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INTRODUCTION OF GENERAL DISTRIBUTIONS ON SPHERE AND TORUS IN VIEW OF TIME SERIES SPECTRA

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SUMMARY

There are various fields where observations are taken on directions in three dimensions, e.g., sphere and torus. Here we will introduce a very general family of distributions on sphere and torus by use of time series spectra, which includes a lot of proposed classical one as special cases. Because time series spectra can be described by a lot of famous parametric models, e.g., AR, ARMA etc., we can develop the systematic model selection in this field by use of AIC, BIC, etc. Applications are very wide.

Keywords: Distributions on sphere and torus; Time series spectra; AR, ARMA models; Model selection.

1. INTRODUCTION

There are various fields where observations are taken on directions in three dimensions, e.g., molecular biology and physics.Examples are directions of palaeomagnetism in rock, directions from the earth to stars, and directions of optical axes in quartz crystals. In the statistical estimations, the family of exponential distributions is the most important and fundamental (see Lehmann *et al.*, 1986, p.56). In what follows, we introduce three typical distributions on sphere. The von Mises-Fisher distribution is very fundamental with probability density function

$$f(\boldsymbol{x}) = c(\varkappa)^{-1} \exp[\varkappa \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{x}], \quad \boldsymbol{x} \in \mathbb{S}^2, \tag{1}$$

where $\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 : ||\boldsymbol{x}|| = 1\}$, $\boldsymbol{\mu} \in \mathbb{S}^2$,$\varkappa$ is the concentration parameter and $c(\cdot)$ is the normalizing constant (e.g., Mardia and Jupp, 2000). The Fisher-Bingham distribution is given by density

$$f(\boldsymbol{x}) = c(\varkappa, \boldsymbol{A})^{-1} \exp[\varkappa \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}], \tag{2}$$

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where $\boldsymbol{A}$ is a symmetric 3×3 matrix (Mardia, 1975). For a special choice of μ and $\boldsymbol{A}$, Kent (1982) derived a polar co-ordinates form of Eq. (2) by

$$f(\theta,\phi)=c(\varkappa,\beta)^{-1}\exp[\varkappa\cos\theta+\beta\sin^2\theta\cos 2\phi]. \tag{3}$$

On $\mathbb{S}^d$, Kato and McCullagh (2020) introduced a Cauchy family of distribution by

$$f(\boldsymbol{x})=\frac{\Gamma((d+1)/2)}{2\pi^{(d+1)/2}}\left[\frac{1-\rho^2}{1+\rho^2-2\rho\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{x}}\right]^d,\quad \boldsymbol{x}\in\mathbb{S}^d. \tag{4}$$

For modelling of torsional angles of molecules, Singh *et al.* (2002) introduced the following distribution on torus by

$$f(\theta,\phi)=c(\varkappa_1,\varkappa_2,\lambda)\exp[\varkappa_1\cos(\theta-\mu_1)+\varkappa_2\cos(\phi-\mu_2)+\lambda\sin(\theta-\mu_1)\sin(\phi-\mu_2)], \tag{5}$$

where $-\pi\le\theta,\phi\le\pi$, $\varkappa_1,\varkappa_2\ge 0$, $-\infty<\lambda<\infty$, $-\pi\le\mu_1,\mu_2\le\pi$ and $c(\cdot)$ is a normalization constant. Also, Kato and Pewsey (2015) introduced the following wrapped Cauchy type distribution by density

$$\begin{aligned}f(\theta,\phi)=&c[c_0-c_1\cos(\theta-\mu_1)-c_2\cos(\phi-\mu_2)-c_3\cos(\theta-\mu_1)\cos(\phi-\mu_2)\\&-c_4\sin(\theta-\mu_1)\sin(\phi-\mu_2)]^{-1},\quad(-\pi\le\theta,\phi\le\pi),\end{aligned} \tag{6}$$

where c, c_i and μ_i are appropriate constants.

For circular data, the wrapped Cauchy density is often used, and is defined by

$$f_{\mathrm{W}}(\omega)=\frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2+2\rho\cos(\omega-\mu)},\quad \omega\in[-\pi,\pi],(0\le\rho<1), \tag{7}$$

where μ is the mean parameter. Time series researchers understand that this is exactly the AR(1) spectral density

$$f_S(\omega)=c\frac{1}{|1+\rho e^{\mathrm{i}(\omega-\mu)}|^2},\quad \omega\in[-\pi,\pi], \tag{8}$$

(see, e.g., Brockwell and Davis, 1991, p.125). Motivated by this, Taniguchi *et al.* (2020) introduced a very general family of joint circular distributions by a higher order spectral density

$$f_S(\omega_1,\omega_2,\ldots,\omega_n),\quad \omega_k\in[-\pi,\pi], \tag{9}$$

which can be decomposed to

$$\prod_{k=1}^{n} f_S(\omega_k), \tag{10}$$

if $\omega_1,\omega_2,\ldots,\omega_n$ are independent, where $f_S(\omega_k)$ is the spectral density of frequency ω_k. Advantage of this approach is that we can introduce time series models for $f_S(\omega_k)$, i.e., AR(p), ARMA(p,q) models etc., then the systematic model selection in this field can be carried out.

