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RELIABILITY ESTIMATION OF DEPENDENCE STRUCTURE SYSTEM FOR HUANG-KOTZ ITERATED FGM WITH LINDLEY MARGINAL

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SUMMARY

Huang and Kotz (1984) proposed a two-parameter extension of the original Fralie-Gumble-Morgenstern (FGM) family to model the higher association between the random variables. In this problem, we develop an iterated FGM (IFGM) based dependent stress-strength reliability model using Lindley marginals. Some important statistical and reliability properties of the proposed distribution are also derived. The prime goal of this study is to investigate the effect of stress-strength reliability parameters with respect to the variation in the dependence parameters α and β. Further, we compared the IFGM stress-strength reliability model with the original FGM using graphical representations to assess whether reliability was over or under-estimated. Finally, we investigated the performance of the proposed estimators through both Monte Carlo simulations as well as real data sets.

Keywords: Lindley distribution; Stress-strength reliability; IFGM copula; Pseudo likelihood estimation; Monte-Carlo simulation.

1. INTRODUCTION

In recent years, research on estimating dependence stress-strength reliability and investigating their dependence structure has received ample amount of attention in the literature. While handling the dependence issue of random variables, researchers faced

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mathematical difficulties in constructing joint distribution in the case of multivariate scenarios. Hence only a few attempts have been made in this direction.

A copula-based approach is more appropriate for modeling associations between two or more random variables and resolving such issues in the construction of FGM family of multivariate life distributions. Sklar (1959) proposed a copula function in that any multidimensional joint distribution function can be split into a copula function and a corresponding number of marginal distribution functions. It acts as a connecting function and illustrates the dependence structure between the random variables. In recent studies, the applications of copulas are spread over many areas of research, like, operational risk management by Arbenz (2013), hydrology data by Bekrizadeh *et al.* (2013), and competing risks models by Louzada *et al.* (2012), etc.

Many families of copula are available in the literature, see, Joe (1997) and Nelsen (2007), which are well utilized as per their feasibility in different domain areas of research. Among them, the FGM copula is one of the most widely considered family due to its simple shape and the exact calculus of polynomial functions. Hence it became more popularized in recent years, and Morgenstern (1956), Gumbel (1960), and Farlie (1960) are the pioneer contributors of FGM copula.

In statistical reliability, the idea of dependent stress-strength reliability (SSR) and its estimation using FGM copula is carried out by several authors in the literature. Domma and Giordano (2013) considered the SSR under FGM and generalized FGM copula when stress and strength follow the Burr system of marginals. Patil and Naik-Nimbalkar (2017) studied the estimation of SSR for FGM and other families of Copula for Pareto marginals. Recently, James *et al.* (2022) considers the estimation of dependence SSR by assuming that both X (strength) and Y (stress) follow Lindley marginal distributions linked with the FGM copula function. James and Chandra (2022) considered the dependence SSR with stress follows Exponential and strength follows Xgamma distribution and see the references therein. Recently, James *et al.* (2023) developed FGM based SSR for two parameter Rayleigh marginals. For recent developments for association measure with SSR one may refer to Pathak and Vellaisamy (2022), Patil *et al.* (2024), Arshad *et al.* (2023) and references therein.

Later, many authors considered the extension and modifications of the FGM copula to improve the correlation coefficient's bound. Huang and Kotz (1984) studied successive iterations to extend the FGM family with an upper bound of correlation increased up to 0.434 with two iterations using uniform marginals. Subsequently, a polynomial type single parameter extension of the FGM family with a positive correlation of 0.39 is proposed by Huang and Kotz (1999).

Further, Bairamov and Kotz (2002) extended the polynomial-based FGM copula proposed by Huang and Kotz (1999) with more wider range of ρ (-0.48,0.502). Amblard and Girard (2009) proposed a symmetric extension of FGM copulas with a correlation range is (-0.75, 1). Bekrizadeh *et al.* (2012) proposed a new family of copula with the bound of the correlation (-0.5, 0.43). Pathak and Vellaisamy (2016) further extended the family given by Bekrizadeh *et al.* (2012) with ρ (-0.48, 0.5318).

In this paper, we focus on iterated based generalization of FGM distribution, introduced by Huang and Kotz (1984), due to its flexible form of PDF and CDF expressions than the other extensions of the FGM family. To the best of our knowledge, there are no articles in the literature studying Iterated FGM and its properties, not even on SSR dependence under IFGM, which motivates us to contribute in this direction. Our research aims to fill this gap by exploring the properties of the iterated FGM family, specifically focusing on dependence SSR under IFGM. By investigating these aspects, we hope to provide valuable insights into the generalization of FGM distributions. This study will contribute to the understanding of the iterated FGM family and its properties, particularly in terms of dependence SSR. Our findings may offer new perspectives on the generalization of FGM distributions and their applications in various fields.

The primary objective of this paper, it is to develop an bivariate IFGM distribution due to Haung and Kotz's(1984) with Lindley marginals followed by investigating its statistical properties including SSR modelling. Further, we compare the impacts of the dependence parameters α and β on $R = P(Y < X)$. Finally, we analyzed dependent stress-strength reliability with the variation of the marginal parameters of stress and strength distributions and also see the effect of adding the additional dependence parameter β on R compared to the case of FGM.

The purpose of this research is to enhance the existing knowledge on IFGM distributions by introducing a novel bivariate model and examining its statistical characteristics. By analyzing the effect of dependence parameters on reliability, we can gain a deeper understanding of how these distributions can be employed in practical situations. Additionally, evaluating the impact of introducing the parameter β on reliability can offer valuable insights into the model's behavior under varying conditions.

The rest of the paper is organized in the following manner. In Section 2, the Iterated FGM Bivariate Lindley (IFGMBL) distribution is introduced, and its properties are presented and discussed. Section 3 discusses the three popular measures of association, including Spearman's rho, Kendall's tau, and Blomqvist's beta. In Section 4, the dependence stress-strength model and the reliability characteristics of the IFGMBL are discussed. Section 5 deals with parameter estimation of dependence stress-strength reliability. Section 6 illustrates the performance of the proposed model estimators using simulations. Real data analysis of the proposed model is presented in Section 7. Finally, concluding remarks and future research perspectives are discussed in Section 8.

2. Iterated FGM bivariate Lindley distribution

Huang and Kotz (1984) extended the original FGM family to cover a wider range of associations between two or more random variables. The joint c.d.f and p.d.f of FGM with two iteration is given by

$$F_{XY}(x,y) = F_X(x)G_Y(y)\left[1 + \alpha\bar{F}_X(x)\bar{G}_Y(y) + \beta F_X(x)G_Y(y)\bar{F}_X(x)\bar{G}_Y(y)\right], \quad (1)$$

$$f_{XY}(x,y)=f_X(x)g_Y(y)\big[1+\alpha(1-2F_X(x))(1-2G_Y(y))+\beta F_X(x)G_Y(y)$$
$$(2-3F_X(x))(2-3G_Y(y))\big], \tag{2}$$

where F_X and G_Y are the c.d.f's, $f(x)$ and $g(y)$ are the p.d.f's and $\bar{F}_X(x)$ and $\bar{G}_Y(y)$ denote the survival functions of X and Y, respectively. The natural parameter space ν is convex, where $\nu=\{(\alpha,\beta):-1\leq\alpha\leq 1;\alpha+\beta\geq -1;\beta\leq\frac{3-\alpha+\sqrt{9-6\alpha-3\alpha^2}}{2}\}$.

We assume that both X and Y follow two parameter Lindley distribution (Shanker *et al.*, 2013). Then, the joint p.d.f and c.d.f of IFGMBL distribution are given as follows

$$\begin{aligned} f_{XY}(x,y)=&\frac{\theta_1^2}{(\theta_1+\alpha_1)}(1+\alpha_1 x)e^{-\theta_1 x}\frac{\theta_2^2}{(\theta_2+\alpha_2)}(1+\alpha_2 y)e^{-\theta_2 y}\\ &\times\Big\{1+\alpha(\frac{2(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x}-1)(\frac{2(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y}-1)\\ &+\beta(1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x})(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y})\\ &\times(\frac{3(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x}-1)(\frac{3(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y}-1)\Big\}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} F_{XY}(x,y)=&(1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x})(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y})\\ &\times\Big\{1+\alpha(\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x})(\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y})\\ &+\beta(1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x})(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y})\\ &\times(\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x})(\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y})\Big\}, \end{aligned} \tag{4}$$

where θ_i and α_i, with $i=1,2$, are the marginal parameters of stress and strength distributions.

2.1. Conditional Distribution

Let (X,Y) be a two-dimensional random variable, then the conditional c.d.f of X given $Y=y$ is given by

$$F_{X/Y}(x/y)=\Big[1-\alpha\Big\{\frac{2(\theta_2+\alpha_2+\theta_2\alpha_2 y)e^{-\theta_2 y}}{(\theta_2+\alpha_2)}-1\Big\}\Big]\Big[1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)e^{-\theta_1 x}}{(\theta_1+\alpha_1)}\Big]$$

$$+\alpha\Big[\frac{2(\theta_2+\alpha_2+\theta_2\alpha_2 y)e^{-\theta_2 y}}{(\theta_2+\alpha_2)}-1\Big]\Big[1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)^2 e^{-2\theta_1 x}}{(\theta_1+\alpha_1)^2}\Big]$$
$$+\beta\frac{(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{\theta_1 x}}{(\theta_1+\alpha_1)}\Big[1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x}\Big]^2$$
$$\times\Big[\frac{2y\theta_2(\theta_2+\alpha_2)+2e^{-\theta_2 y}(\theta_2+3\alpha_2+\alpha_2\theta_2 y)-2\theta_2-4\alpha_2}{\theta_2(\theta_2+\alpha_2)}$$
$$+\frac{3}{\theta_2(\theta_2+\alpha_2)^2}\Big\{(\theta_2+\alpha_2)^2 y\theta_2+2\alpha_2(\theta_2+\alpha_2)(1-\theta_2 y e^{-\theta_2 y}-e^{-\theta_2 y})$$
$$+\alpha_2^2\Big(\frac{2\theta_2^2 y^2 e^{-2\theta_2 y}+2\theta_2 y e^{-2\theta_2 y}+e^{-2\theta_2 y}-1}{4}\Big)-2(\theta_2+\alpha_2)^2(e^{-\theta_2 y}-1)$$
$$-2\alpha_2(\alpha_2+\theta_2)\Big(\frac{1-2\theta_2 y e^{-2\theta_2 y}-e^{-2\theta_2 y}}{4}\Big)+(\theta_2+\alpha_2)^2\big(\frac{e^{-2\theta_2 y}-1}{2}\big)\Big\}\Big], \tag{5}$$

and the conditional p.d.f is given by

$$f_{X/Y}(x/y)=\frac{\theta_1^2(1+\alpha_1 x)e^{-\theta_1 x}}{(\theta_1+\alpha_1)}\Big[1+\alpha\Big\{\frac{2(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x}-1\Big\}$$
$$\times\Big\{\frac{2(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y}-1\Big\}+\beta\Big(\frac{3(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x}-1\Big)$$
$$\times\Big(\frac{3(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y}-1\Big)\Big(1-\frac{\theta_1+\alpha_1+\theta_1\alpha_1 x}{(\theta_1+\alpha_1)}e^{-\theta_1 x}\Big)$$
$$\times\Big(1-\frac{\theta_2+\alpha_2+\theta_2\alpha_2 y}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\Big)\Big]. \tag{6}$$

