropy, the failure extropy is always negative. If the support of the absolutely continuous random variable is unbounded, then the failure extropy is equal to $-\infty$. For more readings on applications and related concepts of extropy one can refer to Gneiting and Raftery (2007), Furuichi and Mitroi (2012),Vontobel (2013), Qiu (2017), Qiu and Jia (2018), Nair and Sathar (2020), Kamari and Buono (2021), Kayal (2021) and the references therein.

Quantile-based methodologies offer a promising avenue to address the challenges posed by extreme events. Unlike mean-based analyses, which may overlook extreme outcomes, quantile-based approaches focus on specific percentiles of the distribution, providing a more nuanced understanding of tail risks. In this paper, we delve into applying quantile-based concepts to failure extropy, seeking to unravel the intricate dynamics of extreme behaviours within complex systems. When modelling and analysing statistical data, quantile functions are effective and comparable substitutes for the distribution function (see Gilchrist, 2000; Nair *et al.*, 2013). For a random variable X with cdf $F(x)$, the quantile function is defined as

$$Q(u) = F^{-1}(u) = \inf\{x|F(x) \geq u\}, \quad 0 \leq u \leq 1. \tag{3}$$

Quantile functions have several characteristics that are not shared with the distribution functions. Nair and Sankaran (2009), Nair *et al.* (2012), and the references therein provide a thorough study on the quantile function and its properties in modelling and analysis.

Quantile-based extropy concepts are of recent origin. Reliability properties of extropy and its related measures using quantile function are discussed in Krishnan *et al.* (2020) and also Kayal (2021) discussed failure extropy, dynamic failure extropy and their weighted versions. Recently, Vijayan and Sathar (2023) studied the quantile-based cumulative past extropy of order statistics. The estimation of cumulative residual extropy and its quantile version is a recent work by Chakraborty and Pradhan (2023). Now, the literature is active in quantile-based studies of extropy.

The primary objective of this paper is to present a comprehensive exploration of quantile-based methodologies in the context of failure extropy. We aim to contribute novel results that incorporate the interesting properties of quantile functions. Furthermore, we introduce a non-parametric estimator tailored to failure extropy, offering a versatile tool for researchers and practitioners.

The subsequent Sections of this paper are organised to unfold the progression of our research. Section 2 introduces our methodology, and Section 3 presents the dynamic quantile-based failure extropy concept, highlighting some interesting properties. In Section 4, we propose a nonparametric estimator for quantile-based failure extropy and carry out a simulation study and the results obtained through our analyses. Finally, in Section 5, we conclude with a real data application of failure extropy in a quantile scenario and concluding remarks on our work and avenues for future research.

2. Quantile-based failure extropy

In this Section, we introduce the quantile-based failure extropy (QFE) concept and some related results. When F is strictly increasing, we have $FQ(u) = u$, where $FQ(u)$ represents the composite function $F(Q(u))$. Parzen (1979) defined the density quantile function by $fQ(u) = f(Q(u))$ and quantile density function by $q(u) = \frac{d}{du}Q(u)$ and $fQ(u) = [q(u)]^{-1}$.

Definition 1. *An equivalent definition of the failure extropy given in Eq. (2) in terms of QF is given by the quantile failure extropy (QFE)*

$$\xi_Q = -\frac{1}{2}\int_0^1 u^2 q(u)du. \tag{4}$$

Example 2. *When X follows the Govindarajulu distribution with a quantile function*

$$Q(u) = \theta + \sigma\{(\beta+1)u^{\beta} - \beta u^{\beta+1}\},\ \theta, \beta, \sigma > 0, \tag{5}$$

the corresponding quantile density function is $q(u) = \sigma\,\beta\,(\beta+1)\,u^{\beta-1}(1-u)$ *(see Govindarajulu, 1977). Then, the QFE is* $\xi_Q = -\frac{\sigma\beta(\beta+1)}{(\beta+2)(\beta+3)}$.

Example 3. *Let X follow Tukey lambda distribution with quantile function* $Q(u) = \frac{u^{\lambda}-(1-u)^{\lambda}}{\lambda}$ *and quantile density function* $q(u) = u^{\lambda-1} + (1-u)^{\lambda-1}$ *as in Freimer et al. (1988), the failure extropy is* $\xi_Q = -\frac{\lambda^2+\lambda+2}{2\lambda(\lambda+1)(\lambda+2)}$, $\lambda > 0$.

Table 1 shows the failure extropy of some life distributions.

TABLE 1
Failure extropy of some well known distributions.

Distribution	$Q(u)$	ξ_Q
Uniform	bu	$-\frac{b}{6}$
Power	$\alpha u^{\frac{1}{\beta}}; \quad \alpha,\beta>0$	$-\frac{\alpha}{2(1+2\beta)}$
Generalized Tukey Lambda	$\lambda_1+\frac{1}{\lambda_2}\left[\frac{u^{\lambda_3}-1}{\lambda_3}-\frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right]$ $\lambda_3,\lambda_4>0$	$-\frac{1}{2\lambda_2(\lambda_3+2)}-\frac{1}{\lambda_2\lambda_4(\lambda_4+1)(\lambda_4+2)}$
Generalized Weibull	$\sigma\left(\frac{1-(1-u)^{\lambda}}{\lambda}\right)^{\alpha}; \quad \sigma,\alpha,\lambda>0$	$-\frac{\sigma}{2\lambda^{\alpha}}\left[1+\frac{\left(\frac{2}{\lambda}\right)!\alpha!}{\left(\frac{2}{\lambda}+\alpha\right)!}-\frac{2\left(\frac{1}{\lambda}\right)!\alpha!}{\left(\frac{1}{\lambda}+\alpha\right)!}\right]$

