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# SOME PROPERTIES OF EXPONENTIATED - EXPONENTIAL LOGISTIC DISTRIBUTION AND ITS RELATED REGRESSION MODEL

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SUMMARY

This paper introduces a new class of logistic distribution, namely Exponentiated logistic distribution, which is derived from type II logistic distribution. We have investigated its properties, discussed parameter estimation, and demonstrated its usefulness in analysing real-life medical data. The developed model provides researchers with valuable tools for accurately modelling and analysing medical phenomena, thereby contributing to advancements in healthcare research and decision-making.

*Keywords*: Logistic distribution; Logistic regression model; Simulation

## 1. INTRODUCTION

The logistic growth function is quite relevant from a practical point of view, and it has been applied as a growth model in several areas of research such as biology (Pearl, 1924; Schultz, 1930; Oliver, 1982), bioassay problems (Pearl, 1940; Emmens, 1940; Wilson and Worcester, 1943; Berkson, 1944, 1951; Finney, 1947, 1978), survival data (Plackett, 1959), public health (Dyke and Patterson, 1952), etc. For a detailed account of the properties and applications of the logistic model see Balakrishnan (2013). A continuous random variable $X$ is said to follow the standard logistic distribution (LD) if its probability density function (PDF) is of the following form:

$$f_1(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \tag{1}$$

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where $x \in R = (-\infty, +\infty)$. Balakrishnan and Leung (1988) proposed two generalised logistic distributions of type I ($LD_I$) and type II ($LD_{II}$), respectively, through the following PDFs $f_2(\cdot)$ and $f_3(\cdot)$, in which $x \in R$, $\alpha > 0$ and $\beta > 0$

$$f_2(x, \alpha, \beta) = \alpha\beta \frac{\alpha e^{-\beta x}}{\left(1 + e^{-\beta x}\right)^{\alpha+1}} \tag{2}$$

and

$$f_3(x, \alpha) = \frac{\alpha e^{-\alpha x}}{(1 + e^{-x})^{\alpha+1}}. \tag{3}$$

The corresponding CDFs of $LD_I$ and $LD_{II}$ are, respectively

$$F_2(x) = \frac{1}{\left(1 + e^{-\beta x}\right)^{\alpha}} \tag{4}$$

and

$$F_3(x) = 1 - \frac{e^{-\alpha x}}{(1 + e^{-x})^{\alpha}}. \tag{5}$$

Some inference aspects of $LD_{II}$ have been investigated in Balakrishnan and Hossain (2007). Recently exponentiated versions of various distributions have been studied in the literature in order to create more flexibility in the respective models. Exponentiated version of $LD_{II}$ is studied by Manju (2016). Also the distribution with same PDF has been studied by Sapkota (2020). Through the present paper we consider a detailed study of the Exponentiated-Exponential Logistic distribution and organised the paper as follows. In Section 2, we present the definition of the Exponentiated-Exponential logistic distribution (ELD) and describe some important properties. A location scale extension along with the maximum likelihood estimation of the parameters of the ELD is considered in Section 3. In Section 4, two real life medical datasets are considered for illustrating the usefulness of the model compared to the LD, the $LD_I$ and the $LD_{II}$. In Section 5, a generalized likelihood ratio test procedure is suggested for testing the significance of the parameters and in Section 6, a simulation study is conducted to examine the performance of the maximum likelihood estimators (MLEs) of the parameters of the ELD. In Section 7, the regression model of this distribution is proposed along with two numerical applications. A brief simulation study is also carried out here.