In this paper, for distributions on sphere and torus, in view of above, we will introduce a very general family of the distributions by time series spectra, whose forms will be

$$f(\theta,\phi)=\sum_{k=-\infty}^{\infty}\{A_k e^{ik(\theta+\phi)}+B_k e^{ik(\theta-\phi)}+C_k e^{ik\theta}\},$$

for $(\theta,\phi)\in\mathbb{S}^2$. $f(\theta,\phi)$ is the sum of time series spectra, hence, we may use ARMA(p,q) modeling etc. We can develop the systematic model selection by use of AIC and BIC etc. The advantages of this form is, by general spectral theory (e.g., Hannan, 1970) it is shown that it is automatically Möbius transformation invariant. From the polar transformation, similarly, we can introduce a very general distribution on torus as

$$g(\theta,\phi)\propto\sum_{k=-\infty}^{\infty}a_k e^{ik\theta}+\sum_{k=-\infty}^{\infty}b_k e^{ik\phi}+\sum_{k=-\infty}^{\infty}c_k e^{ik(\theta+\phi)}+\sum_{k=-\infty}^{\infty}d_k e^{ik(\theta-\phi)},$$

which is of the form of time series spectra. Here $-\pi\le\theta,\phi\le\pi$. The applications are in various fields (e.g., Mardia and Jupp, 2000).

2. Distributions on sphere

In this Section we introduce a very general distribution on sphere in view of time series spectra. Let the 3-dim polar representation be given by

$$\begin{cases} x &= \sin\theta\cos\phi; \\ y &= \sin\theta\sin\phi; \\ z &= \cos\theta. \end{cases} \tag{11}$$

Substituting $\begin{cases} \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i} \end{cases}$ to Eq. (11), we obtain

$$\begin{cases} x &= \frac{e^{i\theta}-e^{-i\theta}}{2i}\times\frac{e^{i\phi}+e^{-i\phi}}{2}=\frac{e^{i(\theta+\phi)}-e^{-i(\theta+\phi)}+e^{i(\theta-\phi)}-e^{-i(\theta-\phi)}}{4i}; & (12) \\ y &= \frac{e^{i\theta}-e^{-i\theta}}{2i}\times\frac{e^{i\phi}-e^{-i\phi}}{2i}=\frac{e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}-e^{i(\theta-\phi)}-e^{i(\phi-\theta)}}{-4}; & (13) \\ z &= \frac{e^{i\theta}+e^{-i\theta}}{2}. & (14) \end{cases}$$

Let $\delta_k = \begin{cases} \frac{1}{4\mathrm{i}} & ,k=1 \\ \frac{-1}{4\mathrm{i}} & ,k=-1 \end{cases}$, $\gamma_k = \begin{cases} -\frac{1}{4} & ,k=1 \\ -\frac{1}{4} & ,k=-1 \end{cases}$, and $\eta_k = \begin{cases} \frac{1}{2} & ,k=1 \\ \frac{-1}{2} & ,k=-1 \end{cases}$. Then, we can see that

$$x = \sum_{k=\pm 1} \delta_k \{\mathrm{e}^{\mathrm{i}k(\phi+\theta)} + \mathrm{e}^{\mathrm{i}k(\phi-\theta)}\}; \tag{15}$$

$$y = \sum_{k=\pm 1} \gamma_k \{\mathrm{e}^{\mathrm{i}k(\phi+\theta)} - \mathrm{e}^{\mathrm{i}k(\phi-\theta)}\}; \tag{16}$$

$$z = \sum_{k=\pm 1} \eta_k \mathrm{e}^{\mathrm{i}k\theta}. \tag{17}$$

Next, we introduce general Fourier coefficients:

(i) pure imaginary coefficient $a_k = -\bar{a}_k$ satisfying $a_{-k} = -a_k$;

(ii) real coefficients b_k, $k \in \mathbb{Z}$, $b_{-k} = b_k$;

(iii) real coefficients c_k, $k \in \mathbb{Z}$, $c_{-k} = c_k$.