The QFE can be reduced to an interesting form as follows

$$\begin{aligned}
\xi_Q &= -\frac{1}{2}\int_0^1 u^2 q(u)du \\
&= \frac{1}{2}\int_0^1 (u-u^2)q(u)du+\frac{1}{2}\int_0^1(1-u)q(u)du-\frac{1}{2}\int_0^1 q(u)du \\
&= \frac{1}{2}L_2+\frac{1}{2}L_1-\frac{1}{2}\int_0^1 dQ(u) \\
&= \frac{L_1+L_2}{2}-\frac{1}{2}Q(1), \qquad (6)
\end{aligned}$$

where $Q(1)$ is the upper support of X and, $L_1 = \int_0^1(1-u)q(u)du$ and $L_2=\int_0^1 u(1-u)q(u)du$ are the first two L-moments of X. For all distributions with infinite supports, ξ_Q is $-\infty$. This does not pose a big issue in estimation problems, since $Q(1)$ is replaced by $X_{(n)}$, the largest order statistics. Further L_1 is the population mean and L_2 is the half of the Gini's mean difference. Thus the estimate of L_1 is the sample mean and that of L_2 is $l_2=\frac{1}{n(n-1)}\sum_{i=1}^{n}(2i-1-n)X_{i:n}$.

3. Quantile-based dynamic failure extropy

In contrast to static approaches, dynamic QFE integrates the concept of quantiles, which provide a robust statistical measure of the distribution of failure events, with the dynamic nature of systems. This innovative approach acknowledges that failure extropy, the measure of disorder or uncertainty associated with failures, is not a static phenomenon but rather a dynamic process influenced by changing conditions, external factors, and system interactions.

For a lifetime random variable X, the dynamic failure extropy is defined as (see

Kayal, 2021)

$$\mathscr{E}_f(X;t) = -\frac{1}{2}\int_0^t \left(\frac{F(x)}{F(t)}\right)^2 dx, t > 0, F(t) > 0.$$

At time t, if the system is down, $\mathscr{E}_f(X;t)$ quantifies the uncertainty about its past life; hence, it is also called past failure extropy. Clearly, $\mathscr{E}_f(X;0) = 0$ and $\mathscr{E}_f(X;\infty) = \mathscr{E}_f(X)$.

DEFINITION 4. *For an absolutely continuous non-negative random variable X, the quantile-based dynamic failure extropy (QDFE) is defined as*

$$\xi_Q(u) = -\frac{1}{2u^2}\int_0^u p^2 q(p) dp. \tag{7}$$

EXAMPLE 5. *If X follow the Govidarajulu distribution with quantile function in Eq. (5), then the QDFE is obtained as $\xi_Q(u) = -\frac{\sigma\beta(\beta+1)u^\beta[\beta+3-u(\beta+2)]}{2(\beta+2)(\beta+3)}$.*

EXAMPLE 6. *Consider a random variable X having Pareto distribution with quantile density function $q(u) = \frac{1}{ac}(1-u)^{-\frac{1}{c}-1}$. Then the corresponding dynamic failure extropy is* $\xi_Q(u) = -\frac{ac}{2u^2}\left[\frac{c}{c+1} - \frac{2c}{2c+1} + \frac{3c}{3c+1} - \frac{c(1-u)^{\frac{1}{c}+1}}{c+1} + \frac{2c(1-u)^{\frac{1}{c}+2}}{2c+1} - \frac{c(1-u)^{\frac{1}{c}+3}}{3c+1}\right]$.

We now discuss various properties and implications of the dynamic extropy. Since the notion of past life is associated with lifetime data analysis, we recall two important concepts from reliability and survival analysis. The reversed hazard rate of X is $\lambda(x) = \frac{f(x)}{\bar{F}(x)}$, whose quantile version is $\Lambda(u) = \lambda(Q(u)) = [uq(u)]^{-1}$ and the mean waiting time (reversed residual life) is $r(x) = \frac{1}{F(t)}\int_0^x F(x)dx$ whose equivalent is $R(u) = r(Q(u)) = \frac{1}{u}\int_0^u pq(p)dp$. We present some identities that enable the calculations of $\xi_Q(u)$ from $\Lambda(u)$ or $R(u)$ and vice-versa. From Eq. (7),

$$\xi_Q(u) = -\frac{1}{2u^2}\int_0^u \frac{p}{\Lambda(p)} dp \tag{8}$$

or

$$\Lambda(u) = -\frac{1}{2}[2\xi_Q(u) + u\xi_Q'(u)]^{-1}. \tag{9}$$

Using the result $(\Lambda(u))^{-1} = R(u) + uR'(u)$ (Nair *et al.*, 2013), where $R(u) = u^{-1}\int_0^u pq(p)dp$ is the reversed mean residual quantile function, the identity in Eq. (9) becomes

$$\begin{aligned}
\xi_Q(u) &= -\frac{1}{2u^2}\int_0^u p(R(p) + pR'(p))dp \\
&= -\frac{1}{2u^2}\left(\int_0^u pR(p)dp + \int_0^u p^2 R'(p)dp\right).
\end{aligned}$$

Upon performing integration by parts in the second term, we get

$$\xi_Q(u) \;=\; -\frac{1}{2\,u^2}\left(u^2R(u) - \int_0^u pR(p)dp\right).$$

Further,

$$R(u) = -\frac{2}{u}\left(\xi_Q(u) + \int_0^u \xi_Q(p)dp\right).$$

Nair and VineshKumar (2010) obtained the first two L—moments of reversed residual life as

$$\theta_1(u) = \frac{1}{u}\int_0^u Q(p)\,dp \tag{10}$$

and

$$\theta_2(u) = \frac{1}{u^2}\int_0^u (2p-u)Q(p)\,dp. \tag{11}$$

We now present an identities connecting $\xi_Q(u)$, $\theta_1(u)$ and $\theta_2(u)$.