2.2. *Moment generating function*

Let (X,Y) be a random vector with joint p.d.f $f_{X,Y}(x,y)$, the moment generating function (m.g.f) of (X,Y) is obtained as

$$M_{(X,Y)}(t_1,t_2)=E(e^{t_1 x}e^{t_2 y}), \tag{7}$$

provided that the expected value exists on $-h_1<t_1<h_1$ and $-h_2<t_2<h_2$, for $h_1>0$ and $h_2>0$. Using Equation (3), the m.g.f of IFGMBL distribution is given by

$$M_{(X,Y)}(t_1,t_2)=M+\alpha\prod_{i=1}^{2}K_i(\theta_i,\alpha_i,t_i)+\beta\prod_{i=1}^{2}L_i(\theta_i,\alpha_i,t_i), \tag{8}$$

where

$$M=\frac{\theta_1^2(\theta_1-t_1+\alpha_1)}{(\theta_1+\alpha_1)(\theta_1-t_1)^2}\frac{\theta_2^2(\theta_2-t_2+\alpha_2)}{(\theta_2+\alpha_2)(\theta_2-t_2)^2},$$

$$K_i(\theta_i,\alpha_i,t_i)=\frac{2\theta_i^2}{(\theta_i+\alpha_i)^2(2\theta_i-t_i)^3}\{t_i^2(\theta_i+\alpha_i)-t_i(4\theta_i^2+6\theta_i\alpha_i+\alpha_i^2)$$
$$+4\theta_i^3+8\alpha_i\theta_i^2+4\theta_i\alpha_i^2\}-\frac{\theta_i^2(\theta_i-t_i+\alpha_i)}{(\theta_i+\alpha_i)(\theta_i-t_i)^2},\quad i=1,2,$$

$$L_i(\theta_i,\alpha_i,t_i)=\frac{\theta_i^2}{(\theta_i+\alpha_i)}\Bigg[\frac{4}{(\theta_i+\alpha_i)}\left\{\frac{(\theta_i+\alpha_i)}{(2\theta_i-t_i)}+\frac{(2\theta_i\alpha_i+\alpha_i^2)}{(2\theta_i-t_i)^2}+\frac{2\alpha_i^2\theta_i}{(2\theta_i-t_i)^3}\right\}$$
$$-\frac{(\theta_i-t_i+\alpha_i)}{(\theta_i-t_i)^2}-\frac{3}{(\theta_i+\alpha_i)^2}\left\{\frac{(\theta_i+\alpha_i)^2}{(3\theta_i-t_i)}+\frac{(6\alpha_i^2\theta_i^2+4\alpha_i^3\theta_i)}{(3\theta_i-t_i)^3}\right.$$
$$\left.+\frac{(2\theta_i\alpha_i(\theta_i+\alpha_i)+\alpha_i(\theta_i+\alpha_i)^2)}{(3\theta_i-t_i)^2}+\frac{6\alpha_i^3\theta_i^2}{(3\theta_i-t_i)^4}\right\}\Bigg],\quad i=1,2. \tag{9}$$

3. Dependence measures

A copula is a tool that can be used to analyze the connection between two variables. Several coefficients of association based on copulas are well described in the literature to describe the dependence scenario between random variables. This Section discusses Spearman's rho, Kendall's tau, and Blomqvist's beta, three popular measures of association.

3.1. *Correlation Structure*

According Huang and Kotz (1984), the correlation coefficient between X and Y is given by

$$\rho=\frac{Cov(X,Y)}{\sigma_x\sigma_y}=\frac{\alpha}{4}\,\frac{\mu_{22}-\mu_{12}}{\sigma_x}\,\frac{\nu_{22}-\nu_{12}}{\sigma_y}+\frac{\beta}{9}\,\frac{\mu_{33}-\mu_{23}}{\sigma_x}\,\frac{\nu_{33}-\nu_{23}}{\sigma_y}, \tag{10}$$

where $F_X(x)$ and $G_Y(y)$ be an absolutely continuous marginals with finite standard deviation σ_x and σ_y, $\mu_{kn}=E[X_{k:n}]$ and $\nu_{kn}=E[Y_{k:n}]$, $X_{k:n}$ and $Y_{k:n}$ are the k^{th} smallest order statistic of sample size n from F and G, respectively. Further, for any marginal choice of F with finite σ(x), the maximum upper bound of Eq. (10) is 0.50, based on optimal α and β values. Using Eq. (10), the expression of the correlation coefficient for IFGML distribution is given by

$$\rho=\frac{\alpha}{4}\prod_{i=1}^{n}\frac{\frac{(3\alpha_i^2+6\theta_i\alpha_i+2\theta_i^2)}{2\theta_i(\theta_i+\alpha_i)^2}}{\sqrt{\frac{\theta_i^2+4\theta_i\alpha_i+2\alpha_i^2}{\theta_i^2(\theta_i+\alpha_i)^2}}}+\frac{\beta}{9}\prod_{i=1}^{n}\frac{\frac{(25\alpha_i^3+75\theta_i\alpha_1^2+72\theta_i^2\alpha_i+18\theta_i^3)}{18\theta_i(\theta_i+\alpha_i)^3}}{\sqrt{\frac{\theta_i^2+4\theta_i\alpha_i+2\alpha_i^2}{\theta_i^2(\theta_i+\alpha_i)^2}}}, \tag{11}$$

when the additional parameter β=0, Eq. (11) will give the correlation coefficient of FGM distribution.

3.2. Kendall's tau

Kendall's tau for a two-dimensional random vector (X,Y) with joint distribution function $F_{XY}(xy)$ is defined as follows:

$$\tau = 4\int_x \int_y F_{XY}(xy) f_{XY}(xy)\mathrm{d}x\mathrm{d}y - 1. \tag{12}$$

Then, the expression of Kendall's tau for IFGMBL distribution is obtained as follows

$$\tau = \frac{2\alpha}{9} + \frac{\beta}{18} + \frac{\alpha\beta}{450}. \tag{13}$$

Admissible range of Kendall's τ of IFGMBL distribution are reported in Table 1.

TABLE 1
Admissible range of Kendall's τ of IFGMBL distribution.

β	α	-1	-0.5	-0.1	0.1	0.5	1
0.1		-0.2169	-0.1057	-0.0167	0.0278	0.1168	0.2280
0.5		-0.1956	-0.0839	0.0054	0.0501	0.1394	0.2511
1		-0.1689	-0.0567	0.0331	0.0780	0.1678	0.2800

When the additional parameter β=0, in Eq. (13) reduces to Kendall's tau coefficient of association of FGM distribution.

3.3. Medial correlation

Median correlation, or Blomqvist's beta, is a measure of association based on the medians. A population version of Blomqvist's beta is given by

$$\beta(F_{XY}(x,y)) = 4F_{XY}(M_X, M_Y) - 1 = 4C(\frac{1}{2}, \frac{1}{2}) - 1. \tag{14}$$

Then the expression of medial correlation for IFGMBL distribution can be obtained as

$$\beta(F_{XY}(x,y)) = \frac{\alpha}{4} + \frac{\beta}{16}. \tag{15}$$

4. Reliability measures

4.1. Dependent Stress-Strength Reliability

Let (X,Y) be a two-dimensional random variable following an IFGMBL distribution, and then the stress-strength reliability R is given by

$$R = P(Y < X) = \int_0^\infty \int_0^x f_{XY}(x,y)\mathrm{d}y\mathrm{d}x. \tag{16}$$

A stress-strength measure can be expressed as a sum of three components based on the joint density function: one in the case of independence ($\alpha = \beta = 0$), one in the case where dependence is a function of α only, and the other in the case where dependence is a function of β only, i.e.,

$$R = R_I + R_\alpha + R_\beta, \tag{17}$$

where

$$R_I = \int_0^\infty \int_0^x f_X(x) g_Y(y) \mathrm{d}x \mathrm{d}y,$$

$$R_\alpha = \alpha \int_0^\infty \int_0^x f_X(x) g_Y(y)(1-2F_X(x))(1-2G_Y(y)) \mathrm{d}x \mathrm{d}y,$$

$$R_\beta = \beta \int_0^\infty \int_0^x f_X(x) g_Y(y) F_X(x) G_Y(y)(2-3F_X(x))(2-3G_Y(y)) \mathrm{d}x \mathrm{d}y.$$

The stress-strength reliability expressions of R_I, R_α and R_β under IFGMBL distribution are obtained as

$$\begin{aligned} R_I =& 1 - \frac{\theta_2 \theta_1^2}{(\theta_1+\alpha_1)(\theta_2+\alpha_2)} \left[\frac{1}{(\theta_1+\theta_2)} + \frac{(\alpha_1+\alpha_2)}{(\theta_1+\theta_2)^2} + \frac{2\alpha_1\alpha_2}{(\theta_1+\theta_2)^3} \right] \\ & - \frac{\alpha_2 \theta_1^2}{(\theta_1+\alpha_1)(\theta_2+\alpha_2)} \left[\frac{1}{(\theta_1+\theta_2)} + \frac{\alpha_1}{(\theta_1+\theta_2)^2} \right], \end{aligned} \tag{18}$$

$$\begin{aligned} R_\alpha =& \frac{\alpha \theta_1^2 \theta_2^2}{(\theta_1+\alpha_1)(\theta_2+\alpha_2)^2} \Bigg[\frac{\left(4(\theta_2+\alpha_2)^2 - 2\theta_2(\theta_2+\alpha_2) - 2\alpha_2\theta_2 + \alpha_2^2(1+\theta_2)\right)(\theta_1+\alpha_1)}{8\theta_1^2\theta_2^2} \\ & - \frac{1}{2\theta_2} \Bigg\{ \frac{(\theta_2+\alpha_2)}{2(\theta_1+\theta_2)} + \frac{1}{4(\theta_1+\alpha_1)(\theta_1+\theta_2)^2} \Big((\theta_1+\alpha_1)(\theta_2+\alpha_2)(\alpha_1+\alpha_2) \\ & + \theta_1\alpha_1(\theta_2+\alpha_2) + \theta_2\alpha_2(\theta_1+\alpha_1) \Big) + \frac{1}{4(\theta_1+\alpha_1)(\theta_1+\theta_2)^3} \Big(\alpha_1\alpha_2(\theta_1+\alpha_1) \\ & \times (\theta_2+\alpha_2) + \alpha_1\theta_1(\theta_2+\alpha_2)(\alpha_1+\alpha_2) + \alpha_2\theta_2(\theta_1+\alpha_1)(\alpha_1+\alpha_2) \\ & + \theta_1\alpha_1\theta_2\alpha_2 \Big) + \frac{3}{8(\theta_1+\alpha_1)(\theta_1+\theta_2)^4} \Big(\alpha_1^2\alpha_2\theta_1(\theta_2+\alpha_2) + \alpha_2^2\alpha_1\theta_2(\theta_1+\alpha_1) \end{aligned}$$