## 2. Definition and properties

A continuous random variable $X$ is said to follow an exponentiated type II logistic distribution if its CDF is of the following form: $x \in R$ , $\alpha > 0$ and $\beta > 0$

$$F(x) = \left[1 - \frac{e^{-\alpha x}}{(1 + e^{-x})^{\alpha}}\right]^{\beta}. \tag{6}$$

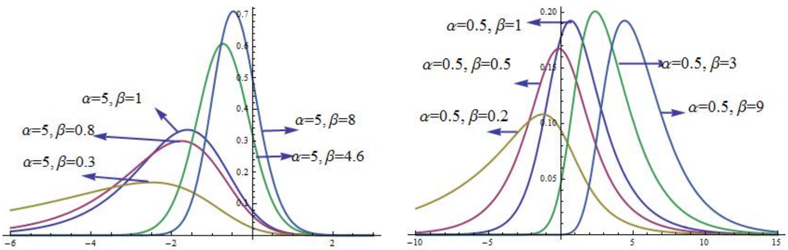
The density function corresponding to Eq. (6) is given by

$$f(x;\alpha,\beta)=\alpha\beta\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha+1}}\left[1-\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right]^{\beta-1}. \tag{7}$$

The distribution of a random variable having CDF in Eq. (6) or PDF in Eq. (7) is hereafter we denoted by $\text{ELD}_{\text{II}}(\alpha,\beta)$. Note that

1. when $\beta=1$, the $\text{ELD}_{\text{II}}$ reduces to the $\text{LD}_{\text{II}}$,
2. when the $\alpha=1$ and $\beta=1$, the $\text{ELD}_{\text{II}}$ reduces to LD.

The PDF plots of $\text{ELD}_{\text{II}}(\alpha,\beta)$ for different choices of $\alpha$ and $\beta$ are given in Figure 1. From Figure 1 it clear that the distribution is positively skewed for $\alpha<1$ , $\beta>1$ and is negatively skewed for $\alpha>1$ , $\beta<1$.




(a) $\alpha>1,\beta<1$ (b) $\alpha<1,\beta>1$

*Figure 1* – Plots of PDF of $\text{ELD}_{\text{II}}$ for varying $\alpha$ and $\beta$.

LEMMA 1. *The characteristic function $\Phi_X(t)$ of $ELD_{II}(\alpha,\beta)$ with PDF in Eq. (7) is the following, for $t\in R$*

$$\Phi_X(t)=\alpha\beta\sum_{k=0}^{\infty}(-1)^k\binom{\beta-1}{k}B(\alpha k+\alpha-it,\,1+it), \tag{8}$$

$Re(\alpha k+\alpha-it)>0$, $Re(1+it)>0$.

PROOF. Let $X$ follows $\text{ELD}_{\text{II}}(\alpha,\beta)$ with PDF in Eq. (7). Then, by the definition of characteristic function, we have the following for any $t\in R$ and $i=\sqrt{-1}$

$$\begin{aligned}\Phi_X(t)&=\int_{-\infty}^{\infty}e^{itx}f(x)dx\\&=\int_{-\infty}^{\infty}e^{itx}\alpha\beta\left(\tfrac{e^{-x}}{1+e^{-x}}\right)^{\alpha}\tfrac{1}{1+e^{-x}}\left(1-\tfrac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right)^{\beta-1}dx.\end{aligned} \tag{9}$$

Put $u = \frac{e^{-x}}{1+e^{-x}}$ in Eq. (9) one get,

$$\Phi_X(t) = \alpha\beta \int_0^1 (1-u)^{it} u^{\alpha-it-1} (1-u^{\alpha})^{\beta-1} du. \tag{10}$$

Now expanding $(1-u^{\alpha})^{\beta-1}$ in Eq. (10) and rearranging the terms to obtain the following

$$\Phi_X(t) = \alpha\beta \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \int_0^1 u^{\alpha k+\alpha-it-1} (1-u)^{it} du,$$

which gives Eq. (8), in light of the definition of beta integral. □

LEMMA 2. *The mean and variance of $ELD_{II}(\alpha,\beta)$ with PDF in Eq. (7) are, respectively,*

$$\begin{aligned} Mean &= \alpha\beta \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \left( \frac{1}{\eta^2_{(0,0)}} - \frac{1}{\eta_{(0,0)}} \left[ \Psi\left(\eta_{(0,1)}\right) - \Psi(1) \right] \right) \\ &= \Lambda(\alpha,\beta)(say), \end{aligned} \tag{11}$$