THEOREM 7. *Let X be an absolutely continuous non-negative random variable with QDFE $\xi_Q(u)$. Then, the following identity holds:*

$$\xi_Q(u) = \frac{\theta_2(u) - u\,\theta_1'(u)}{2}, \tag{12}$$

where $\theta_1'(u)$ is the derivative of $\theta_1(u)$.

PROOF. *From Nair and VineshKumar (2010), we have the identities*

$$R(u) = u\,\theta_1'(u) \quad \textit{and} \quad \theta_2(u) = \frac{1}{u^2}\int_0^u p\,R(p)\,dp. \tag{13}$$

Now, by plugging Eq. (13) in Eq. (10), we get the desired result. □

REMARK 8. *We can equivalently express Eq. (12) as*

$$\xi_Q(u) = -\frac{\theta_2(u) + u\,\theta_2'(u)}{2}, \tag{14}$$

which requires only the second L—moments of reversed residual life to determine $\xi_Q(u)$.

THEOREM 9. *Let X be a non-negative continuous random variable. Then, the identity*

$$\xi_Q(u) = C\,R(u),\; C \in \mathscr{R} \tag{15}$$

holds if and only if X has a power distribution of the form $Q(u) = \alpha\,u^{\frac{1}{\beta}},\;\; \alpha,\beta > 0$.

PROOF. *Assume X follow power distribution. Then, we have*

$$\theta_2(u) = \left(\frac{\alpha\,\beta}{(\beta+1)(2\,\beta+1)}\right)\, u^{\frac{1}{\beta}} \tag{16}$$

and

$$R(u) = \left(\frac{\alpha}{\beta+1}\right)\, u^{\frac{1}{\beta}}. \tag{17}$$

Now Eq. (12) gives

$$\xi_Q(u) = -\left(\frac{\alpha\, u^{\frac{1}{\beta}}}{2(2\,\beta+1)}\right), \tag{18}$$

so that Eq. (15) is satisfied with $C = -\left(\frac{\beta+1}{2(2\,\beta+1)}\right)$. *Conversely since* $\xi_Q(u) = C\,R(u)$, *from Equations (12) and (13), we have*

$$\theta_2(u) = (2\,C+1)\,R(u). \tag{19}$$

Now, using the identity $R(u) = u\,\theta'(u) + 2\,\theta_2(u)$, *Eq. (19) becomes*

$$(3-4C)\theta_2(u) = (2\,C+1)\,u\,\theta_2'(u). \tag{20}$$

The above first-order differential equation is in variable separable form, and it yields the solution as

$$\theta_2(u) = K\; u^{\frac{3-4\,C}{2C+1}}, \tag{21}$$

which is of the form of Eq. (16) with

$$K = \frac{\alpha\,\beta}{(\beta+1)(2\,\beta+1)} \quad \text{and} \quad C = \frac{3\,\beta-1}{4\,\beta+2}. \tag{22}$$

This completes the proof. □

Next theorem discusses a characterization result that the QDFE uniquely determines the distribution. This enables us to identify the lifetime distribution from the functional form of $\xi_Q(u)$.

THEOREM 10. *Let X be an absolutely continuous non-negative random variable with quantile function* $Q(u)$ *and quantile density function* $q(u)$. *Then, the QDFE* $\xi_Q(u)$ *uniquely determines the distribution of* X, *as*

$$\begin{aligned} q(u) &= -2\xi_Q'(u) - \frac{4}{u}\xi_Q(u) \\ &= -\frac{1}{u^2}\frac{d}{d\,u}\left[u^2\xi_Q(u)\right]. \end{aligned} \tag{23}$$

PROOF. The proof is obtained by differentiating Eq. (7). □

Some implications of Theorem 10 are the following.

1. An important observation is that all kinds of functions cannot be quantified as failure extropy of an absolutely continuous non-negative random variable.

   1.1. It is easy to see that ξ_Q is always negative. Further

   $$uq(u) \geq u^2 q(u) \Rightarrow -\frac{1}{2u^2}\int_0^u pq(p)dp \leq -\frac{1}{2u^2}\int_0^u p^2 q(p)dp,$$

   so that

   $$-\frac{1}{2u}R(u) \leq \xi_Q(u).$$

   Thus, QDFE lies in $\left(-\frac{1}{2u}R(u), 0\right)$.

   1.2. The differential Equation (9) means that the function $\xi_Q(u)$ has a change point at $\xi_Q'(u) = 0$ which is at the value of u for which $\xi_Q(u) = -\frac{1}{4\Lambda(u)}$. From Eq. (8), this value of $\xi_Q(u)$ must satisfy

   $$-\frac{1}{4\Lambda(u)} = -\frac{1}{2u^2}\int_0^u \frac{p}{\Lambda(p)}dp.$$

   On differentiation and simplification, this leads to $\Lambda'(u) = 0$ or $\Lambda(u) = k$, a constant. Since there is no constant reversed hazard quantile function for an absolutely continuous non-negative random variable, a change point does not exist for $\xi_Q(u)$. Thus $\xi_Q(u)$ is a decreasing function. Since $u = F(x)$, whenever x increases so does u and hence the monotonicity of $\xi_Q(u)$ and $\xi_F(t)$ are the same. The classification of life distributions into increasing and decreasing $\xi_f(X,t)$ does not hold as claimed in Nair and Sathar (2020). Other results in the same work based on monotonic dynamic failure extropy seem to hold modifications see also Kayal (2021).