$$+2\alpha_1^2\alpha_2\theta_1\theta_2\Big)+\frac{3\alpha_1^2\alpha_2^2\theta_1\theta_2}{4(\theta_1+\alpha_1)(\theta_1+\theta_2)^5}-\frac{(\theta_2+\alpha_2)}{(\theta_1+2\theta_2)}-\frac{\big((\alpha_1+\alpha_2)(\theta_2+\alpha_2)+\alpha_2\theta_2\big)}{(\theta_1+2\theta_2)^2}$$
$$-\frac{2\big(\alpha_1\alpha_2(\theta_2+\alpha_2)+2\alpha_2^2\theta_2\big)}{(\theta_1+2\theta_2)^3}-\frac{6\alpha_1\alpha_2^2\theta_2}{(\theta_1+2\theta_2)^4}\Bigg\}-\frac{1}{4\theta_2^2(\theta_1+\alpha_1)}$$
$$\times\Bigg\{\frac{\alpha_2(\theta_1+\alpha_1)(\theta_2+\alpha_2)}{(\theta_1+\theta_2)}+\frac{\big(\alpha_1\alpha_2(2\theta_1+\alpha_1)(\theta_2+\alpha_2)+\alpha_2^2\theta_2(\theta_1+\alpha_1)\big)}{2(\theta_1+\theta_2)^2}$$
$$+\frac{\big(\alpha_1^2\theta_1\alpha_2(\theta_2+\alpha_2)+\alpha_2^2\theta_2\alpha_1(\theta_1+\alpha_1)+\alpha_1\alpha_2^2\theta_1\theta_2\big)}{2(\theta_1+\theta_2)^3}+\frac{3\alpha_1^2\alpha_2^2\theta_1\theta_2}{4(\theta_1+\theta_2)^4}\Bigg\}$$
$$+\frac{1}{4\theta_2^2}\Bigg\{\frac{2\alpha_2(\theta_2+\alpha_2)}{(\theta_1+2\theta_2)}+\frac{2\big(\alpha_1\alpha_2(\theta_2+\alpha_2)+\alpha_2^2\theta_2\big)}{(\theta_1+2\theta_2)^2}+\frac{4\alpha_1\alpha_2^2\theta_2}{(\theta_1+2\theta_2)^3}\Bigg\}$$
$$-\frac{(\theta_2+\alpha_2)}{\theta_2^2(\theta_1+\alpha_1)}\Bigg\{\frac{(\theta_1+\alpha_1)(\theta_2+\alpha_2)}{(2\theta_1+\theta_2)}+\frac{\big(\alpha_1(\theta_2+\alpha_2)(2\theta_1+\alpha_1)+\alpha_2\theta_2(\theta_1+\alpha_1)\big)}{(2\theta_1+\theta_2)^2}$$
$$+\frac{2\big(\alpha_1^2\theta_1(\theta_2+\alpha_2)+\alpha_1\alpha_2\theta_2(\theta_1+\alpha_1)+\alpha_1\theta_1\alpha_2\theta_2\big)}{(2\theta_1+\theta_2)^3}+\frac{6\alpha_1^2\theta_1\theta_2\alpha_2}{(2\theta_1+\theta_2)^4}\Bigg\}$$
$$+\frac{(\theta_2+\alpha_2)}{\theta_2^2}\Bigg\{\frac{(\alpha_2+\theta_2)}{(\theta_1+\theta_2)}+\frac{\big(\alpha_1(\theta_2+\alpha_2)+\alpha_2\theta_2\big)}{(\theta_1+\theta_2)^2}+\frac{2\alpha_1\alpha_2\theta_2}{(\theta_1+\theta_2)^3}\Bigg\}\Bigg], \tag{19}$$

$$R_\beta=\frac{\beta\theta_1^2}{(\theta_1+\alpha_1)}\Bigg[\frac{2}{\theta_1}+\frac{2\big(\alpha_1\theta_1+\alpha_1^2\big)}{\theta_1^2(\theta_1+\alpha_1)}-\frac{12}{(\theta_1+\alpha_1)(\theta_2+\alpha_2)^2}\Bigg\{\frac{(\theta_1+\alpha_1)(\theta_2+\alpha_2)(1+\alpha_2)}{2(\theta_1+\theta_2)}$$
$$+\frac{1}{4(\theta_1+\theta_2)^2}\Big(\alpha_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)(1+\alpha_2)+\alpha_1\theta_1(\theta_2+\alpha_2)(1+\alpha_2)$$
$$+\alpha_2\theta_2(\theta_1+\alpha_1)(1+\alpha_2)+\alpha_2(\theta_1+\alpha_1)(\theta_2+\alpha_2)\Big)+\frac{1}{4(\theta_1+\theta_2)^3}$$
$$\times\Big(\alpha_1^2\theta_1(\theta_2+\alpha_2)(1+\alpha_2)+\alpha_1\alpha_2\theta_2(\theta_1+\alpha_1)(1+\alpha_2)+\alpha_1\alpha_2(\theta_1+\alpha_1)(\theta_2+\alpha_2)$$
$$+\alpha_1\theta_1\alpha_2\theta_2(1+\alpha_2)+\alpha_1\alpha_2\theta_1(\theta_2+\alpha_2)+\alpha_2^2\theta_2(\theta_1+\alpha_1)\Big)+\frac{3}{8(\theta_1+\theta_2)^4}$$
$$\times\Big(\alpha_1^2\theta_1\alpha_2\theta_2(1+\alpha_2)+\alpha_1^2\alpha_2\theta_1(\theta_2+\alpha_2)+\alpha_2^2\alpha_1\theta_2(\theta_1+\alpha_1)+\alpha_1\theta_1\alpha_2^2\theta_2\Big)$$
$$+\frac{24\alpha_1^2\theta_1\alpha_2^2\theta_2}{32(\theta_1+\theta_2)^5}\Bigg\}-\frac{4}{(\theta_1+\alpha_1)(\theta_2+\alpha_2)^3}\Bigg\{\frac{(\theta_1+\alpha_1)(\theta_2+\alpha_2)^3}{(2\theta_1+3\theta_2)}$$
$$+\frac{1}{(2\theta_1+3\theta_2)^2}\Big(\alpha_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)^3+\alpha_1\theta_1(\theta_2+\alpha_2)^3$$

$$
+3\theta_2\alpha_2(\theta_1+\alpha_1)(\theta_2+\alpha_2)^2\Big)+\frac{1}{(2\theta_1+3\theta_2)^3}\Big(2\alpha_1^2\theta_1(\theta_2+\alpha_2)^3
$$

$$
+6\alpha_1\alpha_2\theta_2(\theta_1+\alpha_1)(\theta_2+\alpha_2)^2+6\alpha_1\theta_1\alpha_2\theta_2(\theta_2+\alpha_2)^2
$$

$$
+6\alpha_2^2\theta_2^2(\theta_2+\alpha_2)(\theta_1+\alpha_1)\Big)+\frac{1}{(2\theta_1+3\theta_2)^4}\Big(18\alpha_1^2\theta_1\alpha_2\theta_2(\theta_2+\alpha_2)^2
$$

$$
+18\alpha_1\alpha_2^2\theta_2^2(\theta_1+\alpha_1)(\theta_2+\alpha_2)+18\alpha_1\theta_1\alpha_2^2\theta_2^2(\theta_2+\alpha_2)+6\alpha_2^3\theta_2^3(\theta_1+\alpha_1)\Big)
$$

$$
+\frac{1}{(2\theta_1+3\theta_2)^5}\Big(72\alpha_1^2\theta_1\alpha_2^2\theta_2^2(\theta_2+\alpha_2)+24\alpha_1\alpha_2^3\theta_2^3(\theta_1+\alpha_1)+24\alpha_1\theta_1\alpha_2^3\theta_2^3\Big)
$$

$$
+\frac{120\alpha_1^2\theta_1\alpha_2^3\theta_2^3}{(2\theta_1+3\theta_2)^6}\bigg\}+\frac{4}{(\theta_1+\alpha_1)(\theta_2+\alpha_2)}\bigg\{\frac{(\theta_1+\alpha_1)(\theta_2+\alpha_2)}{(2\theta_1+\theta_2)}
$$

$$
+\frac{\big(\alpha_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)+\alpha_1\theta_1(\theta_2+\alpha_2)+\alpha_2\theta_2(\theta_1+\alpha_1)}{(2\theta_1+\theta_2)^2\big)}
$$

$$
+\frac{2\big(\alpha_1^2\theta_1(\theta_2+\alpha_2)+\alpha_1\alpha_2\theta_2(\theta_1+\alpha_1)+\alpha_1\theta_1\alpha_2\theta_2\big)}{(2\theta_1+\theta_2)^3}+\frac{6\alpha_1^2\theta_1\alpha_2\theta_2}{(2\theta_1+\theta_2)^4}\bigg\}
$$

$$
-\frac{(\theta_1+\alpha_1)}{\theta_1^2}+\frac{3}{(\theta_2+\alpha_2)^2}\bigg\{\frac{(1+\alpha_2)(\theta_2+\alpha_2)}{(\theta_1+2\theta_2)}
$$

$$
+\frac{\big(\alpha_1(1+\alpha_2)(\alpha_2+\theta_2)+\alpha_2(\alpha_2+\theta_2)+\theta_2\alpha_2(1+\alpha_2)\big)}{(\theta_1+2\theta_2)^2}
$$

$$
+\frac{2\alpha_1\alpha_2(\theta_2+\alpha_2)+2\alpha_1\alpha_2\theta_2(1+\alpha_2)+2\theta_2\alpha_2^2}{(\theta_1+2\theta_2)^3}+\frac{6\alpha_1\alpha_2^2\theta_2}{(\theta_1+2\theta_2)^4}
$$

$$
+\frac{1}{(\theta_2+\alpha_2)^3}\bigg(\frac{(\alpha_2+\theta_2)^3}{(\theta_1+3\theta_2)}+\frac{\alpha_1(\alpha_2+\theta_2)^3+3\alpha_2\theta_2(\alpha_2+\theta_2)^2}{(\theta_1+3\theta_2)^2}
$$

$$
+\frac{\big(6\alpha_1\alpha_2\theta_2(\alpha_2+\theta_2)^2+6\alpha_2^2\theta_2^2(\alpha_2+\theta_2)\big)}{(\theta_1+3\theta_2)^3}+\frac{\big(18\alpha_1\alpha_2^2\theta_2^2(\alpha_2+\theta_2)+6\alpha_2^3\theta_2^3\big)}{(\theta_1+3\theta_2)^4}
$$

$$
+\frac{24\alpha_1\alpha_2^3\theta_2^3}{(\theta_1+3\theta_2)^5}\bigg)\bigg\}-\frac{1}{(\theta_2+\alpha_2)}\bigg\{\frac{(\theta_2+\alpha_2)}{(\theta_1+\theta_2)}+\frac{(\alpha_1(\theta_2+\alpha_2)+\theta_2\alpha_2)}{(\theta_1+\theta_2)^2}
$$