It is seen that

$$\sum_{k=\pm 1} a_k \{\mathrm{e}^{\mathrm{i}k(\phi+\theta)} + \mathrm{e}^{\mathrm{i}k(\phi-\theta)}\} + \sum_{k=\pm 1} b_k \{\mathrm{e}^{\mathrm{i}k(\phi+\theta)} - \mathrm{e}^{\mathrm{i}k(\phi-\theta)}\} + \sum_{k=\pm 1} c_k \mathrm{e}^{\mathrm{i}k\theta} \tag{18}$$

corresponds to the part $\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{x}$ in the von Mises-Fisher distribution of Eq. (1) and Kato-McCullagh distribution of Eq. (4). Evidently, if we take $a_k = \mu_1 \delta_k$, $b_k = \mu_2 \gamma_k$ and $c_k = \mu_3 \eta_k$, then Eq. (18) becomes $\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{x}$. Because $\mathrm{e}^{\mathrm{i}k(\phi+\theta)}$, $\mathrm{e}^{\mathrm{i}k(\phi-\theta)}$ and $\mathrm{e}^{\mathrm{i}k(\theta)}$ are Fourier basis at frequency $\phi+\theta$, $\phi-\theta$ and θ, it is natural to think of the strength of these frequencies by the spectral densities

$$f_S(\theta,\phi) = \sum_{k=-\infty}^{\infty} a_k \{\mathrm{e}^{\mathrm{i}k(\phi+\theta)} + \mathrm{e}^{\mathrm{i}k(\phi-\theta)}\} + \sum_{k=-\infty}^{\infty} b_k \{\mathrm{e}^{\mathrm{i}k(\phi+\theta)} - \mathrm{e}^{\mathrm{i}k(\phi-\theta)}\} + \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}k\theta}. \tag{19}$$

Mardia (1975) introduced the Fisher-Bingham distribution in Eq. (2), which includes $\kappa\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$. We already checked $\boldsymbol{\mu}^{T}\boldsymbol{x}$ part. Without loss of generality we assume $\boldsymbol{A}$ is a diagonal matrix, i.e., $\boldsymbol{A} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Hence $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$. We just check $\lambda_1 x^2$ part. Note that $x^2 =$ a linear combination of $\mathrm{e}^{2\mathrm{i}(\pm)(\phi+\theta)}$, $\mathrm{e}^{2\mathrm{i}(\pm)(\phi-\theta)}$, $\mathrm{e}^{\mathrm{i}(\pm)2\phi}$, $\mathrm{e}^{\mathrm{i}(\pm)2\theta}$ and constant. Then, we can see that this is a part of

$$f_S(\theta,\phi) + \sum_{k=-\infty}^{\infty} d_k \mathrm{e}^{\mathrm{i}k\phi}. \tag{20}$$

The other parts are similar. Therefore we can introduce our very general family of distributions on sphere as follows. For a smooth function $H[\cdot]$, we propose

$$F_S(\theta,\phi) = H\left[\sum_{k=-\infty}^{\infty} (a_k \mathrm{e}^{\mathrm{i}k(\theta+\phi)} + b_k \mathrm{e}^{\mathrm{i}k(\theta-\phi)} + c_k \mathrm{e}^{\mathrm{i}k\theta} + d_k \mathrm{e}^{\mathrm{i}k\phi})\right], \tag{21}$$

as a probability density on sphere, where a_k, b_k, c_k and d_k are complex-valued, and $F_S(\theta,\phi) \geq 0$ so that $\int F_S(\theta,\phi)\,d\theta\,d\phi = 1$. Fernández-Durán and Gregorio-Domínguez (2014) introduced a Fourier series approach for sphere distributions. Their approach uses the quadratic form of Fourier series. But our approach use the spectral form of Eq. (19) and Eq. (20). We can evaluate the normalizing constant for the density of Eq. (20). In our case, the density form is in Eq. (20) $\times \sin\phi$, where $\sin\phi$ is the Jacobian. Then, it is easy to see

$$
\begin{aligned}
&\int_0^\pi\int_0^{2\pi} \sum_{k=-\infty}^{\infty} \sin\phi \cdot a_k\{e^{ik(\phi+\theta)} + e^{ik(\phi-\theta)}\}\,d\theta\,d\phi \\
&\quad + \int_0^\pi\int_0^{2\pi} \sum_{k=-\infty}^{\infty} \sin\phi \cdot b_k\{e^{ik(\phi+\theta)} - e^{ik(\phi-\theta)}\}\,d\theta\,d\phi \\
&\quad + \int_0^\pi\int_0^{2\pi} \sum_{k=-\infty}^{\infty} \{\sin\phi \cdot c_k e^{ik\theta} + \sin\phi \cdot d_k e^{ik\phi}\}\,d\theta\,d\phi \\
&\quad = 2\pi a_0 \int_0^\pi 2\sin\phi\,d\phi + 2\pi c_0 \int_0^\pi \sin\phi\,d\phi + d_1(\frac{-1}{2i}) + d_{-1}\frac{1}{2i} \\
&\quad = 8\pi a_0 + 4\pi c_0.
\end{aligned}
$$

Hence, if we take $4\pi(2a_0 + c_0) = 1$ and $d_1 = d_{-1}$, then Eq. (20) becomes a sphere distribution.