   1.3. Certain functions of u are not admissible as $\xi_Q(u)$. For example, $\xi_Q(u)$ cannot be a constant; otherwise, it would mean that

   $$q(u) = \frac{4K}{u}, \quad K > 0$$

   or $Q(u) = 4K \log u$, taking $\log 0 = 0$. This gives $F(x) = e^{\frac{x}{4K}}$, which is not a distribution function in $(0, \infty)$.

1.4. Theorem 10 can be used to generate new distributions with convenient forms of $\xi_Q(u)$. Since $\xi(u)$ is a decreasing function, two popular decreasing functional forms are linear and exponential. Taking $\xi_Q(u) = au + b, a \leq 0$ and $b \leq 0$, we get after some algebra

$$q(u) = -6\,b - \frac{4a}{u}$$

or

$$Q(u) = -6\,b\,u - 4a\log u,$$

which subsumes the power distribution as $a = 0$. Note that this distribution has no tractable distribution function to make use of its analytical properties using definition $\xi_f(X,t) = -\frac{1}{2}\int_0^t \left[\frac{F^2(x)}{F(t)}\right]^2 dx$ in the distribution function approach. Now, considering $\xi_Q(u) = ae^{-bu}, b > 0$, we get

$$q(u) = \frac{2a}{u^2}(b\,u^2 - 2u)e^{-bu},$$

which does not produce a proper quantile function. Hence, $\xi_Q(u)$ does not produce an exponential decay. It must be mentioned here that for most standard distributions $\xi_Q(u)$ has complicated forms that may not be suitable in real data problems. Empirically, we get an idea of the functional form of $\xi_Q(u)$, and this can be the starting point for choosing the model.

2. There are several advantages arising from the quantile-based approach discussed in the present work, of which some illustrations are given below.

2.1. The affine transformation $Y = AX + B$, takes the quantile function of Y to $Q_Y(u) = AQ_X(u) + B$, so that $q_Y(u) = Aq_X(u)$. Thus $\xi_{Q_Y}(u) = A\xi_{Q_X}(u)$. In data analysis problems, one can add or subtract any constant from all the observations without affecting the value of the DQFE.

2.2. If $Q_1(u)$ and $Q_2(u)$ are two quantile functions, their sum $Q(u) = Q_1(u) + Q_2(u)$ is also a quantile function. This leads to

$$\xi_Q(u) = \xi_{Q_1}(u) + \xi_{Q_2}(u).$$

Further, the mean inactivity times also satisfy

$$R_Q(u) = R_{Q_1}(u) + R_{Q_2}(u),$$

so that the bounds of $\xi_Q(u)$ becomes $(-\frac{2}{u}R_Q(u), 0)$.

In modelling problems using quantile functions when a candidate model $Q_1(u)$ does not provide a satisfactory fit, it is a practice to add an appropriate quantile function $Q_2(u)$ band as the fitted values of $Q_1(u)$ to resolve the model adequacy issue. Many quantile functions available in literature can be derived in this fashion. For example, the well known generalized lambda distribution

$$Q(u) = \lambda_1 + \frac{u^{\lambda_3} + (1-u)^{\lambda_4}}{\lambda_2},$$

arises as a result of the affine transformation in (a) and the addition of the quantile functions of the power distribution $Q_1(u) = \frac{u^{\lambda_3}}{\lambda_2}$ and the pareto distribution $Q_2(u) = \frac{(1-u)^{\lambda_4}}{\lambda_2}$. To find the corresponding failure entropies, one needs to apply the addition formula mentioned above and avoid fresh calculations for the new model.

2.3. If $Q_1(u)$ and $Q_2(u)$ are two positive quantile functions, then we have $Q(u) = Q_1(u)Q_2(u)$ is also a quantile functions. Here, we obtain, $\xi_Q(u) = -\frac{1}{2u^2}\int p^2(q_1(p)Q_2(p) + q_2(p)Q_1(p))dp$. If ϕ is non-negative, increasing function of X, then

$$\xi_{Q_{\phi(X)}}(u) = -\frac{1}{2u^2}\int_0^u p^2 q_X(p)\phi' Q(p)dp.$$

2.4. If X has quantile function $Q(u)$, then its reciprocal $\frac{1}{X}$ has quantile function $\frac{1}{Q(1-u)}$, the corresponding failure extropy can be defined as $\xi_{Q_{\frac{1}{X}}}(u) = -\frac{1}{2u^2}\int_0^u \frac{p^2 q(1-p)}{Q^2(1-p)}dp$.

2.5. If $g(y)$ is the pdf of $Y = \phi(X)$, ϕ is a monotonic function, then $g(y) = \frac{f(\phi^{-1}(y))}{|\phi'\phi^{-1}(y)|} \Rightarrow g(Q(u)) = \frac{1}{q_X(u)\phi' Q(u)}$, or simply we can use the fact that $\phi(Q_X(u))$ is the quantile function of Y. Now,

$$\xi_{Q_Y}(u) = -\frac{1}{2u^2}\int_0^u p^2 q_Y(p)dp \tag{24}$$

$$= -\frac{1}{2u^2}\int_0^u p^2 q_X(p)\phi' Q(p)dp. \tag{25}$$

2.6. We can define the quantile density in terms of hazard quantile function (see Nair *et al.*, 2013) $q(u) = \frac{1}{(1-u)H(u)}$. Now, we can express the definition of failure extropy in terms of hazard quantile function as

$$\xi_Q(u) = -\frac{1}{2u^2}\int_0^u p^2 q(p)dp \tag{26}$$

$$= -\frac{1}{2u^2}\int_0^u p^2 \frac{1}{(1-p)H(p)}dp \tag{27}$$

$$= \frac{1}{2u^2}\int_0^u \frac{(p+1)}{H(p)}dp - \frac{1}{2u^2}\int_0^u \frac{1}{(1-p)H(p)}dp. \tag{28}$$