$$
+\frac{2\alpha_1\alpha_2\theta_2}{(\theta_1+\theta_2)^3}\bigg\}-\frac{3}{(\theta_1+\alpha_1)^2}\bigg\{\frac{(\theta_1+\alpha_1)^2}{3\theta_1}+\frac{\big(\alpha_1(\theta_1+\alpha_1)^2+2\alpha_1\theta_1(\theta_1+\alpha_1)\big)}{9\theta_1^2}
$$

$$
+\frac{\big(4\alpha_1^2\theta_1(\theta_1+\alpha_1)+2\alpha_1^2\theta_1^2\big)}{27\theta_1^3}+\frac{6\alpha_1^3\theta_1^2}{81\theta_1^4}\bigg\}+\frac{9}{(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)^2}
$$

$$
\times\bigg\{\frac{(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)(1+\alpha_2)}{(3\theta_1+2\theta_2)}+\frac{1}{(3\theta_1+2\theta_2)^2}\Big(\alpha_1(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)(1+\alpha_2)
$$

$$+2\alpha_1\theta_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)(1+\alpha_2)+\alpha_2\theta_2(\theta_1+\alpha_1)^2(1+\alpha_2)+\alpha_2(\theta_1+\alpha_1)^2$$

$$\times(\theta_2+\alpha_2)\Big)+\frac{1}{(3\theta_1+2\theta_2)^3}\Big(4\alpha_1^2\theta_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)(1+\alpha_2)+2\alpha_1\alpha_2\theta_2(\theta_1+\alpha_1)^2$$

$$\times(1+\alpha_2)+2\alpha_1\alpha_2(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)+2\alpha_1^2\theta_1^2(\theta_2+\alpha_2)(1+\alpha_2)+4\alpha_1\alpha_2\theta_1$$

$$\times(\theta_1+\alpha_1)(\theta_2+\alpha_2)+4\alpha_1\theta_1\alpha_2\theta_2(\theta_1+\alpha_1)(1+\alpha_2)+2\alpha_2^2\theta_2(\theta_1+\alpha_1)^2\Big)$$

$$+\frac{1}{(3\theta_1+2\theta_2)^4}\Big(6\alpha_1^3\theta_1^2(\theta_2+\alpha_2)(1+\alpha_2)+12\alpha_1^2\theta_1\alpha_2\theta_2(\theta_1+\alpha_1)(1+\alpha_2)+12\alpha_1^2\theta_1\alpha_2$$

$$\times(\theta_1+\alpha_1)(\theta_2+\alpha_2)+6\alpha_1\alpha_2^2\theta_2(\theta_1+\alpha_1)^2+6\alpha_1^2\theta_1^2\alpha_2\theta_2(1+\alpha_2)$$

$$+6\alpha_1^2\theta_1^2\alpha_2(\theta_2+\alpha_2)+12\alpha_1\theta_1\alpha_2^2\theta_2(\theta_1+\alpha_1)\Big]+\frac{1}{(3\theta_1+2\theta_2)^5}\Big[24\alpha_1^3\theta_1^2\alpha_2\theta_2(1+\alpha_2)$$

$$+24\alpha_1^3\theta_1^2\alpha_2(\theta_2+\alpha_2)+48\alpha_1^2\theta_1\alpha_2^2\theta_2(\theta_1+\alpha_1)+24\alpha_1^2\theta_1^2\alpha_2^2\theta_2\Big]+\frac{120\alpha_1^3\theta_1^2\alpha_2^2\theta_2}{(3\theta_1+2\theta_2)^6)}\Big\}$$

$$+\frac{3}{(\theta_1+\alpha_1)^2\theta_2+\alpha_2)^3}\Big\{\frac{(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)^3}{3(\theta_1+\theta_2)}+\frac{1}{9(\theta_1+\theta_2)^2}\Big[\alpha_1(\theta_1+\alpha_1)^2$$

$$\times(\theta_2+\alpha_2)^3+2\theta_1\alpha_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)^3+3\theta_2\alpha_2(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)^2\Big)$$

$$+\frac{1}{27(\theta_1+\theta_2)^3}\Big(4\alpha_1^2\theta_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)^3+6\alpha_1\alpha_2\theta_2(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)^2$$

$$+12\alpha_1\alpha_2\theta_1\theta_2(\theta_1+\alpha_1)(\theta_2+\alpha_2)^2+2\theta_1^2\alpha_1^2(\theta_2+\alpha_2)^3+6\theta_2^2\alpha_2^2(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)\Big)$$

$$+\frac{1}{81(\theta_1+\theta_2)^4}\Big(6\alpha_1^3\theta_1^2(\theta_2+\alpha_2)^3+36\alpha_1^2\theta_1\alpha_2\theta_2(\theta_1+\alpha_1)(\theta_2+\alpha_2)^2+18\alpha_1\alpha_2^2\theta_2^2$$

$$\times(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)+18\alpha_1^2\theta_1^2\alpha_2\theta_2(\theta_2+\alpha_2)^2+36\alpha_1\theta_1\alpha_2^2\theta_2^2(\theta_1+\alpha_1)(\theta_2+\alpha_2)$$

$$+6\alpha_2^3\theta_2^3(\theta_1+\alpha_1)^2\Big)+\frac{1}{243(\theta_1+\theta_2)^5}\Big(72\alpha_1^3\alpha_2\theta_1^2\theta_2(\theta_2+\alpha_2)^2+144\alpha_1^2\alpha_2^2$$

$$\times\theta_1\theta_2^2(\theta_1+\alpha_1)(\theta_2+\alpha_2)+24\alpha_1\alpha_2^3\theta_2^3(\theta_1+\alpha_1)^2+48\alpha_1\theta_1\alpha_2^3\theta_2^3(\theta_1+\alpha_1)$$

$$+72\theta_1^2\alpha_1^2\theta_2^2\alpha_2^2(\theta_2+\alpha_2)\Big)+\frac{1}{729(\theta_1+\theta_2)^6}\Big(360\alpha_1^3\alpha_2^2\theta_1^2\theta_2^2(\theta_2+\alpha_2)$$

$$+240\alpha_1^2\alpha_2^3\theta_1\theta_2^3(\theta_1+\alpha_1)+120\alpha_1^2\theta_1^2\alpha_2^3\theta_2^3\Big)+\frac{720\alpha_1^3\alpha_2^3\theta_1^2\theta_2^3}{2187(\theta_1+\theta_2)^7}\Big\}$$

$$
\begin{aligned}
&-\frac{3}{(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)}\Bigg\{\frac{(\theta_1+\alpha_1)^2(\theta_2+\alpha_2)}{(3\theta_1+\theta_2)}+\frac{1}{(3\theta_1+\theta_2)^2}\Bigg(\alpha_1(\theta_1+\alpha_1)^2\\
&\times(\theta_2+\alpha_2)+2\alpha_1\theta_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)+\alpha_2\theta_2(\theta_1+\alpha_1)^2\Bigg)+\frac{1}{(3\theta_1+\theta_2)^3}\\
&\times\Bigg(4\alpha_1^2\theta_1(\theta_1+\alpha_1)(\theta_2+\alpha_2)+2\alpha_1\alpha_2\theta_2(\theta_1+\alpha_1)^2+2\alpha_1^2\theta_1^2(\theta_2+\alpha_2)\\
&+4\alpha_1\theta_1\alpha_2\theta_2(\theta_1+\alpha_1)\Bigg)+\frac{1}{(3\theta_1+\theta_2)^4}\Bigg(6\alpha_1^3\theta_1^2(\theta_2+\alpha_2)\\
&+12\alpha_1^2\theta_1\theta_2\alpha_2(\theta_1+\alpha_1)+6\alpha_1^2\theta_1^2\alpha_2\theta_2\Bigg)+\frac{24\alpha_1^3\theta_1^2\alpha_2\theta_2}{(3\theta_1+\theta_2)^5}\Bigg\}\Bigg].
\end{aligned}
\tag{20}
$$

We first obtain the estimates of $\theta_1, \alpha_1, \theta_2, \alpha_2, \alpha$ and β by choosing an appropriate estimation method, then we substitute these estimates in Eq. (20), we get the reliability estimate. Further, when $\alpha_1 = \alpha_2 = 1$, in Eq. (20), then R will reduce to the reliability function of IFGM based one parameter Lindley distribution.

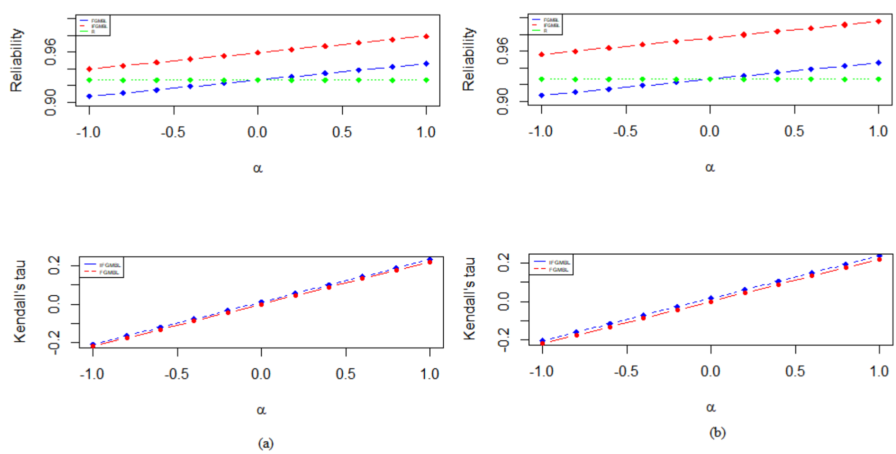


Figure 1 – Reliability function and corresponding Kendall's tau for some combinations of marginal and copula parameters (a) θ_1=0.1, α_1=0.2, θ_2=0.3, α_2=0.2 and $\beta = 0.1$ and (b) λ_1=0.2, α_1=0.3, θ_2=0.5, α_2=0.1 and $\beta = 0.3$.

A graphical comparison of R_{IFGM}, R_{FGM} and R is shown in Figure 1. The Figure displays the reliability function and corresponding Kendall's tau for some combinations of marginal and copula parameters θ_1, α_1, θ_2, α_2 and β in relation to the dependence parameter α. It is evident that assuming independence between stress and strength results in R having higher or lower values of reliability than the actual case. Let R_l, τ_l and R_u,

τ_u represent the values of the reliability function and Kendall"s tau for the lower and upper values of α, respectively. Subfigure (b) show that when X and Y are independent, R is 0.9266, but if linked by IFGM copula, R_u= 0.9952 (R_l=0.9560) with correlation τ_u=0.2396. When it linked by FGM copula R_u=0.9461 with correlation τ_u=0.2222. Therefore, IFGM is more relevant to R as it models for higher dependence between X and Y than FGM.

