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NONPARAMETRIC ESTIMATION OF QUANTILE-BASED MEAN INACTIVITY TIME FUNCTION

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SUMMARY

In this article, we propose non-parametric estimators for mean inactivity time function for complete and censored data. The asymptotic properties of the estimators are established using suitable regularity conditions. Monte Carlo simulation studies are used to study the efficiency of the estimators. Three real data sets are used to demonstrate the usefulness of the estimation procedure.

Keywords: Mean inactivity time function; Kernel density estimation; Quantile function; Mean squared error.

1. INTRODUCTION

Let X be an absolutely continuous random variable with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The inactivity time was defined by Ruiz and Navarro (1996) as a conditional random variable, in parallel with the residual life as a conditional random variable $X_{(t)} = t - X|X \leq t$, in parallel with the residual life $X_t = (X - t|X > t)$. The random variable $X_{(t)}$ is also known as reversed residual life or inactivity time. It represents the time elapsed since the failure of a unit given that its lifetime is at most t. The mean of inactivity time (MIT) or expected inactivity time (EIT) or Mean Past Life function (MPL) of $X_{(t)}$ denoted by $\widetilde{m}(t)$, is defined as

$$\widetilde{m}(t) = E\left(X_{(t)}\right) = \frac{1}{F(t)}\int_0^t (t-u)f(u)du = \frac{1}{F(t)}\int_0^t F(u)du, \quad t > 0, \tag{1}$$

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$\widetilde{m}(t)$ measures the average (expected) time elapsed when failure of the device or organism already occurs at time t. Suppose at a specific checking time $t > 0$, and the system has stopped working, then Eq. (1) can be used to compute the average inactivity time of the system. Apart from applications in reliability, MIT plays a vital role in engineering, survival studies, economics, forensic studies, astronomy etc. Furthermore, in actuarial science, the MIT is used to calculate the optimum premiums for life insurance policies.

Mean inactivity time has generated considerable interest among different researchers. Navarro *et al.* (1997) introduced a new stochastic order, namely the reversed mean residual life order, and studied its usefulness in the comparison of two random variables. Finkelstein (2002) used MIT for describing different maintenance policies in reliability. Kayid and Ahmad (2004) and Ahmad *et al.* (2005) have established several properties of stochastic comparison using MIT order. Asadi (2006) examined the importance of MIT of components in parallel systems. Kundu *et al.* (2010) obtained certain characterizing properties of probability models using MIT, while Kundu and Nanda (2010) derived some reliability properties of it. Khan *et al.* (2021) have studied various properties, boundaries, and asymptotic behaviors of mean inactivity time functions and have developed certain characterization theorems. Kayid (2024) have studied multivariate quantile inactivity time and its different applications.

In recent times, there has been a growing interest in the study of probability distributions and reliability measures using quantile functions, as an alternative to the traditional distribution function approach. The quantile function of X with cdf $F(x)$ is defined as

$$Q(u) = F^{-1}(x) = \inf\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1. \tag{2}$$

If $f(\cdot)$ is the probability density function of X, then $f(Q(u))$ is known as the density quantile function and $q(u) = \frac{d}{du}Q(u)$ is termed as the quantile density function of X. Differentiating $F(Q(u)) = u$ we get,

$$f(Q(u))q(u) = 1. \tag{3}$$

The quantile approach has many advantages. There are some distributions, or class of distributions, that do not have a tractable distribution function while the quantile function exists, wherein the quantile approach is a better alternative to work with. Furthermore, there are certain properties of the quantile function that are not shared by the distribution function approach. For more details on quantile functions, one could refer to Parzen (1979), Gilchrist (2000), Nair *et al.* (2013), and the references therein. This makes the quantile-based study on various measures of importance.

Nair and Sankaran (2009) defined basic reliability functions in terms of quantile functions. The quantile-based reversed mean residual life function or quantile-based MIT

(QMIT) is defined by Nair and Sankaran (2009) as

$$R(u) = \frac{1}{u}\int_0^u (Q(u) - Q(t))\,dt. \tag{4}$$

$R(u)$ can also be expressed as

$$R(u) = \frac{1}{u}\int_0^u p q(p)\,dp, \tag{5}$$

$Q(u)$ can be uniquely determined from $R(u)$ by the relation

$$Q(u) = R(u) + \int_0^u \frac{R(p)}{p}\,dp. \tag{6}$$

The mean inactivity time is related to reversed hazard quantile function (Nair and Sankaran, 2009) by the relation $\frac{1}{\Lambda(u)} = R(u) + uR'(u)$, where reversed hazard rate is given by $\Lambda(u) = \frac{1}{uq(u)}$ and $R'(u)$ is the derivative of $R(u)$ with respect to u. As mentioned earlier, certain family of distributions do not have closed form for $\widetilde{m}(t)$, but have simple forms for $R(u)$. For example, Midhu *et al.* (2013) studied the class of distributions with linear mean residual quantile function for X_t, with quantile function

$$Q(u) = -(c+\mu)\log(1-u) - 2cu, \quad \mu > 0, -\mu \leq c < \mu, 0 \leq u \leq 1 \tag{7}$$