REMARK 11. *Since the failure extropy is a special case of the dynamic version when $u \longrightarrow 1$, all the results in this Section subsume those of the ξ_Q.*

4. NON-PARAMETRIC ESTIMATION

In this Section, we focus on deriving an estimator for the quantile-based measure ξ_Q, leveraging L-moments, which are robust alternatives to conventional moments. As pointed out in the earlier Sections, we have the identity connecting ξ_Q and the first two L-moments as

$$\xi_Q = \frac{L_1+L_2}{2} - \frac{1}{2}Q(1), \tag{29}$$

where L_1 and L_2 are the first two $L-$moments are given by Hosking and Wallis (1990). Using the above relation with $L-$moments, we propose a distribution-free estimator of ξ_Q based on a random sample of size n from the population by replacing L_1 and L_2 by their sample counterparts. The first and second sample $L-$moments are expressed as

$$\hat{L}_1 = l_1 = \frac{1}{n}\sum_{i=1}^{n} X_{(i)} \quad \text{and} \tag{30}$$

$$\hat{L}_2 = l_2 = \frac{1}{n(n-1)}\sum_{i=1}^{n}(2i-1-n)X_{(i)}, \tag{31}$$

where, $X_{(i)}$ is the i^{th} order statistic.

Using Eq. (30) and Eq. (31), the non-parametric estimator for $\xi_Q(X)$ becomes

$$\hat{\xi}_Q(X) = \frac{\hat{L}_1 + \hat{L}_2 - X_{(n)}}{2}. \tag{32}$$

Note that $\hat{\xi}_Q(X)$ can be expressed as a linear function of the order statistics in the form

$$\hat{\xi}_Q(X) = \sum_{i=1}^{n} C_i X_{(i)}, \quad \text{where} \quad C_i = \begin{cases} \frac{i-1}{n(n-1)} & i = 1, 2, \cdots n-1 \\ \frac{2-n}{2n} & i = n \end{cases}. \tag{33}$$

We invoke the following Theorem 12 to write the asymptotic distribution of $\hat{\xi}_Q(X)$.

THEOREM 12 (SEE SECTION 4 IN STIGLER, 1969). *Assume that*

(i) the baseline distribution has a probability density function that is continuous and positive throughout its support and with smooth tails,

(ii) the variance of the distribution has the same order of the variance of our linear function with positive coefficients and

(iii) the largest order statistics do not unduly influence the statistic.

Under these conditions, the statistic

$$S_n = \sum_{i=1}^{n} C_i X_{(i)}$$

is asymptotically normal $N(ES_n, Var(S_n))$.

The distribution function of the r^{th} order statistic is known to be

$$\begin{aligned} F_{X_{(r)}}(x) &= \sum_{k=1}^{r} \binom{n}{k} F,^{k}(x)(1-F(x))^{n-k} \\ &= I_{u_r}(r, n-r+1) \end{aligned} \quad (34)$$

where

$$I_{u_r}(m,n) = \frac{\int_0^{u_r} t^{m-1}(1-t)^{n-1} dt}{\int_0^1 t^{m-1}(1-t)^{n-1} dt}$$

is the incomplete beta ratio. Accordingly, the quantile function of $X_{(r)}$ is

$$Q_r(u_r) = Q(I_{u_r}^{-1}(r, n-r+1), \quad (35)$$

where $u_r = F_{X(r)}(x)$ and $I_{u_r}^{-1}$ is the inverse of I_r. When no assumption about F is made, $Q_r(\cdot)$ in Eq. (35) is replaced by the empirical quantile function

$$\bar{Q}_n(u) = F_n^{-1}(u) = \inf[x | F_n(x) \geq u],$$

where F_n is the empirical distribution function.

$$\bar{Q}_n(u) = X_{(t)}, \quad \frac{t-1}{n} \leq u < \frac{t}{n}.$$

The mean and variance of the estimator S_n has the form

$$E(S_n) = \sum_{i=1}^{n} C_i E(X_{(i)})$$

and

$$\text{Var}(S_n) = \sum_{i=1}^{n} C_i^2 \text{Var}(X_{(i)}) + 2 \sum_{i,j=1(i \neq j)}^{n} C_i C_j \text{Cov}(X_{(i)}, X_{(j)}).$$

These are evaluated from the sample using

$$E(X_{(i)}) = \int_0^1 Q(I_{u_i}^{-1}(i,n-i+1))du_i, \tag{36}$$

$$E(X_{(i)}^2) = \int_0^1 Q^2(I_{u_i}^{-1}(i,n-i+1))du_i \tag{37}$$

and

$$E(X_{(i)}X_{(j)}) = \int_0^1 \int_0^1 Q(I_{u_i}^{-1}(i,n-i+1))\,Q(I_{u_j}^{-1}(j,n-j+1))du_i du_j. \tag{38}$$

We now consider few examples by assuming specific distributions from which the sample arises.