4.2. Survival Function

Let (X,Y) be a two-dimensional random variable, then the bivariate survival function can be defined as

$$S(x,y)=P(X>x,Y>y). \tag{21}$$

The survival function of IFGMBL distribution is given by

$$\begin{aligned} S(X,Y)=&\left(\frac{\theta_1+\alpha_1+\alpha_1\theta_1 x}{(\theta_1+\alpha_1)}e^{-\theta_1 x}\right)\left(\frac{\theta_2+\alpha_2+\alpha_2\theta_2 y}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\right)\\ &\times\left[1+\alpha\left(1-\frac{\theta_1+\alpha_1+\alpha_1\theta_1 x}{(\theta_1+\alpha_1)}e^{-\theta_1 x}\right)\left(1-\frac{\theta_2+\alpha_2+\alpha_2\theta_2 y}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\right)\right.\\ &\left.+\beta\left(1-\frac{\theta_1+\alpha_1+\alpha_1\theta_1 x}{(\theta_1+\alpha_1)}e^{-\theta_1 x}\right)^2\left(1-\frac{\theta_2+\alpha_2+\alpha_2\theta_2 y}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\right)^2\right]. \end{aligned} \tag{22}$$

4.3. Hazard Function

The bivariate hazard rate function due to Basu (1971)

$$h(x,y)=\frac{f(x,y)}{S(x,y)}, \tag{23}$$

$$\begin{aligned} h(x,y)=&\frac{1}{C(\alpha,\theta)}\left[\theta_1^2(1+\alpha_1 x)\theta_2^2(1+\alpha_2 y)\left\{1+\alpha\left(\frac{2(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{\theta_1+\alpha_1}e^{-\theta_1 x}-1\right)\right.\right.\\ &\times\left(\frac{2(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{\theta_2+\alpha_2}e^{-\theta_2 y}-1\right)+\beta\left(1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{\theta_1+\alpha_1}e^{-\theta_1 x}\right)\\ &\times\left(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{\theta_2+\alpha_2}e^{-\theta_2 y}\right)\left(\frac{3(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{\theta_1+\alpha_1}e^{-\theta_1 x}-1\right)\\ &\left.\left.\times\left(\frac{3(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{\theta_2+\alpha_2}e^{-\theta_2 y}-1\right)\right\}\right], \end{aligned} \tag{24}$$

where

$$C(\alpha,\theta)=(\theta_1+\alpha_1+\alpha_1\theta_1 x)(\theta_2+\alpha_2+\alpha_2\theta_2 y)\Big[1+\alpha\Big(1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{\theta_1+\alpha_1}e^{-\theta_1 x}\Big)$$
$$\times\Big(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{\theta_2+\alpha_2}e^{-\theta_2 y}\Big)+\beta\Big(1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x)}{\theta_1+\alpha_1}e^{-\theta_1 x}\Big)^2$$
$$\times\Big(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)}{\theta_2+\alpha_2}e^{-\theta_2 y}\Big)^2\Big].$$

Further, Johnson and Kotz (1975) defined a hazard rate function in a vector form, as shown below

$$h_V(x,y)=\left(\frac{-\partial\, lnS(x,y)}{\partial x},\ \frac{-\partial\, lnS(x,y)}{\partial y}\right). \tag{25}$$

Then, the vector components of hazard rate function using Eq. (25) can be obtained as follows

$$\begin{aligned}-\frac{\partial \ln S(x,y)}{\partial x}=&\theta_1-\frac{\alpha_1\theta_1}{(\alpha_1+\theta_1+\alpha_1\theta_1 x)}\\ &-\frac{1}{K_3(\theta,\alpha)}\Big\{\frac{\alpha\theta_1^2}{(\theta_1+\alpha_1)}\Big(1-\frac{\theta_2+\alpha_2+\alpha_2\theta_2 y}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\Big)e^{-\theta_1 x}\\ &\times(1+\alpha_1 x)+\frac{2\beta\theta_1^2}{(\theta_1+\alpha_1)}\Big(1-\frac{\theta_2+\alpha_2+\alpha_2\theta_2 y}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\Big)^2 e^{-\theta_1 x}\\ &\times(1+\alpha_1 x)\Big(1-\frac{\theta_1+\alpha_1+\alpha_1\theta_1 x}{(\theta_1+\alpha_1)}e^{-\theta_1 x}\Big)\Big\},\end{aligned} \tag{26}$$

where

$$\begin{aligned}K_3(\theta,\alpha)=&1+\alpha\Big(1-\frac{\theta_1+\alpha_1+\alpha_1\theta_1 x}{(\theta_1+\alpha_1)}e^{-\theta_1 x}\Big)\Big(1-\frac{\theta_2+\alpha_2+\alpha_2\theta_2 y}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\Big)\\ &+\beta\Big(1-\frac{\theta_1+\alpha_1+\alpha_1\theta_1 x}{(\theta_1+\alpha_1)}e^{-\theta_1 x}\Big)^2\Big(1-\frac{\theta_2+\alpha_2+\alpha_2\theta_2 y}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\Big)^2.\end{aligned} \tag{27}$$

A similar form of expression for $-\frac{\partial \ln S(x,y)}{\partial y}$ is obtained by replacing x, α_1, α_2, θ_1, θ_2 with y, α_2, α_1, θ_2 and θ_1, respectively, in Eq. (26).

4.4. *Mean Time to Failure*

The mean time to failure (MTTF) of a bivarite random vector (X,Y) can be defined as

$$\mathrm{MTTF}=\int_{\mu_1}^{\infty}\int_{\mu_2}^{\infty} S_{(X,Y)}(x,y)\mathrm{d}y\mathrm{d}x. \tag{28}$$

Then, using Equations (22) and (28), MTTF for IFGMBL distribution is obtained as

$$\text{MTTF} = \frac{(\theta_1+2\alpha_1)(\theta_2+2\alpha_2)}{\theta_1\theta_2(\theta_1+\alpha_1)(\theta_2+\alpha_2)} + \alpha K_4(\alpha_i,\theta_i) + \beta K_5(\alpha_i,\theta_i), \tag{29}$$

where

$$K_4(\alpha_i,\theta_i) = \prod_{i=1}^{2}\left(\frac{\theta_i+2\alpha_i}{\theta_i(\theta_i+\alpha_i)} - \frac{2\theta_i^2+6\theta_i\alpha_i+5\alpha_i^2}{4\theta_i(\theta_i+\alpha_i)^2}\right), \quad i=1,2, \tag{30}$$

$$\begin{aligned} K_5(\alpha_i,\theta_i) = \prod_{i=1}^{2}\Bigg(&\frac{(\theta_i+2\alpha_i)}{\theta_i(\theta_i+\alpha_i)} - \frac{2\theta_i^2+6\theta_i\alpha_i)+5\alpha_i^2}{2\theta_i(\theta_i+\alpha_i)^2} \\ &+ \frac{9(\theta_i+\alpha_i)^2(\theta_i+2\alpha_i)+6\alpha_i^2(\theta_i+\alpha_i)+2\alpha_i^3}{27\theta_i(\theta_i+\alpha_i)^3}\Bigg), \quad i=1,2. \end{aligned} \tag{31}$$

4.5. *Mean residual life function*

The mean residual life function (m.r.l) of IFGMBL distribution using the bivariate m.r.l proposed by Shanbhag and Kotz (1987) is defined as

$$r(x,y) = (r_1(x,y), r_2(x,y)), \tag{32}$$

where

$$r_1(x,y) = E(X-x|X \geq x, Y \geq y), \tag{33}$$

and

$$r_2(x,y) = E(Y-y|X \geq x, Y \geq y). \tag{34}$$

For IFGMBL distribution $r_1(x,y)$ and $r_2(x,y)$can be obtained as

$$
\begin{aligned}
r_1(x,y) =& \frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)e^{-\theta_2 y}}{(\theta_2+\alpha_2)S(x,y)}\Bigg[\frac{(\theta_1+2\alpha_1+\theta_1\alpha_1 x)e^{-\theta_1 x}}{\theta_1(\theta_1+\alpha_1)} \\
&+\alpha\Bigg\{\bigg(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)e^{-\theta_2 y}}{(\theta_2+\alpha_2)}\bigg)\bigg(\frac{(\theta_1+2\alpha_1+\theta_1\alpha_1 x)e^{-\theta_1 x}}{\theta_1(\theta_1+\alpha_1)}\bigg) \\
&-\frac{e^{-2\theta_1 x}}{2\theta_1(\theta_1+\alpha_1)^2}\bigg((\theta_1+\alpha_1+\theta_1\alpha_1 x)^2+(\theta_1+\alpha_1+\theta_1\alpha_1 x)\alpha_1 \\
&+\frac{\alpha_1^2}{2}\bigg)\Bigg\}+\beta\bigg(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y)e^{-\theta_2 y}}{(\theta_2+\alpha_2)}\bigg)^2\Bigg\{\frac{(\theta_1+2\alpha_1+\theta_1\alpha_1 x)e^{-\theta_1 x}}{\theta_1(\theta_1+\alpha_1)} \\
&-\frac{e^{-2\theta_1 x}}{\theta_1(\theta_1+\alpha_1)^2}\bigg((\theta_1+\alpha_1+\theta_1\alpha_1 x)^2+(\theta_1+\alpha_1+\theta_1\alpha_1 x)\alpha_1+\frac{\alpha_1^2}{2}\bigg) \\
&+\frac{e^{-3\theta_1 x}}{27\theta_1(\theta_1+\alpha_1)^3}\bigg(9(\theta_1+\alpha_1+\alpha_1\theta_1 x)^3+9(\theta_1+\alpha_1+\alpha_1\theta_1 x)^2 \\
&+6\alpha_1^2(\theta_1+\alpha_1+\alpha_1\theta_1 x)+2\alpha_1^3\bigg)\Bigg\}\Bigg],
\end{aligned} \tag{35}
$$

where S(x,y) is the survival function of IFGMBL distribution defined in Eq. (22). A similar form of expression for $r_2(x,y)$ is obtained by replacing x, α_1, α_2, θ_1 and θ_2 with y, α_2, α_1, θ_2 and θ_1, respectively, in Eq. (35).

4.6. *Vitality Function*

The bivariate vitality function proposed by Sankaran and Nair (1991) is given by

$$V(x,y)=(V_1(x,y),V_2(x,y)), \tag{36}$$

where

$$V_1(x,y)=E[X\mid X\geq x, Y\geq y], \tag{37}$$

and

$$V_2(x,y)=E[Y\mid X\geq x, Y\geq y]. \tag{38}$$

Further, the bivariate vitality function $V_i(x,y)$ is related to the mean residual life function $r(x,y)$ with the following relation as

$$V_i(x,y)=x+r_i(x,y), \quad i=1,2. \tag{39}$$

Then the vector components $V_1(x,y)$ and $V_2(x,y)$ for IFGMBL distribution is given by

$$V_1(x,y)=x+r_1(x,y), \tag{40}$$

and

$$V_2(x,y) = y + r_2(x,y), \tag{41}$$

where the bivariate vitality function of IFGMBL distribution can be obtained by combining Equations (36), (40) and (41).