For Eq. (7), the QMIT based on Eq. (5) is obtained as

$$R(u) = \frac{(c+\mu)\log(1-u) - u(c+\mu+cu)}{u}. \tag{8}$$

That is, for the class of distributions that does not have a closed form expression for $f(t), F(t)$ or $\widetilde{m}(t)$, the QMIT is a useful measure to compute the average life of a device when it has failed at time $t > 0$ and its modelling and analysis are also important. A recent study on QMIT, we refer to Kayal *et al.* (2018).

In survival studies, at many situations we may not have complete information about the lifetime of all the individuals, wherein the lifetimes are said to be censored. There are different censoring mechanisms adopted by the experimenters; however, the more commonly encountered one is the random right censoring. In random right censoring, the individuals start at random times such that both the lifetimes and the censoring times are random, and it occurs when a subject leaves the study before an event occurs, or the study ends before the event has occurred. The estimation of the quantile function using the kernel density approach under right censoring was first suggested in Padgett (1986). Sankaran and Nair (2009) discussed the non-parametric estimation of the hazard quantile function under right censoring. The non-parametric estimation of the quantile density function for censored and uncensored data was discussed in Soni *et al.* (2012). Sankaran and Midhu (2017) discussed the non-parametric estimation of the mean residual quantile function under right censoring. The non-parametric estimation of the cause-specific hazard quantile function is studied by Sankaran and Dileep Kumar (2019). Silpa *et al.* (2021) have studied the non-parametric estimation of quantile-based entropy function.

The objective of the present study is to introduce non-parametric estimators of QMIT. The proposed study has many advantages. The proposed estimators will be useful in situations where the event has already happened at a time point and average lifetime is to be estimated. The non-parametric estimation of QMIT is also important for computing the mean lifetime when parametric models are inadequate or not available, or no closed form distribution function but their quantile function exists. The article is organized as follows. Section 2 discusses non-parametric estimator for QMIT using the quantile density function estimator of Jones (1992). We also propose estimators of QMIT using the quantile density estimators mentioned in Section 2. Asymptotic properties of the estimators are discussed in Section 3. We carry out extensive simulation studies to assess the performance of the estimators in Section 4. The method is applied to three real data sets in Section 5. Finally, Section 6 provides a brief conclusion of the study.

2. Non-parametric estimation

In this Section we consider non-parametric estimation of quantile-based mean inactivity time.

2.1. *Estimation of quantile density function*

Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables representing the lifetime for n components or devices with a common distribution function $F(x)$ and quantile function $Q(u)$. Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding order statistics. Parzen (1979) introduced the emperical quantile function as

$$Q(u) = X_{r:n} \quad for \quad \frac{r-1}{n} < u < \frac{r}{n}, \quad r = 1, 2, \ldots, n, \tag{9}$$

which is a step function with jump $\frac{1}{n}$. A smoothed version of Eq. (9) is also available in Parzen (1979) and given by

$$Q_n(u) = n\left(\frac{r}{n} - u\right) X_{r-1:n} + n\left(u - \frac{r-1}{n}\right) X_{r:n}, \tag{10}$$

for $\frac{r-1}{n} \leq u \leq \frac{r}{n}, r = 1, 2, ..., n$. The corresponding empirical quantile density function estimator is

$$q_n(u) = \frac{d}{du} Q_n(u) = n\left(X_{r:n} - X_{r-1:n}\right), \quad for \quad \frac{r-1}{n} < u < \frac{r}{n}. \tag{11}$$

Later, Jones (1992) proposed a smooth estimator of the quantile density function $q(u)$ given by

$$\hat{q}_n(u) = \frac{1}{f_n(Q_n(u))}, \tag{12}$$

where $f_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{x-X_i}{h_n}\right)$ is a kernel density estimator of $f(x)$, h_n is the bandwidth parameter and $K(\cdot)$ is the kernel function which satisfies the following conditions:

1. $K(u) \geq 0$ for all u,
2. $\int_{-\infty}^{\infty} K(u) du = 1$,
3. $K(u)$ has finite support, that is $K(u) = 0$ for $|u| > c$, where $c > 0$ is some constant,
4. $K(u)$ is symmetric about zero,
5. $K(u)$ satisfies Lipschitz condition, that there exists a positive constant M such that $|K(u) - K(v)| \leq M|u - v|$,

and $Q_n(u) = \inf\{x : F_n(x) \geq u\}, 0 \leq u \leq 1$ is the emperical estimator of the quantile function $Q(u)$. The estimator defined in Eq. (12) is consistent and asymptotically normally distributed.