EXAMPLE 13. *Consider $X_1, X_2, \ldots, X_n$ as a random sample from a Uniform(0,1) distribution. Now, since $X_{(i)}$ follows beta distribution with parameters i and $n-i+1$, we have*

$$E[X_{(i)}] = \frac{i}{n+1}, \quad Var(X_{(i)}) = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

Applying this into Theorem 12, estimator in Eq. (33) follows a normal distribution with

$$E[\hat{\xi}_Q(X)] = \sum_{i=1}^{n} C_i E[X_{(i)}] = \sum_{i=1}^{n} C_i \left(\frac{i}{n+1}\right),$$

which simplifies to

$$E[\hat{\xi}_Q(X)] = \frac{1}{n+1}\sum_{i=1}^{n} C_i i = \frac{2-n}{6(n+1)},$$

and for variance,

$$Var(\hat{\xi}_Q(X)) = \sum_{i=1}^{n} C_i^2 \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

Upon taking the limit, we obtain

$$\lim_{n\to\infty} E[\hat{\xi}_Q(X)] = -\frac{1}{6}, \quad \lim_{n\to\infty} Var(\hat{\xi}_Q(X)) = 0.$$

This shows the consistency of $\hat{\xi}_Q(X)$ in the case of Uniform distribution .

The histogram in Figure 1 shows the sampling distribution of the proposed estimator based on 1000 Monte Carlo replications for Example 13. The red curve represents the normal density fitted using the sample mean and variance of the estimator. The approximate bell shape and close alignment with the normal curve provide visual evidence supporting the asymptotic normality of the estimator, as stated in Theorem 12.

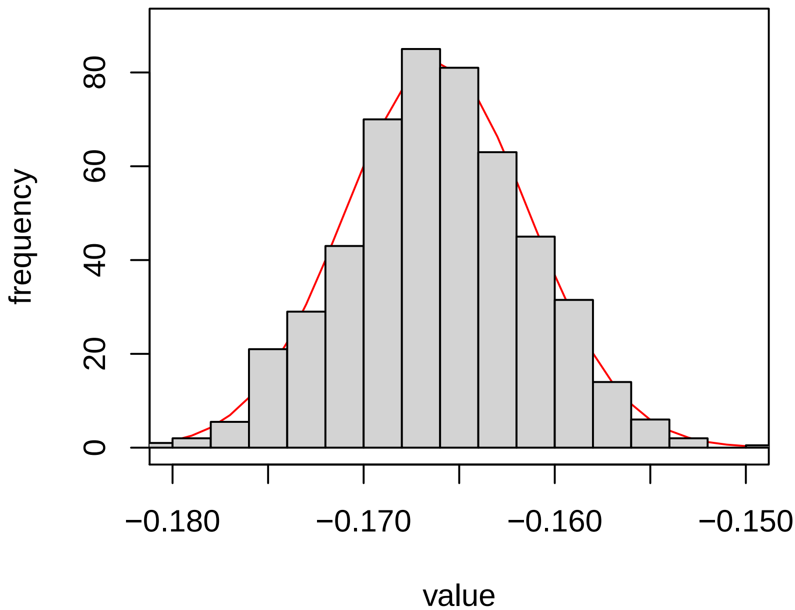

Figure 1 – Histogram of the Estimator in Example 4.1 with Normal Density Overlay Illustrating Asymptotic Normality.

EXAMPLE 14. *Consider $X_1, X_2, \dots, X_n$ as a random sample from a Govindarajulu distribution with parameters $\theta = 0$, $\sigma = 1$ and $\beta = 1$ in Eq. (5), quantile function $Q(u) = 2u - u^2$. Then,*

$$E(X_{(r)}) = \frac{2nr + 3r - r^2}{(n+1)(n+2)}$$

and

$$Var(X_{(r)}) = \frac{4r(r+1)}{(n+1)(n+2)}\left(1 - \frac{r+2}{n+3}\left[1 - \frac{r+3}{4(n+4)}\right]\right) - \frac{(2nr+3r-r^2)^2}{(n+1)^2(n+2)^2}.$$

Applying this into Theorem 12, estimator in Eq. (33) follows a normal distribution with

$$E[\hat{\xi}_Q(X)] = \frac{1}{(n+1)(n+2)} \sum_{i=1}^{n} C_i (2ni + 3i - i^2)$$

and variance

$$Var(\hat{\xi}_Q(X)) = \sum_{i=1}^{n} C_i^2 \left\{ \frac{4i(i+1)}{(n+1)(n+2)} \left[1 - \frac{i+2}{n+3} \left(1 - \frac{i+3}{4(n+4)} \right) \right] - \frac{(2ni+3i-i^2)^2}{(n+1)^2(n+2)^2} \right\}.$$

Now, as $n \to \infty$, $E[\hat{\xi}_Q(X)] = 0$ *and* $Var(\hat{\xi}_Q(X)) = 0$, *shows the consistency.*

The histogram given in Figure 2 displays the empirical distribution of the estimator computed from 1000 repeated samples of the model considered in Example 14. The proposed normal curve has been overlaid on the histogram and this confirms that the estimator behaves approximately normally for large samples.