5. Parameter estimation

A two-phase estimation method is presented in this Section for estimating marginal and dependence parameters. Phase one involves estimating the dependence parameters α and β using Blomqvist's beta and Kendall's tau measures of association, given by

$$M_{XY} = \frac{\alpha}{4} + \frac{\beta}{16}, \tag{42}$$

$$\tau_{XY} = \frac{2\alpha}{9} + \frac{\beta}{18} + \frac{\alpha\beta}{450}. \tag{43}$$

We get the sample estimates of the dependence parameters by solving the equations $M_{XY} = \tilde{M}_{XY}$ and $\tau_{XY} = \tilde{\tau}_{XY}$ simultaneously, where $\tilde{M}_{XY}$ and $\tilde{\tau}_{XY}$ are the sample versions of Blomqvist's beta and Kendall's tau respectively.

Suppose (x_i, y_i), $i = 1, 2, \ldots, n$ be the pairs of n independently generated bivariate random samples from the joint distribution function $F_{XY}(x,y)$. We first obtain the moment estimates $\tilde{\alpha}$ and $\tilde{\beta}$ of dependence parameters α and β, then the marginal parameters are estimated by maximizing the following pseudo-likelihood function.

$$\begin{aligned}
\ln l = \ln \sum_{i=1}^{n} \Bigg\{ & \frac{\theta_1^2}{(\theta_1+\alpha_1)}(1+\alpha_1 x_i)e^{-\theta_1 x_i} \frac{\theta_2^2}{(\theta_2+\alpha_2)}(1+\alpha_2 y_i)e^{-\theta_2 y_i} \\
& \times \Bigg(1 + \tilde{\alpha}\Big(\frac{2(\theta_1+\alpha_1+\theta_1\alpha_1 x_i)}{(\theta_1+\alpha_1)}\Big)e^{-\theta_1 x_i} - 1\Big)\Big(\frac{2(\theta_2+\alpha_2+\theta_2\alpha_2 y_i)}{(\theta_2+\alpha_2)}\Big)e^{-\theta_2 y_i} - 1\Big) \\
& + \tilde{\beta}\Big(1 - \frac{(\theta_1+\alpha_1+\theta_1\alpha_1 x_i)}{(\theta_1+\alpha_1)}\Big)e^{-\theta_1 x_i}\Big)\Big(1 - \frac{(\theta_2+\alpha_2+\theta_2\alpha_2 y_i)}{(\theta_2+\alpha_2)}\Big)e^{-\theta_2 y_i}\Big) \\
& \times \Big(\frac{3(\theta_1+\alpha_1+\theta_1\alpha_1 x_i)}{(\theta_1+\alpha_1)}\Big)e^{-\theta_1 x_i} - 1\Big)\Big(\frac{3(\theta_2+\alpha_2+\theta_2\alpha_2 y_i)}{(\theta_2+\alpha_2)}\Big)e^{-\theta_2 y_i} - 1\Big)\Bigg)\Bigg\}
\end{aligned} \tag{44}$$

Next, the normal equations are given as follows:

$$
\begin{aligned}
\frac{\partial \ln l}{\partial \theta_1} =& \frac{2\,n}{\theta_1} - \frac{n}{\theta_1+\alpha_1} - \sum_{i=1}^{n} x_i \\
&+ \sum_{i=1}^{n} \tilde{\alpha}\Big(\frac{2(\theta_2+\alpha_2+\theta_2\alpha_2 y_i)}{\theta_2+\alpha_2} e^{-\theta_2 y_i} - 1\Big) D(\theta_1,\alpha_1,x_i) \\
&+ \sum_{i=1}^{n} \frac{1}{(1+\frac{P_2}{P_1})} \Big\{ P_1 \tilde{\beta} G_Y(y_i)\Big(\frac{3(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2)} e^{-\theta_2 y} - 1\Big) \\
&\times \Big[\Big(\frac{(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{-\theta_1 x} - (\theta_1+\alpha_1)[(1+\alpha_1 x)e^{-\theta_1 x} - x(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{-\theta_1 x}]}{(\theta_1+\alpha_1)^2}\Big) \\
&\times \Big(\frac{3(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1)} e^{-\theta_1 x} - 1\Big) + \Big(\frac{1-(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1} e^{-\theta_1 x}\Big) \\
&\times \Big(\frac{3(\theta_1+\alpha_1)[(1+\alpha_1 x)e^{-\theta_1 x} - x(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{-\theta_1 x} - 3(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{-\theta_1 x}]}{(\theta_1+\alpha_1)^2}\Big)\Big] \\
&- P_2\Big[g_Y(y)\Big(\frac{(\theta_1+\alpha_1)(1+\alpha_1 x)(2\theta_1 e^{-\theta_1 x} - x\theta_1^2 e^{-\theta_1 x} - (1+\alpha_1 x)\theta_1^2 e^{-\theta_1 x}}{(\theta_1+\alpha_1)^2}\Big) \\
&+ \tilde{\alpha} g_Y(y)\Big[1 - 2(1 - \frac{(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1)} e^{-\theta_1 x}\Big] \\
&\times \frac{1}{(\theta_1+\alpha_1)^2}\Big(2(\theta_1+\alpha_1)\Big[(1+\alpha_1 x)e^{-\theta_1 x} - x(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{-\theta_1 x}\Big] \\
&- 2(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{-\theta_1 x}\Big) \\
&\times f_X(x) + \Big(\frac{(\theta_1+\alpha_1)(1+\alpha_1 x)(2\theta_1 e^{-\theta_1 x} - x\theta_1^2 e^{-\theta_1 x}) - (1+\alpha_1 x\theta_1^2 e^{-\theta_1 x}}{(\theta_1+\alpha_1)^2}\Big) \\
&\times \Big[1 - 2F_X(x)\Big]\Big]\Big\} \frac{1}{P_1^2} = 0,
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
\frac{\partial \ln l}{\partial \theta_2} =& \frac{2\,n}{\theta_2} - \frac{n}{\theta_2+\alpha_2} - \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \tilde{\alpha}\Big(\frac{2(\theta_1+\alpha_1+\theta_1\alpha_1 x_i)}{\theta_1+\alpha_1} e^{-\theta_1 x_i} - 1\Big) D(\theta_2,\alpha_2,y_i) \\
&+ \sum_{i=1}^{n} \frac{1}{(1+\frac{P_2}{P_1})} \Big\{ P_1 \tilde{\beta} F_X(x)\Big(\frac{3(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1)} e^{-\theta_1 x} - 1\Big) \\
&\times \frac{1}{(\theta_2+\alpha_2)^2}\Big[\Big((\theta_2+\alpha_2+\alpha_2\theta_2 y)e^{-\theta_2 y} - (\theta_2+\alpha_2) \\
&\times \Big[(1+\alpha_2 y)e^{-\theta_2 y} - y(\theta_2+\alpha_2+\alpha_2\theta_2 y)e^{-\theta_2 y}\Big]\Big)
\end{aligned}
$$

$$\times\Big(\frac{3(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2}e^{-\theta_2 y}-1\Big)+\Big(\frac{1-(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1}e^{-\theta_1 x}\Big)$$

$$\times\frac{1}{(\theta_2+\alpha_2)^2}\Big(3(\theta_2+\alpha_2)\big[(1+\alpha_2 y)e^{-\theta_2 y)}-y(\theta_2+\alpha_2+\alpha_2\theta_2 y)$$

$$\times e^{-\theta_2 y)}-3(\theta_2+\alpha_2+\alpha_2\theta_2 y)e^{-\theta_2 y)}\big]\Big)\Big]$$

$$-P_2\Big[f_X(x)\Big(\frac{(\theta_2+\alpha_2)(1+\alpha_2 y)(2\theta_2 e^{-\theta_2 y}-x\theta_2^2 e^{-\theta_2 y}-(1+\alpha_2 y)\theta_2^2 e^{-\theta_2 y}}{(\theta_2+\alpha_2)^2}\Big)$$

$$+\tilde{\alpha}f_X(x)\big[1-2(1-\frac{(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y}\big]$$

$$\times\frac{1}{(\theta_2+\alpha_2)^2}\Big(2(\theta_2+\alpha_2)\big[(1+\alpha_2 y)e^{-\theta_2 y)}-y(\theta_2+\alpha_2+\alpha_2\theta_2 y)e^{-\theta_2 y)}\big]$$

$$-2(\theta_2+\alpha_2+\alpha_2\theta_2 y)e^{-\theta_2 y)}\Big)$$

$$\times g_Y(y)+\Big(\frac{(\theta_2+\alpha_2)(1+\alpha_2 y)(2\theta_2 e^{-\theta_2 y}-y\theta_2^2 e^{-\theta_2 y})-(1+\alpha_2 y\theta_2^2 e^{-\theta_2 y}}{(\theta_2+\alpha_2)^2}\Big)$$

$$\big[1-2G_Y(y)\big]\Big]\Big\}\frac{1}{P_1^2}=0. \tag{46}$$

$$\frac{\partial\ln l}{\partial\alpha_1}=-\frac{n}{\theta_1+\alpha_1}+\sum_{i=1}^{n}\frac{x_i}{(1+\alpha_1 x_i)}+\sum_{i=1}^{n}\tilde{\alpha}\big(\frac{2(\theta_2+\alpha_2+\theta_2\alpha_2 y_i)}{(\theta_2+\alpha_2)}e^{-\theta_2 y_i}-1\big)$$

$$\times Q(\theta_1,\alpha_1,x_i)+\sum_{i=1}^{n}\frac{1}{(1+\frac{P_2}{P_1})}\Big\{P_1\tilde{\beta}G_Y(y)\Big(\frac{3(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y}-1\Big)$$

$$\times\Big[\Big(\frac{(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{-\theta_1 x}-(\theta_1+\alpha_1)(1+\theta_1 x)e^{-\theta_1 x}}{(\theta_1+\alpha_1)^2)}\Big)$$

$$\times\Big(\frac{3(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x}-1\Big)+\Big(\frac{1-(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1}e^{-\theta_1 x}\Big)$$

$$\times\Big(\frac{3(\theta_1+\alpha_1)(1+\theta_1 x)e^{-\theta_1 x}-3(\theta_1+\alpha_1+\alpha_1\theta_1 x)e^{-\theta_1 x}}{(\theta_1+\alpha_1)^2}\Big)\Big]$$

$$-P_2\Big[\frac{g_Y(y)\theta_1^2 e^{-\theta_1 x}(\theta_1 x-1)}{(\theta_1+\alpha_1)^2}+\tilde{\alpha}g_Y(y)\Big(\frac{2(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y}-1\Big)$$

$$\times\Big(\frac{\theta_1^2 e^{-\theta_1 x}(\theta_1 x-1)}{(\theta_1+\alpha_1)^2}\big(\frac{2(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x}-1\big)$$

$$+\big(\frac{2(\theta_1+\alpha_1)(1+\theta_1 x)e^{-\theta_1 x}-2(\theta_1+\alpha_1+\theta_1\alpha_1 x)e^{-\theta_1 x}}{(\theta_1+\alpha_1)^2}\big)$$