2.2. *Non-parametric estimation of mean inactivity time*

In this Section, we propose non-parametric estimators for QMIT in Eq. (4) based on the estimators mentioned in Section 2.1. Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with corresponding order statistics $X_{1:n}, X_{2:n}, ..., X_{n:n}$. First, we propose a plug-in integral estimator for mean inactivity time based on Jones (1992), given by

$$\hat{R}_1(u) = \frac{1}{u} \int_0^u F_n(Q_n(p)) \hat{q}_n(p) dp, \tag{13}$$

where $\hat{q}_n(p)$ is defined in Eq. (12).

We then develop a non-parametric estimator for quantile mean inactivity time under right censoring using a smoothing approach. Let X be a non-negative random variable representing the lifetime of an individual. Suppose that X is right censored by a non-negative random variable Z. We observe only (T,δ), where $T = Min(X,Z)$ and $\delta = I(X \leq Z)$ with $I(\cdot)$ denoting the indicator function. Let the $X_i's$ and $Z_i's$ be i.i.d. observations of X and Z. Assume that X and Z are independent. If all the $Z_i's$ are fixed constants, the observations are time censored. Otherwise, if the $Z_i's$ are equal to the same constant, then we have type-I censoring. Let $Z_i's$ be random samples with distribution function $G(x)$. Then we have (T,δ) as a randomly right-censored sample. Let us denote the distribution function of T as $H(x)$. If we assume that X and T are independent, then

$$1-H(x) = (1-F(x))(1-G(x)). \tag{14}$$

In the present study, we develop a smooth non-parametric estimation of $R(u)$ under right censoring using kernel density method. Based on right censored samples, a non-parametric estimate of $Q(u)$ is given by $\widehat{Q_n}(u) = \inf\{x : F_n(x) \geq u\}$ where $\widehat{F}(x) = 1 - S_n(x)$ with $S_n(x)$ as the Kaplan-Meier estimator of $S(x)$ for the ordered failure times $X_{1:n}, X_{2:n}, ..., X_{n:n}$, corresponding to $X_i, i = 1,2,...,n$ is given by

$$S_n(x) = \prod_{k:X_{k:n}<x} \left(1 - \frac{d_k}{n_k}\right), \tag{15}$$

where d_k is the number of failures at $X_{k:n}$ and n_k be the number of subjects at risks in $X_{k:n}; k = 1,2,...,n$. Let $\{h_n\}$ be a bandwidth sequence of positive numbers such that $h(n) \to 0$ as $n \to \infty$. Parzen (1979) proposed a smooth kernel estimator for $Q(u)$ in the uncensored case as

$$Q^*(u) = \int_0^u K\left(\frac{u-p}{h}\right)\widehat{Q}(p)dp, \tag{16}$$

where $\widehat{Q}(u)$ is the empirical quantile estimator. Sheather and Marron (1990) have modified the estimator in Eq. (16) by approximating the integral to a sum which is given by

$$Q_n^*(u) = \frac{\sum_{i=1}^n K\left(\frac{1-1/2}{n} - u\right) T_{i:n}}{\sum_{i=1}^n K\left(\frac{1-1/2}{n} - u\right)}. \tag{17}$$

From Eq. (7), we suggest a simple non-parametric estimator of $R(u)$ as

$$\widehat{R}_2(u) = \frac{1}{u}\int_0^u (Q_n^*(u) - Q_n^*(t))dt. \tag{18}$$

In the presence of censoring, Eq. (17) reduces to

$$\widetilde{Q}_n(u) = \frac{\sum_{i=1}^n K(F_n(T_{i:n}) - u) T_{i:n}}{\sum_{i=1}^n K(F_n(T_{i:n}) - u)}. \tag{19}$$

Using Eq. (19), the non-parametric estimator in Eq. (18) modifies in the censored case to

$$\widehat{R}_3(u) = \frac{1}{u}\int_0^u (\widetilde{Q}_n(u) - \widetilde{Q}_n(t)dt. \tag{20}$$

We observe that the plug-in integral estimator $\widehat{R}_1(u)$ in Eq. (13) is not capable of considering censored observations, however, the smoothed estimators $\widehat{R}_2(u)$ and $\widehat{R}_3(u)$ can accommodate both censored and uncensored observations respectively.