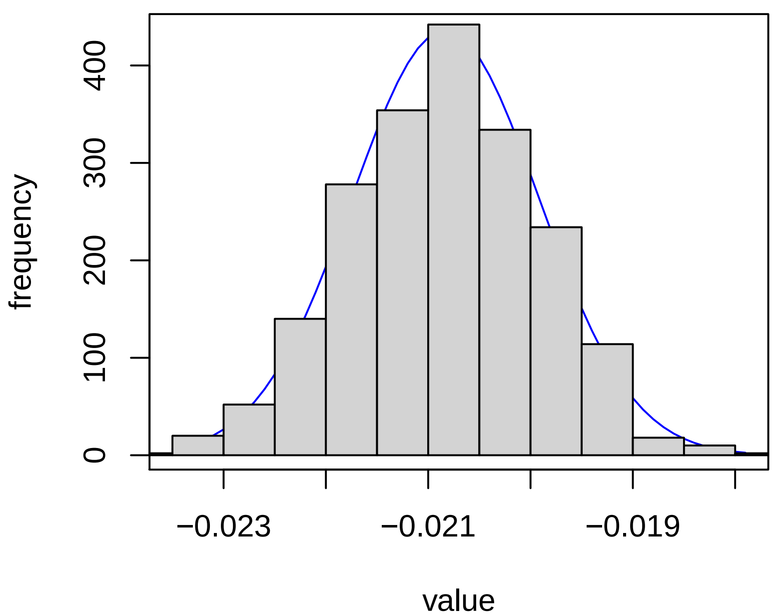

Figure 2 – Histogram of the Estimator in Example 4.2 with Normal Density Overlay Illustrating Asymptotic Normality.

To examine the performance of the proposed estimator for failure extropy an extensive simulation study has been carried out for different parametric models. Specifically, we consider three families of distributions:

(i) the Govindarajulu distribution (Govindarajulu, 1977),

(ii) the Generalized Tukey Lambda distribution (Freimer *et al.*, 1988),

(iii) the Generalized Weibull model (Mudholkar *et al.*, 1996),

each with varying parameter combinations. For each model, we assess the absolute bias and mean squared error (MSE) of the estimator based on random samples of different sizes ($n = 50, 100, 200, 500, 1000$). These measures provide insights into the accuracy and efficiency of the estimator as the sample size increases and under varying distributional characteristics. The simulation results demonstrate the robustness and effectiveness of the proposed non-parametric estimator across various parametric models and sample sizes. As expected, the absolute bias and MSE decrease consistently with increased sample size, highlighting the estimator's asymptotic properties. Among the considered models, the performance varies depending on the underlying distribution parameters, but the estimator remains reliable and well-suited for practical applications. The results are shown through Table 2 to 4. In all these cases, the asymptotic results were met for samples of size 50.

We acknowledge that the simulated MSE values appear to be exceptionally low; however, the MATHEMATICA code used to generate them has been thoroughly verified by both ourselves and the editor, and no errors were identified. The code is publicly available on GitHub at: `https://github.com/silpasubhash112/Quantile-based-Failure-extropy.git`.

TABLE 2
Absolute bias and MSEs of failure extropy for Govindarajulu model.

Model Govindarajulu(σ, β)	n	Ab. bias	MSE
$\left(\frac{1}{8}, \frac{1}{16}\right)$	20	20.2×10^{-5}	65.1×10^{-9}
	50	12.4×10^{-5}	24.7×10^{-9}
	100	8.95×10^{-5}	12.7×10^{-9}
	200	6.31×10^{-5}	6.39×10^{-9}
	500	4.18×10^{-5}	2.75×10^{-9}
	1000	2.9×10^{-5}	1.37×10^{-9}
$\left(\frac{1}{8}, \frac{1}{8}\right)$	20	39.9×10^{-5}	252.4×10^{-9}
	50	24.6×10^{-5}	96.0×10^{-9}
	100	17.7×10^{-5}	49.6×10^{-9}
	200	12.5×10^{-5}	25.0×10^{-9}
	500	8.25×10^{-5}	10.7×10^{-9}
	1000	5.74×10^{-5}	5.30×10^{-9}
$\left(\frac{1}{16}, \frac{1}{8}\right)$	20	19.9×10^{-5}	63.10×10^{-9}
	50	12.3×10^{-5}	24.0×10^{-9}
	100	8.8×10^{-5}	12.4×10^{-9}
	200	6.24×10^{-5}	6.25×10^{-9}
	500	4.13×10^{-5}	2.69×10^{-9}
	1000	2.87×10^{-5}	1.34×10^{-9}
$\left(\frac{1}{32}, \frac{1}{8}\right)$	20	9.97×10^{-5}	157.75×10^{-10}
	50	6.14×10^{-5}	60.01×10^{-10}
	100	4.43×10^{-5}	31.00×10^{-10}
	200	3.12×10^{-5}	15.63×10^{-10}
	500	2.06×10^{-5}	6.71×10^{-10}
	1000	1.43×10^{-5}	3.34×10^{-10}

TABLE 3
Absolute bias and MSEs of failure extropy for Generalized Tukey Lambda model.

Model Tukey$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	n	Ab. bias	MSE
	20	52.9×10^{-4}	378.53×10^{-7}
	50	25.4×10^{-4}	97.92×10^{-7}
	100	14.8×10^{-4}	35.58×10^{-7}
$(5, 4, 5, 4)$	200	9.3×10^{-4}	13.52×10^{-7}
	500	5.3×10^{-4}	4.26×10^{-7}
	1000	3.4×10^{-4}	1.74×10^{-7}
	20	121.2×10^{-4}	205.31×10^{-6}
	50	60.08×10^{-4}	55.36×10^{-6}
	100	37.29×10^{-4}	22.32×10^{-6}
$(1, 2, 3, 4)$	200	25.26×10^{-4}	9.97×10^{-6}
	500	15.02×10^{-4}	3.50×10^{-6}
	1000	10.23×10^{-4}	1.64×10^{-6}
	20	125.8×10^{-4}	223.62×10^{-6}
	50	64.3×10^{-4}	63.41×10^{-6}
	100	40.87×10^{-4}	26.71×10^{-6}
$(2, 2, 3, 3)$	200	28.1×10^{-4}	12.38×10^{-6}
	500	17.0×10^{-4}	4.49×10^{-6}
	1000	11.66×10^{-4}	2.13×10^{-6}
	20	57.9×10^{-3}	484.32×10^{-5}
	50	30.5×10^{-3}	146.34×10^{-5}
	100	20.4×10^{-3}	66.53×10^{-5}
$(5, 1, 2, 1)$	200	14.3×10^{-3}	32.88×10^{-5}
	500	8.9×10^{-3}	12.39×10^{-5}
	1000	6.2×10^{-3}	5.98×10^{-5}

TABLE 4
Absolute bias and MSEs of failure extropy for Generalized Weibull model.