$$\begin{aligned} Variance &= 2\alpha\beta \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i=1}^{j} (-1)^k \binom{\beta-1}{k} \left[ \frac{1}{i(j+1)\eta_{(j,1)}} + \frac{1}{\eta^3_{(0,0)}} + \frac{1}{\eta_{(0,0)}\eta^2_{(j,0)}} - \right. \\ & \left. \frac{\Psi\left(\eta_{(0,1)}\right) - \Psi(1)}{\eta^2_{(0,0)}} \right] - \Lambda^2(\alpha,\beta), \end{aligned} \tag{12}$$

*where*

$$\Psi(a) = \frac{d \log \Gamma a}{da}$$

*and*

$$\eta_{(a,b)} = \alpha + \alpha k + a + b.$$

PROOF.

$$\begin{aligned} \mu_1' &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \alpha\beta \int_{-\infty}^{\infty} x \left( \frac{e^{-x}}{1+e^{-x}} \right)^{\alpha} \frac{1}{1+e^{-x}} \left( 1 - \frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}} \right)^{\beta-1} dx \\ &= \alpha\beta \int_0^1 \ln\left( \frac{1-u}{u} \right) u^{\alpha-1} (1-u^{\alpha})^{\beta-1} du, \quad \text{where} \quad u = \frac{e^{-x}}{1+e^{-x}}. \end{aligned} \tag{13}$$

By applying binomial expansion in $(1-u^{\alpha})^{\beta-1}$, Eq. (13) reduces to

$$\mu_1' = \alpha\beta \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \left( \int_0^1 \ln(1-u)\, u^{\alpha+\alpha k+1} du - \int_0^1 \ln u \; u^{\alpha+\alpha k+1} du \right). \tag{14}$$

From Gradshteyn and Ryzhik (2000) we have

$$\int_0^1 x^{\mu-1} \ln(1-x) dx = -\frac{1}{\mu} [\Psi(\mu+1) - \Psi(1)], \qquad [Re(\mu) > -1]. \tag{15}$$

Applying Eq. (15) in the first integral and using product rule in the second integral of Eq. (14) one can obtain Eq. (11)

$$\text{Variance} = \int_{-\infty}^{\infty} x^2 f(x) dx - \Lambda^2(\alpha, \beta), \tag{16}$$

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 f(x) dx &= \int_{-\infty}^{\infty} x^2 \alpha\beta \left( \frac{e^{-x}}{1+e^{-x}} \right)^{\alpha} \frac{1}{1+e^{-x}} \left[ 1 - \frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}} \right]^{\beta-1} dx \\ &= \alpha\beta \int_0^1 [\ln(1-u) - \ln u]^2 u^{\alpha-1} (1-u^{\alpha})^{\beta-1} du, \end{aligned} \tag{17}$$

in which $u = \frac{e^{-x}}{1+e^{-x}}$. Applying binomial expansion in $(1-u^{\alpha})^{\beta-1}$, Eq. (17) reduces to

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 f(x) dx &= \alpha\beta \sum_{k=0}^{\infty} \binom{\beta-1}{k} (-1)^k \int_0^1 [\ln(1-u) - \ln(u)]^2 \, u^{\alpha+\alpha k-1} du \\ &= \alpha\beta \sum_{k=0}^{\infty} \binom{\beta-1}{k} (-1)^k [I_1 + I_2 + I_3], \end{aligned} \tag{18}$$

where

$$I_1 = \int_0^1 [\ln(1-u)]^2 u^{\alpha+\alpha k-1} du, \tag{19}$$

$$I_2 = -2 \int_0^1 \ln(u) \ln(1-u)\, u^{\alpha+\alpha k-1} du \tag{20}$$

and

$$I_3 = \int_0^1 [\ln(u)]^2 u^{\alpha+\alpha k-1} du. \tag{21}$$

From Gradshteyn and Ryzhik (2000) we have

$$[\ln(1 \pm u)]^2 = 2 * \sum_{j=1}^{\infty} \frac{(\mp)^{j+1} u^{j+1}}{j+1} \sum_{i=1}^{j} \frac{1}{i} \quad \text{for} \quad u^2 < 1. \tag{22}$$