3. Asymptotic properties

In this Section, we establish the asymptotic properties of $\widehat{R}_1(u)$, $\widehat{R}_2(u)$ and $\widehat{R}_3(u)$.

THEOREM 1. *Let $q(u)$ be the quantile density function corresponding to a density function $f(x)$ and $\widehat{q}_n(u)$ be the estimator of $q(u)$ due to Jones (1992). Then, the estimator $\widehat{R}_1(u)$ is uniformly strong consistent.*

PROOF. From Eq. (13), we have

$$\begin{aligned}
\widehat{R}_1(u) &= \frac{1}{u}\int_0^u F_n(Q_n(p))\widehat{q}_n(p)dp \\
&= \frac{1}{u}\int_0^u F_n(Q_n(p))\widehat{q}_n(p)dp - \frac{1}{u}\int_0^u F(Q(p))\widehat{q}_n(p)dp + \frac{1}{u}\int_0^u F(Q(p))\widehat{q}_n(p)dp \\
&= \frac{1}{u}\int_0^u \widehat{q}_n(p)[F_n(Q_n(p)) - F(Q(p))]dp + \frac{1}{u}\int_0^u F(Q(p))\widehat{q}_n(p)dp. \qquad (21)
\end{aligned}$$

Since $\sup_p |F_n(p) - F(p)| \to 0$ almost surely, Eq. (21) is asymptotically equal to

$$\widehat{R}_1(u) = \frac{1}{u}\int_0^u F(Q(p))\widehat{q}_n(p)dp.$$

Now, we consider

$$\begin{aligned}
\widehat{R}_1(u) - R(u) &= \frac{1}{u}\int_0^u F(Q(p))\widehat{q}_n(p)dp - \frac{1}{u}\int_0^u F(Q(p))q(p)dp \qquad (22)\\
&= \frac{1}{u}\int_0^u F(Q(p))[\widehat{q}_n(p) - q(p)]dp. \qquad (23)
\end{aligned}$$

We have

$$\begin{aligned}\widehat{q}_n(u) &= \frac{1}{f_n(Q_n(u))}\\ &= \frac{1}{f_n(Q_n(u))-f(Q(u))+f(Q(u))}\\ &= \frac{1}{f(Q(u))}\left[\frac{1}{1+\frac{f_n(Q_n(u))-f(Q(u))}{f(Q(u))}}\right].\end{aligned}$$

Using Binomial theorem,

$$\widehat{q}_n(u)=\frac{1}{f(Q(u))}\left[1-\frac{(f_n(Q_n(u))-f(Q(u)))}{f(Q(u))}+\left(\frac{f_n(Q_n(u))-f(Q(u))}{f(Q(u))}\right)^2-\cdots\right]. \tag{24}$$

Thus

$$\widehat{q}_n(u)-q(u)=\frac{-f_n(Q_n(u))+f(Q(u))}{f^2(Q(u))}+\left(\frac{f_n(Q_n(u))-f(Q(u))^2}{f^3(Q(u))}\right)-\cdots$$

Writing Taylor series expansion of $f_n(Q_n(u))$ about $Q(u)$, we have by assuming higher derivatives of $f_n(Q_n(u))$ exists.

$$f_n(Q_n(u))=f_n(Q(u))+(Q_n(u)-Q(u))f_n'(Q(u))+\frac{(Q_n(u)-Q(u))^2 f_n''(Q(u))}{2!}+\cdots,$$

Hence

$$\begin{aligned}f_n(Q_n(u))-f(Q(u))=f_n(Q(u)) \quad &-f(Q(u))+(Q_n(u)-Q(u))f_n'(Q(u))\\ &+\frac{(Q_n(u)-Q(u))^2 f_n''(Q(u))}{2!}+\cdots\end{aligned}$$

As $n\to\infty, \sup_u|Q_n(u)-Q(u)|\to 0$ (Serfling, 2009) and $\sup_u|f_n(u)-f(u)|\to 0$ (Rao, 2014). Thus $\sup_u|f_n(Q_n(u))-f(Q(u))|\to 0$ (Soni *et al.*, 2012), so that Eq. (22) reduces to $\sup_u\left|\widehat{R}_1(u)-R(u)\to 0\right|$, which completes the proof. □