Model Gen. Weibull$(\sigma, \lambda, \alpha)$	n	Ab. bias	MSE
	20	36.4×10^{-5}	207.9×10^{-9}
	50	22.7×10^{-5}	80.1×10^{-9}
	100	16.5×10^{-5}	42.4×10^{-9}
$(1,4,3)$	200	11.6×10^{-5}	21.4×10^{-9}
	500	7.53×10^{-5}	9.00×10^{-9}
	1000	5.27×10^{-5}	4.5×10^{-9}
	20	16.9×10^{-7}	444.8×10^{-14}
	50	10.5×10^{-7}	172.0×10^{-14}
	100	7.64×10^{-7}	91.2×10^{-14}
$(1,5,6)$	200	5.39×10^{-7}	46.2×10^{-14}
	500	3.47×10^{-7}	19.3×10^{-14}
	1000	2.44×10^{-7}	9.6×10^{-14}
	20	15.0×10^{-6}	350.6×10^{-12}
	50	9.41×10^{-6}	137.7×10^{-12}
	100	6.89×10^{-6}	73.9×10^{-12}
$(2,4,6)$	200	4.84×10^{-6}	37.4×10^{-12}
	500	3.13×10^{-6}	15.5×10^{-12}
	1000	2.20×10^{-6}	7.67×10^{-12}
	20	67.42×10^{-7}	71.2×10^{-12}
	50	42.07×10^{-7}	27.5×10^{-12}
	100	30.55×10^{-7}	14.6×10^{-12}
$(4,5,6)$	200	21.54×10^{-7}	7.39×10^{-12}
	500	13.89×10^{-7}	3.08×10^{-12}
	1000	9.76×10^{-7}	1.5×10^{-12}

5. Data analysis

In this Section, we present the results of a real data analysis using two datasets: lifetimes of blood cancer patients and spike times observed in trials on a single neuron. These datasets provide valuable insights into the behaviour of systems subject to failure and are essential for evaluating the performance of failure extropy estimators.

(i) Lifetimes of Blood Cancer Patients:

The lifetimes of blood cancer patients, as reported by Abu-Youssef (2002), are listed below:

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

We first compute the QFE using our proposed estimator. For this dataset, we obtain a QFE of −219.297. A higher value of QFE indicates more variability or uncertainty in the failure times near the extremes. This is crucial for identifying systems with unpredictable or erratic behaviour in high-risk scenarios. Similarly, a small value reflects more concentrated or predictable behaviour in failure times, which can suggest robustness or stability in reliability applications. QFE can aid in the variability of survival times for patients, especially in clinical decision-making. It helps characterize the uncertainty associated with the tail distribution of lifetimes, which is critical in evaluating treatment efficacy and planning follow-up care.

(ii) Spike Times of a Neuron:

The spike times observed in 8 trials on a single neuron, as documented by Kass *et al.* (2003), are as follows:

136.842, 145.965, 155.088, 175.439, 184.561, 199.298, 221.053, 231.579, 246.316, 263.158, 274.386, 282.105, 317.193, 329.123, 347.368, 360.702, 368.421, 389.474, 392.982, 432.281, 449.123, 463.86, 503.86, 538.947, 586.667, 596.491, 658.246, 668.772, 684.912.

We compute the QFE for this dataset and obtain a value of −111.474. Comparing our result with the findings of Kayal (2021), who utilized their estimators in the distributional approach and reported failure extropies as −222.752 and −113.135 respectively for the same dataset, which is almost the same in our analysis.

The data analysis reveals insights into the failure behaviour of blood cancer patients and the firing patterns of neurons. The QFE estimate indicates a high level of asymmetry and variability in the distribution of lifetimes among blood cancer patients. Clinically, this emphasizes the importance of individualized patient care and the need to account for variability in treatment planning and prognosis. The magnitude of QFE highlights the uncertainty in predicting survival outcomes in this patient cohort. By estimating the QFE, we gain valuable information about the reliability and risk associated with these systems.

6. Conclusion

In this article, we have introduced a quantile version of failure extropy along with its dynamic form and its important properties. We have proved that the QDFE uniquely determine the distribution of the random variable. We proposed a non-parametric estimator for QFE and comprehensively evaluated its performance through simulation studies and real-life data analysis. Our simulation study involved generating samples from various parametric combinations of three distributions: the Govindarajulu distribution, the Generalized Weibull distribution, and the Generalized Tukey Lambda distribution. By computing the absolute bias and mean squared errors (MSEs) of the failure extropy estimators across different sample sizes, we assessed the accuracy and robustness of our proposed estimator.

As future work, we aim to extend the estimation of failure extropy to its dynamic version, addressing cases involving censored data and left-truncated and right-censored (LTRC) scenarios. These extensions are crucial for enhancing the applicability and robustness of our quantile-based failure extropy framework in real-world situations. Additionally, we plan to investigate the stochastic ordering of failure extropy, which will involve studying its properties under different stochastic orders. This line of inquiry has the potential to reveal deeper insights into the comparative behaviour of different distributions in terms of their failure extropy, offering valuable applications in reliability analysis and risk assessment.

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