Applying Eq. (22) in Eq. (19), $I_1$ reduces to

$$I_1 = 2 \sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{1}{i(j+1)(\alpha+\alpha k+j+1.)} \tag{23}$$

Applying product rule of integration in Equations (20) and (21), $I_2$ and $I_3$ reduce to the following forms:

$$I_2 = 2 \sum_{j=1}^{\infty} \frac{1}{(\alpha+\alpha k)(\alpha+\alpha k+j)^2} - \frac{\Psi(\alpha+\alpha k+1)-\Psi(1)}{(\alpha+\alpha k)^2} \tag{24}$$

and

$$I_3 = \frac{2}{(\alpha+\alpha k)^3}. \tag{25}$$

Equations (16) and (18) yield Eq. (12) in the light of Equations (23), (24) and (25). □

LEMMA 3. *Measure of skewness $g_a$ of $ELD_{II}(\alpha, \beta)$ with PDF in Eq. (7) is given by*

$$g_a = \log\left(\frac{\rho_{0.5}-1}{\rho_{0.8}-1}\right)\left[\log\left(\frac{\rho_{0.2}-1}{\rho_{0.8}-1}\right)\right]^{-1}, \tag{26}$$

*in which* $\rho_c^{-1} = [1-(1-c^{1/\beta})^{1/\alpha}]$.

PROOF. Galton (1896) introduced the percentile oriented measure of skewness as

$$g_a = \frac{x_{0.8}-x_{0.5}}{x_{0.5}-x_{0.2}}, \quad 0 < g_a < \infty, \tag{27}$$

where $g_a = 1$ indicates symmetry, $g_a < 1$ is interpreted as skewness to the left and $g_a > 1$ is interpreted as skewness to the right. From Eq. (6), one can write the quantile of $\mathrm{ELD}_{\mathrm{II}}$ as

$$X_\gamma = -\log\left[\left(1-\left(1-\gamma^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right)^{-1} - 1\right]. \tag{28}$$

Substituting Eq. (28) in Eq. (27) yields Eq. (26). From Eq. (26) it is evident that skewness depends on both $\alpha$ and $\beta$. The Galton's percentile measure of skewness for different values of $\alpha$ and $\beta$ are given in Table 1. □

*TABLE 1*
*Galton's percentile measure of skewness of $ELD_{II}$ for varying values of $\alpha$ and $\beta$.*

| $\alpha$ \ $\beta$ | 0.5000 | 0.5299 | 0.6000 | 0.8041 | 2.0000 | 2.0372 | 4.6193 | 5.0000 | 7.0000 | 7.0351 | 9.0000 | 9.1201 | 10.0000 | 10.8884 | 11.6005 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.9742 | **1.0000** | 1.0541 | 1.1706 | 1.3714 | 1.3731 | 1.3951 | 1.3941 | 1.3875 | 1.3874 | 1.3816 | 1.3813 | 1.3791 | 1.3772 | 1.3757 |
| 0.8 | 0.8464 | 0.8656 | 0.9066 | **1.0000** | 1.2147 | 1.2176 | 1.3044 | 1.3092 | 1.3248 | 1.3249 | 1.3325 | 1.3328 | 1.3350 | 1.3367 | 1.3378 |
| 2 | 0.7249 | 0.7371 | 0.7636 | 0.8259 | 0.9971 | **1.0000** | 1.1104 | 1.1191 | 1.1528 | 1.1532 | 1.1747 | 1.1758 | 1.1831 | 1.1897 | 1.1943 |
| 5 | 0.6788 | 0.6883 | 0.7090 | 0.7580 | 0.8980 | 0.9006 | **1.0000** | 1.0083 | 1.0414 | 1.0419 | 1.0638 | 1.0650 | 1.0727 | 1.0796 | 1.0846 |
| 10 | 0.6638 | 0.6725 | 0.6912 | 0.7357 | 0.8644 | 0.8667 | 0.9600 | 0.9679 | 0.9996 | **1.0000** | 1.0212 | 1.0222 | 1.0297 | 1.0365 | 1.0413 |
| 20 | 0.6564 | 0.6646 | 0.6824 | 0.7247 | 0.8475 | 0.8497 | 0.9396 | 0.9472 | 0.9779 | 0.9784 | 0.9989 | **1.0000** | 1.0073 | 1.0139 | 1.0186 |
| 50 | 0.6520 | 0.6599 | 0.6771 | 0.7181 | 0.8373 | 0.8395 | 0.9272 | 0.9347 | 0.9647 | 0.9652 | 0.9854 | 0.9864 | 0.9936 | **1.0000** | 1.0047 |
| 100 | 0.6505 | 0.6584 | 0.6754 | 0.7159 | 0.8339 | 0.8361 | 0.9230 | 0.9305 | 0.9603 | 0.9607 | 0.9808 | 0.9818 | 0.9889 | 0.9953 | **1.0000** |