THEOREM 2. *Let $F(\cdot)$ be continuous. Assume that $K(\cdot)$ satisfies conditions 1-4 in Section 2.1. Then for fixed $u\in(0,1)$, $\sqrt{n}(\widehat{R}_1(u)-R(u))$ is asymptotically normal with mean zero and variance $\sigma_1^2(u)$ given by*

$$\sigma_1^2=\frac{n}{u^2}E\left(\int_0^u F_n(Q_n(p))\widehat{q}_n(p)dp\right)^2. \tag{25}$$

PROOF. From Eq. (22), we have

$$\begin{aligned}\widehat{R}_1(u)-R(u) &= \frac{1}{u}\int_0^u F(Q(u))[\widehat{q}_n(p)-q(p)]dp\\ &= \frac{1}{u}\int_0^u F(Q(u))\left[\frac{1}{f_n(Q_n(u))}-\frac{1}{f(Q(n))}\right]dp.\end{aligned}$$

$$\sqrt{n}\left(\widehat{R}_1(u)-R(u)\right)=\frac{\sqrt{n}}{u}\int_0^u F(Q(u))\left[\frac{1}{f_n(Q_n(u))}-\frac{1}{f(Q(n))}\right]dp. \tag{26}$$

Using asymptotic normality of $f_n(Q_n(u))$ (Soni *et al.*, 2012) and Slutsky's theorem (Serfling, 2009), the proof is complete. □

THEOREM 3. *Let $F(\cdot)$ be continuous. Assume that $K(\cdot)$ satisfies conditions 1-4 in Section 2.1. Then the estimator $\widehat{R}_2(u)$ defined in Eq. (20) is uniformly strong consistent.*

PROOF. From Eq. (20) we have

$$\widehat{R}_2(u)=\frac{1}{u}\int_0^u (Q_n^*(u)-Q_n^*(t))dt.$$

Thus

$$\widehat{R}_2(u)-R(u)=\frac{1}{u}\int_0^u (Q_n^*(u)-Q(u))-(Q_n^*(t)-Q(t))\,dt. \tag{27}$$

Since $\sup_u |Q_n^*(u)-Q(u)|\to 0$ almost surely by Sheather and Marron (1990), then (27) gives $\sup_u \left|\widehat{R}_2(u)-R(u)\right|\to 0$ as $n\to\infty$. Hence the proof. □

THEOREM 4. *Let $F(\cdot)$ be continuous. Assume that $K(\cdot)$ satisfies conditions $1-4$ in Section 2.1. Then for fixed $u\in(0,1)$, $\sqrt{n}(\widehat{R}_2(u)-R(u))$ is asymptotically normal with mean zero and variance $\sigma_2^2(u)$ given by*

$$\sigma_2^2(u)=\frac{1}{u^2}E\left[\int_0^u (Q_n^*(u)-Q_n^*(t))\right]^2. \tag{28}$$

PROOF. We have

$$\sqrt{n}(\widehat{R}_2(u)-R(u))=\frac{\sqrt{n}}{u}\int_0^u \sqrt{n}[(Q_n^*(u)-Q(u))-(Q_n^*(t)-Q(t))]dt. \tag{29}$$

From Sheather and Marron (1990), it follows that for fixed $u(0<u<1)$, $\sqrt{n}(Q_n^*(u)-Q(u))$ is asymptotically normal with mean zero and variance

$$\sigma_2^2(u)=n^{-1}u(1-u)(q(u))^2-n^{-1}h(n)q(u)^2\int_{-\infty}^{\infty} pK(p)K^{(-1)}(p)\mathrm{d}p+o\left(n^{-1}\right)h(n),$$

where $K^{-1}(u)$ is anti-derivative of $K(\cdot)$.

Using the functional delta method given in Andersen *et al.* (1993) and by Slutsky's theorem given in Serfling (2009), (29) provides for fixed $u\in(0,1)$, $\sqrt{n}(\widehat{R}_2(u)-R(u))$ is asymptotically normal with mean zero and variance $\sigma_2^2(u)$ given in Eq. (28). This completes the proof. □

The asymptotic properties of $\widehat{R}_3(u)$ is similar to that of $\widehat{R}_2(u)$ and hence omitted.