$$\times f_X(x)\Big)\Big]\Big\}\frac{1}{P_1^2}=0, \tag{47}$$

$$\begin{aligned}
\frac{\partial \ln l}{\partial \alpha_2} =& -\frac{n}{\theta_2+\alpha_2}+\sum_{i=1}^{n}\frac{y_i}{1+\alpha_2 y_i}\\
&+\sum_{i=1}^{n}\tilde{\alpha}\Big(\frac{2(\theta_1+\alpha_1+\theta_1\alpha_1 x_i)}{\theta_1+\alpha_1}e^{-\theta_1 x_i}-1\Big)Q(\theta_2,\alpha_2,y_i)\\
&+\sum_{i=1}^{n}\frac{1}{(1+\frac{P_2}{P_1})}\Big\{P_1\tilde{\beta}F_X(x)\Big(\frac{3(\theta+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1}e^{-\theta_1 x}-1\Big)\\
&\times\Big[\Big(\frac{(\theta_2+\alpha_2+\alpha_2\theta_2 y)e^{-\theta_2 y}-(\theta_2+\alpha_2)(1+\theta_2 y)e^{-\theta_2 y}}{(\theta_2+\alpha_2)^2}\Big)\\
&\times\Big(\frac{3(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2}e^{-\theta_2 y}-1\Big)+\Big(\frac{1-(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2}e^{-\theta_2 y}\Big)\\
&\times\Big(\frac{3(\theta_2+\alpha_2)(1+\theta_2 y)e^{-\theta_2 y}-3(\theta_2+\alpha_2+\alpha_2\theta_2 y)e^{-\theta_2 y}}{(\theta_2+\alpha_2)^2}\Big)\Big]\\
&-P_2\Big[\frac{f_X(x)\theta_2^2 e^{-\theta_2 y}(\theta_2 y-1)}{(\theta_2+\alpha_2)^2}+\tilde{\alpha}f_X(x)\Big(\frac{2(\theta_1+\alpha_1+\alpha_1\theta_1 x)}{(\theta_1+\alpha_1)}e^{-\theta_1 x}-1\Big)\\
&\times\Big(\frac{\theta_2^2 e^{-\theta_2 y}(\theta_2 y-1)}{(\theta_2+\alpha_2)^2}\Big(\frac{2(\theta_2+\alpha_2+\alpha_2\theta_2 y)}{(\theta_2+\alpha_2)}e^{-\theta_2 y}-1\Big)
\end{aligned}$$

$$\begin{aligned}
&+\Big(\frac{2(\theta_2+\alpha_2)(1+\theta_2 y)e^{-\theta_2 y}-2(\theta_2+\alpha_2+\theta_2\alpha_2 y)e^{-\theta_2 y}}{(\theta_2+\alpha_2)^2}\Big)\\
&\times g_Y(y)\Big)\Big]\Big\}\frac{1}{P_1^2}=0,
\end{aligned} \tag{48}$$

where

$$Q(\theta_1,\alpha_1,x_i)=\frac{((1+\theta_1 x_i)(\theta_1+\alpha_1)-(\theta_1+\alpha_1+\theta_1\alpha_1 x_i))2e^{-\theta_1 x_i}}{(\theta_1+\alpha_1)^2 L(x_i,y_i)},$$

$$Q(\theta_2,\alpha_2,y_i)=\frac{((1+\theta_2 y_i)(\theta_2+\alpha_2)-(\theta_2+\alpha_2+\theta_2\alpha_2 y_i))2e^{-\theta_2 y_i}}{(\theta_2+\alpha_2)^2 L(x_i,y_i)},$$

$$D(\theta_1,\alpha_1,x_i)=\frac{((1+\alpha_1 x_i)(\theta_1+\alpha_1)-(\theta_1+\alpha_1+\theta_1\alpha_1 x_i)(\theta_1(\theta_1+\alpha_1)+1))2e^{-\theta_1 x_i}}{(\theta_1+\alpha_1)^2 L(x_i,y_i)},$$

$$D(\theta_2,\alpha_2,y_i)=\frac{((1+\alpha_2y_i)(\theta_2+\alpha_2)-(\theta_2+\alpha_2+\theta_2\alpha_2y_i)(\theta_2(\theta_2+\alpha_2)+1))2e^{-\theta_2y_i}}{(\theta_2+\alpha_2)^2L(x_i,y_i)},$$

$$\begin{aligned}P_1=&\frac{\theta_1^2}{(\theta_1+\alpha_1)}(1+\alpha_1x)e^{-\theta_1x}\frac{\theta_2^2}{(\theta_2+\alpha_2)}(1+\alpha_2y)e^{-\theta_2y}\\&\times\Big(1+\tilde{\alpha}(\frac{2(\theta_1+\alpha_1+\theta_1\alpha_1x)}{(\theta_1+\alpha_1)})e^{-\theta_1x}-1)(\frac{2(\theta_2+\alpha_2+\theta_2\alpha_2y)}{(\theta_2+\alpha_2)})e^{-\theta_2y}-1)\Big),\end{aligned}$$

$$\begin{aligned}P_2=&\tilde{\beta}(1-\frac{(\theta_1+\alpha_1+\theta_1\alpha_1x)}{(\theta_1+\alpha_1)})e^{-\theta_1x})(1-\frac{(\theta_2+\alpha_2+\theta_2\alpha_2y)}{(\theta_2+\alpha_2)})e^{-\theta_2y})\\&\times(\frac{3(\theta_1+\alpha_1+\theta_1\alpha_1x)}{(\theta_1+\alpha_1)})e^{-\theta_1x}-1)(\frac{3(\theta_2+\alpha_2+\theta_2\alpha_2y)}{(\theta_2+\alpha_2)})e^{-\theta_2y}-1),\end{aligned}$$

$$L(x_i,y_i)=\Big[1+\tilde{\alpha}(2\frac{(\theta_1+\alpha_1+\theta_1\alpha_1x_i)}{\theta_1+\alpha_1}e^{-\theta_1x_i}-1)(2\frac{(\theta_2+\alpha_2+\theta_2\alpha_2y_i)}{\theta_2+\alpha_2}e^{-\theta_2y_i}-1)\Big].$$

Due to the nonlinear nature of these pseudo-likelihood equations, given in Eq. (45) to Eq. (48) and hence as an alternative, they can be solved numerically by adopting Newton-Raphson or any other technique.

6. SIMULATION STUDY

In this Section, we employed a two-phase estimation procedure and analyzed the effect of dependence parameters α and β on the reliability parameter R for different combinations of stress and strength parameter values with varying sample sizes on the basis of mean square error (MSE) criterion. We compare the dependence SSR parameter estimates numerically based on Monte-Carlo samples generated from IFGMBL distribution for two different sample sizes $n=20,50,100$ and $n=200$, under the assumption that α_1 and α_2 are known parameters. Following are the steps we follow to generate bivariate samples (x_i, y_i), i=1, 2, ..., n from $F_{XY}(x,y)$.

1. Generate two independent random samples u_i and t_i for $i=1,2,\dots,n$ from $U(0,1)$ distribution.

2. Compute v_i using $C(v_i|u_i)=t_i$ and where $C(v_i|u_i)$ represents the conditional copula of FGMBL distribution.

3. The simulated pairs of data, say (x_i,y_i) for $i=1,2,\dots,n$ is obtained by using the following quantiles function of Lindley distributions:

$$x_i = -\frac{(\theta_1+\alpha_1)}{\theta_1\alpha_1} - \frac{1}{\theta_1} W\left[-\frac{1}{\alpha_1}(1-u_i)(\theta_1+\alpha_1)e^{-\frac{\theta_1+\alpha_1}{\alpha_1}}\right], \quad i = 1,2,\dots,n$$

$$y_i = -\frac{(\theta_2+\alpha_2)}{\theta_2\alpha_2} - \frac{1}{\theta_2} W\left[-\frac{1}{\alpha_2}(1-v_i)(\theta_2+\alpha_2)e^{-\frac{\theta_2+\alpha_2}{\alpha_2}}\right], \quad i = 1,2,\dots,n$$

where $W(\cdot)$ is the Lambert's W function (Corless *et al.*, 1996).

With the help of R software, we perform numerical analysis of the derived reliability estimators. The reliability estimates are numerically compared based on the average estimates and MSEs using simulation results. The purpose of such an attempt is to investigate the pattern of dependence between stress and strength variables and also to observe their effects on R. An average-based estimation of stress-strength reliability parameters and its MSE results for 10000 simulated samples are presented in Table 2 and 3.

It is observed that as the dependence parameter increases, the reliability R also increases as well, i.e. the higher value of the association parameter results in a higher value of reliability. This relationship between the dependence parameter and reliability suggests that there is a strong positive correlation between the two variables. This implies that when there is a greater level of dependence, the system or process being measured is more likely to be reliable. Also, as the sample size increases, the MSEs of the estimates gradually decreases. This indicates that larger sample sizes lead to more accurate and precise estimates. which justify the consistency of reliability estimators.

However, the additional parameter β increases, the reliability R is decreases. This suggests that there is an inverse relationship between the additional parameter β and the reliability of the system or process being measured. Therefore, it can be concluded that as the value of β increases, the system becomes less reliable and vice versa.

Further, the value of R is attaining maximum (or minimum) while the dependence parameter α is maximum ($\alpha = 0.9$) or minimum ($\alpha = -0.9$). This indicates that there is a strong correlation between the dependence parameter α and the reliability R. As α approaches its maximum or minimum value, the reliability of the system reaches its highest or lowest point, respectively. Therefore, it can be inferred that the dependence parameter plays a crucial role in determining the reliability of the system or process being measured.

TABLE 2

Estimates of R when $\beta = 0.3$ with different combinations of $(\theta_1, \theta_2, \alpha_1, \alpha_2)$ with varying values of α.

		α					
n	$(\theta_1,\alpha_2, \theta_2, \alpha_2)$	-0.9	-0.5	-0.1	0.5	0.1	0.9
	(0.1,0.2,0.6,0.3)	0.81	0.83	0.85	0.86	0.89	0.90
		0.81	0.82	0.85	0.86	0.87	0.89
		0.006	0.005	0.004	0.007	0.004	0.005
20	(0.2,0.3,0.7,0.2)	0.75	0.78	0.82	0.83	0.86	0.89
		0.75	0.79	0.81	0.81	0.85	0.89
		0.008	0.007	0.006	0.004	0.007	0.005
	(0.3,0.2.0.8,0.1)	0.68	0.73	0.78	0.80	0.85	0.90
		0.67	0.70	0.77	0.80	0.84	0.88
		0.007	0.005	0.007	0.006	0.005	0.007
	(0.1,0.2,0.6,0.3)	0.81	0.83	0.85	0.86	0.89	0.90
		0.81	0.83	0.85	0.87	0.88	0.91
		0.005	0.004	0.003	0.005	0.004	0.004
50	(0.2,0.3,0.7,0.2)	0.75	0.78	0.82	0.83	0.86	0.89
		0.75	0.79	0.81	0.82	0.86	0.92
		0.006	0.006	0.004	0.004	0.005	0.004
	(0.3,0.2.0.8,0.1)	0.68	0.73	0.78	0.80	0.85	0.90
		0.68	0.71	0.77	0.81	0.85	0.90
		0.006	0.004	0.005	0.006	0.004	0.006
	(0.1,0.2,0.6,0.3)	0.81	0.83	0.85	0.86	0.89	0.90
		0.82	0.84	0.84	0.86	0.88	0.90
		0.004	0.004	0.003	0.002	0.003	0.003
100	(0.2,0.3,0.7,0.2)	0.75	0.78	0.82	0.83	0.86	0.89
		0.75	0.79	0.81	0.82	0.86	0.89
		0.004	0.004	0.004	0.003	0.005	0.003
	(0.3,0.2.0.8,0.1)	0.68	0.73	0.78	0.80	0.85	0.91
		0.69	0.72	0.79	0.81	0.85	0.91
		0.004	0.003	0.004	0.005	0.003	0.005
	(0.1,0.2,0.6,0.3)	0.8110	0.83	0.85	0.86	0.89	0.90
		0.81	0.84	0.86	0.87	0.89	0.91
		0.003	0.003	0.002	0.002	0.002	0.002
200	(0.2,0.3,0.7,0.2)	0.75	0.78	0.82	0.83	0.86	0.89
		0.75	0.78	0.82	0.83	0.87	0.91
		0.003	0.002	0.003	0.002	0.004	0.002
	(0.3,0.2.0.8,0.1)	0.68	0.73	0.78	0.80	0.85	0.91
		0.68	0.71	0.79	0.82	0.86	0.90
		0.004	0.003	0.003	0.004	0.002	0.004

The values in the first row are true value for R, the values in the second row are estimates of R and the values in the third row are MSE for R.