3. Distributions on Torus

In this Section we introduce a very general distribution on torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ in view of time series spectra. Let the 3-dim polar representation be given by

$$
\begin{cases}
x &= \cos\theta + \cos\phi\cos\theta \\
y &= \sin\theta + \cos\phi\sin\theta \\
z &= \sin\phi
\end{cases}, \qquad (0 \leq \theta, \phi \leq 2\pi). \tag{22}
$$

Write $\begin{cases} \cos\theta & = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) \\ \sin\theta & = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \end{cases}$, then we have

$$\begin{aligned} x &= \frac{1}{2}(e^{i\theta}+e^{-i\theta}) + \frac{1}{4}(e^{i\phi}+e^{-i\phi})(e^{i\theta}+e^{-i\theta}) \\ &= \frac{1}{2}(e^{i\theta}+e^{-i\theta}) + \frac{1}{4}(e^{i(\theta+\phi)}+e^{-i(\theta+\phi)}+e^{-i(\phi-\theta)}+e^{i(\phi-\theta)}); \qquad (23) \\ y &= \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) + \frac{1}{4i}(e^{i\phi}+e^{-i\phi})(e^{i\theta}-e^{-i\theta}) \\ &= \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) + \frac{1}{4i}(e^{i(\theta+\phi)}-e^{-i(\theta+\phi)}+e^{-i(\phi-\theta)}-e^{i(\phi-\theta)}); \qquad (24) \\ z &= \frac{1}{2i}(e^{i\phi}-e^{-i\phi}). \qquad (25) \end{aligned}$$

Similarly as in Section 2, we propose a very general family of distributions on $\mathbb{T}$ by

$$g(\theta,\phi) \propto \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} + \sum_{k=-\infty}^{\infty} b_k e^{ik\phi} + \sum_{k=-\infty}^{\infty} c_k e^{ik(\theta+\phi)} + \sum_{k=-\infty}^{\infty} d_k e^{ik(\theta-\phi)}. \qquad (26)$$

$\{a_k\}$, $\{b_k\}$, $\{c_k\}$, $\{d_k\}$ are complex-valued coefficients, and are chosen so that $g(\theta,\phi)$ is real-valued.

Singh *et al.* (2002) introduced the distribution on $\mathbb{T}$ by Eq. (5). We can see that the exponent is of our form, expressed by Eq. (26). Also Kato and Pewsey (2015) introduced the distribution on $\mathbb{T}$ by Eq. (6). It is seen that the inside of the reverse function is of our form, i.e. Eq. (26).

Hence, we introduce our very general family of distributions on $\mathbb{T}$ as follows. For a smooth function $G[\cdot]$, we propose

$$F_T(\theta,\phi) \equiv G\left[\sum_{k=-\infty}^{\infty}(A'_k e^{ik(\theta+\phi)} + B'_k e^{ik(\theta-\phi)} + C'_k e^{ik\theta} + D'_k e^{ik\phi})\right], \qquad (27)$$

as a probability density on $\mathbb{T}$, where A'_k, B'_k, C'_k and D'_k are complex-valued, and $F_T(\theta,\phi) \geq 0$ so that $\int F_T(\theta,\phi)\,d\theta\,d\phi = 1$.

4. Summary and Concluding Remarks

We could introduce a very general distributions on $\mathbb{S}^2$ and $\mathbb{T}$ by time series spectra in Equations (21) and (27), respectively. The advantage is that we can develop the problem of model selection systematically because the time series spectra have a lot of famous finite parametric models, e.g., AR, MA, and ARMA models, i.e., for modeling, we can use

$$F(\theta,\phi) = H[s_1 f^{(1)}_{\text{ARMA}}(\theta+\phi) + s_2 f^{(2)}_{\text{ARMA}}(\theta-\phi) + s_3 f^{(3)}_{\text{ARMA}}(\theta) + s_4 f^{(4)}_{\text{ARMA}}(\phi)],$$

where $s_1, \dots, s_4$ are real constants and $f_{\mathrm{ARMA}}^{(j)}$ are ARMA spectral densities (e.g., Taniguchi and Kakizawa, 2000). Hence we can use AIC, BIC etc to select the model, which enriches applications for data from $\mathbb{S}^2$ and $\mathbb{T}$. Also, the systematic asymptotic estimation theory will be possible (e.g., Taniguchi and Kakizawa, 2000).

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