4. SIMULATIONS

In the present Section, we carry out extensive simulations to assess the performance of the estimators. We take different samples of sizes 50, 100, 200, and 500. We generate 1000 data sets for each scenario. We carry out extensive simulation studies to calculate the bias and mean squared error (MSE) of the estimators $\widehat{R}(u)$ for the uncensored as well as the censored case. We have employed triangular, uniform, Gaussian, and Epanechnikov kernel functions in the simulation studies. However, results based on the Gaussian kernel with $K(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ are being reported since it provides the smallest MSE. We generate random samples of different sizes from two quantile models:

1. Exponential distribution with quantile function,

$$Q(u)=\frac{-\log(1-u)}{\lambda};\lambda>0, \tag{30}$$

and corresponding QMIT,

$$R(u)=-\left[\frac{u+\log(1-u)}{\lambda u}\right].$$

2. Class of distributions with linear hazard quantile function (Midhu *et al.*, 2014), with quantile function

$$Q(u) = \frac{\log\left(\frac{a+bu}{a(1-u)}\right)}{a+b}, \quad a > 0,\ a+b > 0,\ 0 \le u \le 1. \tag{31}$$

and corresponding QMIT,

$$R(u) = -\frac{a\log\left(\frac{bu}{a}+1\right) + b\log(1-u)}{bu(a+b)}.$$

TABLE 1
Bias and MSE of the estimators for exponential distribution.

u	n	$\widehat{R}_1(u)$		$\widehat{R}_2(u)$		$\widehat{R}_3(u)$ (censored)	
		Bias	MSE	Bias	MSE	Bias	MSE
0.2	50	0.026	0.0008	-0.015	0.0003	-0.019	0.0004
	100	0.017	0.0004	-0.013	0.0002	-0.011	0.0002
	200	0.014	0.0002	-0.009	0.0001	-0.009	0.0001
	500	0.011	0.0001	-0.005	0.0001	-0.001	0.0000
0.4	50	0.047	0.0029	-0.008	0.0006	-0.014	0.0007
	100	0.043	0.0021	-0.006	0.0004	-0.005	0.0003
	200	0.039	0.0017	-0.004	0.0002	-0.005	0.0002
	500	0.037	0.0014	-0.002	0.0001	0.001	0.0001
0.6	50	0.103	0.0156	0.005	0.0026	-0.018	0.0018
	100	0.095	0.0111	0.005	0.0013	0.005	0.0011
	200	0.089	0.0091	-0.001	0.0008	0.004	0.0006
	500	0.086	0.0079	0.004	0.0003	-0.009	0.0003
0.8	50	0.187	0.4367	0.0309	0.0094	-0.028	0.0062
	100	0.141	0.0313	0.037	0.0061	-0.030	0.0045
	200	0.131	0.0228	0.027	0.0034	0.017	0.0026
	500	0.123	0.0175	0.011	0.0012	-0.009	0.0011

For the construction of a kernel-type estimator of a quantile function, Padgett (1986) has considered separate bandwidths for different regions of $u \in (0,1)$ in such a way that the mean squared error (MSE) is minimized. We calculate the optimum bandwidths corresponding to different values of u, such as $0.2, 0.4, 0.6$, and 0.8.

To assess the performance of the estimators in both uncensored and censored cases, we apply different parameter combinations of different distributions. First we consider the exponential in Eq. (30) with $\lambda = 0.5$. The bias and MSE of the estimators $\widehat{R}_1(u)$, $\widehat{R}_2(u)$, and $\widehat{R}_3(u)$ are reported in Table 1. To introduce censoring in $\widehat{R}_3(u)$, we use uniform $U(0, b)$ as the censoring distribution where b is chosen in such a way that 20% of the observations are censored. From Table 1, it is evident that both bias and MSE of the estimators decrease as the sample size increases. We notice that among the estimators $\widehat{R}_2(u)$ and $\widehat{R}_3(u)$ have minimum bias and MSE as compared to $\widehat{R}_1(u)$.

TABLE 2

Bias and MSE of the estimators for the class of distributions in Eq. (31) for $a = 0.5, b = 2$ and for different values of n.

u	n	$\widehat{R}_1(u)$		$\widehat{R}_2(u)$		$\widehat{R}_3(u)$ (censored)	
		Bias	MSE	Bias	MSE	Bias	MSE
0.2	50	0.072	0.0087	-0.079	0.0064	-0.087	0.0076
	100	0.061	0.0051	-0.075	0.0057	-0.078	0.0061
	200	0.052	0.0033	-0.069	0.0048	-0.061	0.0038
	500	0.043	0.0021	-0.058	0.0034	-0.044	0.0019
0.4	50	0.080	0.0115	-0.077	0.0064	-0.072	0.0055
	100	0.064	0.0062	-0.063	0.0043	-0.050	0.0031
	200	0.053	0.0038	-0.051	0.0027	-0.037	0.0019
	500	0.043	0.0023	-0.033	0.0012	-0.026	0.0008
0.6	50	0.063	0.0092	-0.051	0.0042	-0.042	0.0031
	100	0.048	0.0044	-0.029	0.0018	-0.027	0.0019
	200	0.035	0.0023	-0.016	0.0008	-0.006	0.0009
	500	0.025	0.0016	-0.009	0.0003	-0.020	0.0003
0.8	50	-0.017	0.0058	-0.095	0.0143	-0.077	0.0108
	100	-0.036	0.0037	-0.048	0.0055	-0.056	0.0062
	200	-0.051	0.0038	-0.013	0.0018	-0.008	0.0023
	500	-0.062	0.0044	0.016	0.0009	-0.008	0.0009