TABLE 3

Estimates of R when $\beta = 0.8$ with different combinations of (θ_1,θ_2, α_1, α_2) with varying values of α.

		α					
n	(θ_1,α_2, θ_2, α_2)	-0.9	-0.5	-0.1	0.5	0.1	0.9
	(0.1,0.2,0.6,0.3)	0.69	0.71	0.73	0.74	0.76	0.78
		0.68	0.73	0.73	0.75	0.78	0.78
		0.008	0.006	0.005	0.008	0.007	0.007
20	(0.2,0.3,0.7,0.2)	0.68	0.71	0.74	0.76	0.79	0.82
		0.70	0.72	0.73	0.75	0.78	0.84
		0.006	0.008	0.007	0.005	0.008	0.006
	(0.3,0.2.0.8,0.1)	0.65	0.70	0.75	0.77	0.82	0.87
		0.65	0.72	0.76	0.77	0.83	0.87
		0.007	0.006	0.007	0.008	0.006	0.007
	(0.1,0.2,0.6,0.3)	0.69	0.71	0.73	0.74	0.76	0.78
		0.69	0.72	0.75	0.76	0.76	0.78
		0.006	0.004	0.003	0.005	0.006	0.006
50	(0.2,0.3,0.7,0.2)	0.68	0.71	0.74	0.76	0.79	0.82
		0.68	0.72	0.76	0.77	0.79	0.82
		0.006	0.007	0.006	0.004	0.008	0.006
	(0.3,0.2.0.8,0.1)	0.65	0.70	0.75	0.77	0.82	0.87
		0.67	0.72	0.75	0.79	0.82	0.89
		0.006	0.004	0.005	0.006	0.005	0.006
	(0.1,0.2,0.6,0.3)	0.69	0.71	0.73	0.74	0.76	0.78
		0.70	0.72	0.73	0.75	0.77	0.80
		0.005	0.004	0.002	0.005	0.004	0.006
100	(0.2,0.3,0.7,0.2)	0.68	0.71	0.74	0.76	0.79	0.82
		0.69	0.73	0.76	0.76	0.77	0.82
		0.004	0.006	0.005	0.003	0.006	0.005
	(0.3,0.2.0.8,0.1)	0.65	0.70	0.75	0.77	0.82	0.87
		0.65	0.71	0.74	0.78	0.82	0.89
		0.004	0.002	0.004	0.006	0.003	0.004
	(0.1,0.2,0.6,0.3)	0.69	0.71	0.73	0.74	0.76	0.78
		0.70	0.71	0.74	0.75	0.76	0.79
		0.003	0.002	0.002	0.003	0.003	0.005
200	(0.2,0.3,0.7,0.2)	0.68	0.71	0.74	0.76	0.79	0.82
		0.68	0.71	0.74	0.75	0.79	0.82
		0.003	0.004	0.004	0.002	0.005	0.003
	(0.3,0.2.0.8,0.1)	0.65	0.70	0.75	0.77	0.82	0.87
		0.65	0.70	0.75	0.78	0.82	0.87
		0.003	0.001	0.002	0.004	0.002	0.003

The values in the first row are true value for R, the values in the second row are estimates of R and the values in the third row are MSE for R.

7. Real data analysis

The data set initially reported by McGilchrist and Aisbett (1991) is analyzed in this Section. The data showed recurrence times of infections among kidney disease patients using portable dialysis equipment. The data sets stress (Y) and strength (X) were transformed by taking the square root of the data and dividing it by 0.1. We observe that our model fits well with the resultant data (Table 4). The correlation coefficient and test of correlation for the real data are reported in Table 5.

TABLE 4
Goodness of fit test for data.

	X				Y			
	D	P-value	AIC	BIC	D	P-value	AIC	BIC
Lindley	0.14	0.64	325.98	328.78	0.11	0.84	318.79	321.60
Weibull	0.15	0.55	326.11	329.80	0.15	0.54	317.11	319.92
Gen.Exp	0.15	0.56	325.69	328.50	0.13	0.68	316.18	318.99

TABLE 5
The correlation coefficient and test of correlation for real data.

Correlation measure	Correlation	P-value
Pearson's	0.09	0.64
Kendall's	0.11	0.39

As a first step, we fitted the Lindley distribution separately to X and Y and determined its validity. Figures 2 and 3 show the empirical and theoretical c.d.f and P–P plots for Lindley with other univariate distributions.

After fitting the bivariate Lindley distribution based on the IFGM copula, the results are compared to those of other FGMB distributions (Table 6).

TABLE 6
The estimates of the parameters of FGM distributions for data.

	$\hat{\theta}_1$	$\hat{\alpha}_1$	$\hat{\theta}_2$	$\hat{\alpha}_2$	$\hat{\alpha}$	$\hat{\beta}$	AIC	BIC
IFGM-Lindley	0.02	0.10	0.02	0.18	0.59	0.27	643.79	652.19
FGM-Lindley	0.02	4.33	0.02	2.37	0.44	-	646.33	653.33
FGM-Weibull	0.21	1.45	0.22	2.47	0.42	-	864.16	871.16
FGM-Gen.Exp	0.41	0.006	0.58	0.009	0.45	-	685.40	692.40

Figure 2 – The plot of empirical and theoretical c.d.f.'s and P–P plot for data X.

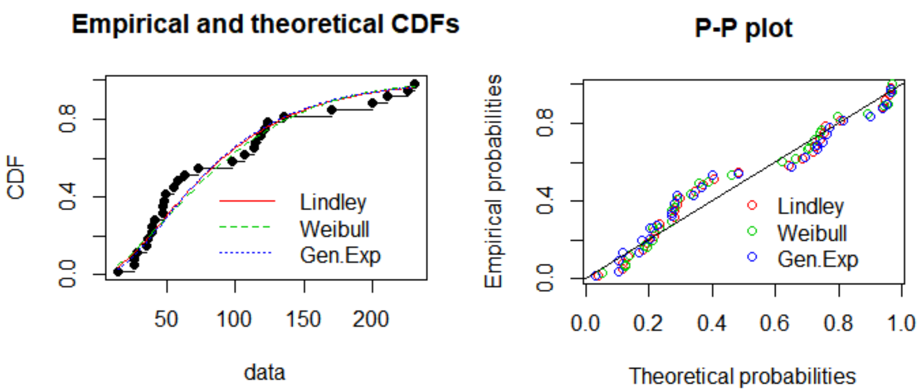
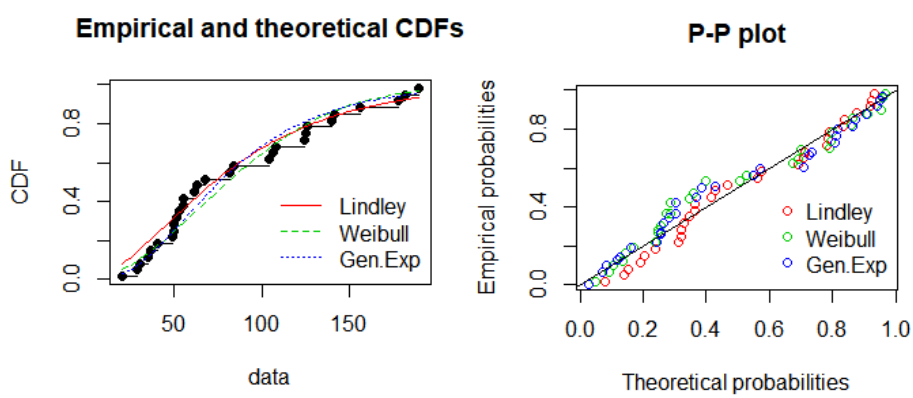


Figure 3 – The plot of empirical and theoretical c.d.f's and P–P plot for data Y



8. Concluding remarks

Huang and Kotz (1984) investigated the single iterated FGM distribution and found that the maximum positive correlation is higher than the usual FGM distribution. Also, they showed that a single iteration can triple the covariance for certain marginal distributions. By examining the impact of iteration on covariance and correlation, researchers can gain a deeper understanding of the behavior of the single iterated FGM distribution in different contexts. This knowledge could lead to more accurate modeling and analysis in fields where a higher positive correlation is important. This present study is an attempt in this direction.

In this article, we consider Iterated FGM bivariate distribution with Lindley marginals. The dependence stress-strength reliability model under IFGMBL distribution is proposed. However, the classical FGM bivariate distribution restricts the association measure between (-0.33, 0.33). As an alternatives, to model higher-order dependence measure between random variables, many authors proposed several extensions and generalizations of the classical FGM in the literature.

In this study, we consider the two parameter extension of FGM family proposed by Huang and Kotz (1984) viz., Iterated FGM, is capable to capture more relationships between X and Y. Based on the expression for the correlation coefficient τ, we found that the IFGMBL distribution is more appropriate for modeling higher positive associations than the FGMBL distribution. Moreover, we found that increasing β enhances the upper bound of τ.

However, the dependence parameter α and the additional parameter β are also compared graphically for SSR using IFGM and FGM with Lindley distribution. The graphical representation clearly shows that if the association between X and Y is neglected, reliability could be either over or under estimated.

We used the Monte-Carlo simulation to study the performance of dependence stress-strength reliability. Further, an investigation on R in relation to the variation in dependence parameter α as well as additional parameter β is performed.

Finally, a real data set is considered to demonstrate the application of the proposed model, and it shows that the IFGMBL distribution is more suitable model as compared to other existing FGM and IFGM distributions based on AIC and BIC. Therefore, the proposed IFGM-based bivariate Lindley is the best choice for dependence stress-strength modeling.

As for future research perspectives, asymptotic properties of IFGMBL distribution may be considered further. Moreover, different families of stress and strength distributions as well as other types of copulas may be used, and the corresponding value of R and their associated properties can be derived. Further, such approaches can be extended in a Bayesian point of view in future studies.

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