Similarly, we carry out the simulation study for the class of distributions given in Eq. (31). The censoring has been introduced in the same manner as described in the above. Table 2 gives the bias and MSE of the estimators for different sample sizes. From Table 2, we note that as the sample sizes increases, the bias and MSE of all the estimators decrease, which confirms the asymptotic properties of the estimators.

We note that the estimators $\widehat{R}_1(u)$, $\widehat{R}_2(u)$, and $\widehat{R}_3(u)$ perform well in terms of bias and MSE for the two quantile models, but $\widehat{R}_3(u)$ accommodates the censoring, which is an essential component in survival studies. We have also examined the performance of these estimators using the quantile density function estimator due to Soni *et al.* (2012); however, the performances were relatively less compared to the proposed estimators, hence not reported.

5. Data analysis

The illustration of the estimation procedure is carried out using two real data sets. We have computed the quantile-based mean inactivity time for the proposed estimators using the Gaussian kernel. The optimal bandwidth parameter h_n is taken based on the plug-in method due to Sheather and Jones (1991).

The first data set is taken from Kleinbaum *et al.* (2012). The data set corresponds to survival time in days from a clinical trial on gastric carcinoma involving 90 patients. The survival time alone is considered as fitting the distribution. For the estimation of parameters, we use the method of L—moments. The data has been fitted with the class of distributions studied by Midhu *et al.* (2014). The estimated parameters were obtained as $\widehat{a} = 1.17$ and $\widehat{b} = 0.89$. Thus, a parametric estimate of $R(u)$ is given as

$$R^*(u) = \frac{0.545(-0.890\log(1-u) - 1.170(1+0.761u))}{u}.$$

We calculate $\widehat{R}_1(u)$ and $\widehat{R}_2(u)$. Figure 1 presents $R^*(u), \widehat{R}_1(u)$ and $\widehat{R}_2(u)$. It follows from the Figure 1 that $\widehat{R}_1(u)$ and $\widehat{R}_2(u)$ are smooth curves close to the parametric estimate of the $R^*(u)$ for the most value of u. Hence, for this data set, we see that both the estimators perform well.

The second data is taken from Bekker *et al.* (2000), which gives the survival times (in years) of a group of 45 patients given chemotherapy treatment alone. Out of 45 observations, eight observations are right censored. This data was earlier studied by Sankaran and Dileep Kumar (2019), and they fitted the class of distributions with a quadratic mean residual quantile function. They employed the method of L—moments for the estimation procedure. The estimates were obtained as

$$\widehat{\alpha} = 1.014 \quad \widehat{\beta} = 0.949 \quad \text{and} \quad \widehat{\gamma} = 3.11. \tag{32}$$

Thus the parametric estimate of $R(u)$, using $\widehat{\alpha}, \widehat{\beta}$ and $\widehat{\gamma}$, is given by

$$R^*(u) = -0.105 + 0.949u + 1.081u^2 - 1.014(u + \log(u-1)).$$

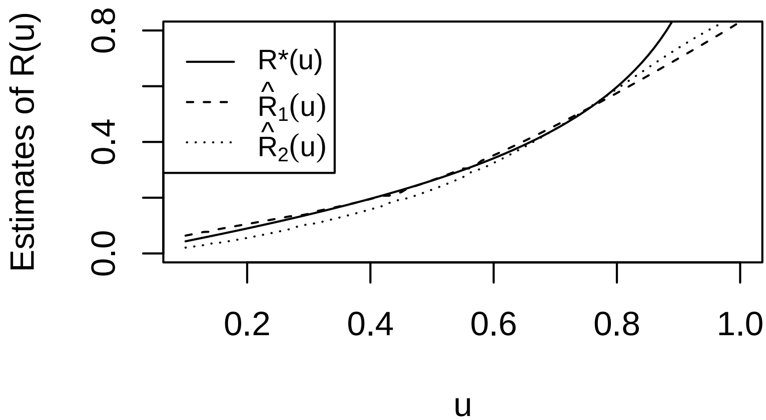

Figure 1 – Plot of $R^*(u), \widehat{R}_1(u)$ and $\widehat{R}_2(u)$ based on the data given in Kleinbaum *et al.* (2012).

Figure 2 shows $R^*(u)$ and $\widehat{R}_3(u)$ for different values of u. From Figure 2, we infer that $\widehat{R}_3(u)$ is a smooth curve close to the parametric estimate of $R^*(u)$ up to $u = 0.6$, however, it shows a boundary bias at the tail points.

The third data set, reported in Helsel (2005), represents the cadmium concentrations in fish for two regions of the rocky mountains. The objective is to determine if concentrations are the same or different in fish livers of the two regions. There are four detection limits, at 0.2, 0.3, 0.4, and 0.6 ug/L. This data is taken from the "Cadmium" data set in the `NADA` package in `R`. There are 19 observations, of which 3 observations are left censored. We know that the Kaplan-Meier estimator is not appropriate for left-censored data. Therefore, for finding out the estimated survival function by the method of Kaplan-Meir, we make the left-censored data into right-censored data by fixing a large time τ and we define new times by τ minus the original times. The data set based on these reverse times is now right-censored, and the Kaplan-Meier estimator can be used for finding the survival function. This method of finding a survival function for a left-censored case is illustrated in Ware and Demets (1976) and is also given in Klein and Moeschberger (2003). In the case of cadmium data, we take $\tau = 100$. Then the non-parametric estimate of $R(u)$ for different values of u is plotted in Figure 3. We note that there are only 19 observations, of which 3 of them are censored. So, this data can be estimated only by a non-parametric method and cannot be compared with a parametric

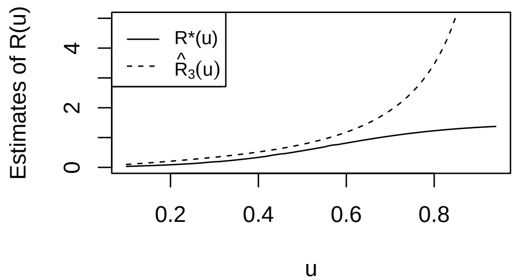

Figure 2 – Plot of $R^{*}(u)$ and $\widehat{R}_3(u)$ using the data given in Bekker *et al.* (2000)

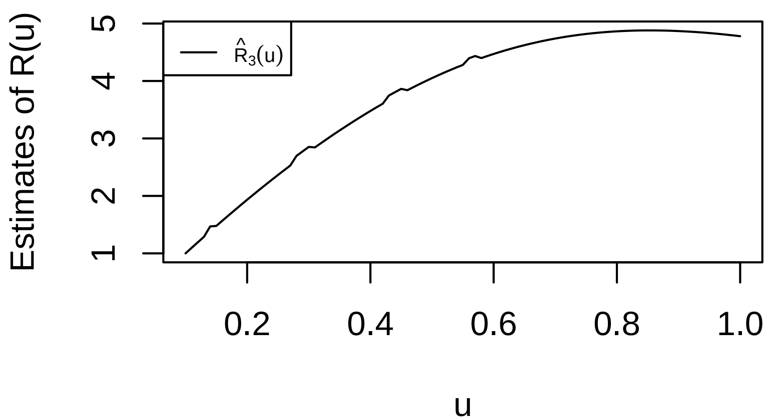

Figure 3 – Plot of $\widehat{R}_3(u)$ for the data given in Helsel (2005).

model for validation purposes.

The three data sets demonstrate real-life applications of the reversed mean residual quantile function. The first data set shows that the proposed estimators perform comparably to parametric estimation methods. The second data set highlights that, for censored observations, the Sheather and Marron (1990) plug-in estimator performs the better, although it faces some boundary bias issues that need further investigation. In the third data set, the relevance of the proposed estimators in the censored case is underscored. The challenge of estimating the mean inactivity time function with censored data has been extensively examined through real-life examples. In all three cases, the data sets were analyzed using quantile functions with simpler forms, allowing for the application of many interesting properties of the quantile function model in these scenarios.

6. Conclusions

The present study provided three non-parametric estimators for quantile-based mean inactivity time functions using plug-in estimators for the analysis of lifetime data. Large sample properties of the proposed estimators were derived. Simulation studies proved that the estimators have small bias and MSE. The performance and usefulness of the estimators were examined through real data analysis. In many practical situations, the problem of estimating the mean inactivity time function deals with censored data, which is extensively studied in the present work and is explained with the aid of a real-life example. We observed that the optimum choice of the bandwidth also depends on the distribution of the lifetime random variable X. More simulation studies are required to investigate this, which is a topic for future research.

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