LEMMA 4. *The PDF of the $k^{th}$ order statistics $X_{k:n}$ of $ELD_{II}(\alpha,\beta)$ is*

$$f_{k:n}(x) = \frac{\alpha\beta}{B(k,n-k+1)} \frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha+1}} \left[1-\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right]^{\beta k-1} \left[1-\left(1-\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right)^{\beta}\right]^{n-k}. \tag{29}$$

PROOF. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the $\mathrm{ELD_{II}}(\alpha,\beta)$ and let $X_{k:n}$ be the $k^{\text{th}}$ order statistics for k = 1, 2, …, n. Let $F_{X_{k:n}}(x)$ and $f_{X_{k:n}}(x)$ denote CDF and PDF of $X_{k:n}$, respectively,

$$f_{k:n}(x) = \frac{1}{B(k,n-k+1)}[F(x)]^{k-1}[1-F(x)]^{n-k} f(x), \tag{30}$$

for $x \in$ R. Applying Equations (6) and (7) in Eq. (30) gives Eq. (29). □

From Lemma 4, we have the following Remarks.

REMARK 5. *The distribution of the largest order statistic $X_{n:n}$ taken from a population following $ELD_{II}(\alpha,\beta)$ is $ELD_{II}(\alpha,n\beta)$.*

REMARK 6. *The PDF of the smallest order statistics is*

$$f_{1:n}(x) = n\alpha\beta \frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha+1}} \left[1-\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right]^{\beta-1} \left[1-\left(1-\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right)^{\beta}\right]^{n-1}. \tag{31}$$

LEMMA 7. *The Renyi entropy of $ELD_{II}(\alpha,\beta)$ is given by*

$$I_R(\gamma) = \frac{1}{1-\gamma}\left\{\gamma\log(\alpha\beta) + \log\left[\sum_{k=0}^{\infty}(-1)^k \binom{\beta\gamma-\gamma}{k} B(\alpha\gamma+\alpha k,\gamma)\right]\right\}. \tag{32}$$

PROOF. For $\gamma > 0$ and $\gamma \neq 1$ the Renyi entropy is defined by

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)\,dx\right\}, \\ &= \frac{1}{1-\gamma}\log\left\{(\alpha\beta)^{\gamma}\int\left(\frac{e^{-x}}{1+e^{-x}}\right)^{\alpha\gamma}\frac{1}{(1+e^{-x})^{\gamma}}\left(1-\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right)^{\gamma(\beta-1)}dx\right\}. \end{aligned} \tag{33}$$

On substituting $\frac{e^{-x}}{1+e^{-x}} = u$ in Eq. (33),

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma}\log\left\{(\alpha\beta)^{\gamma}\int_0^1 u^{\alpha\gamma-1}(1-u)^{\gamma-1}(1-u^{\alpha})^{\gamma(\beta-1)}\,du\right\} \\ &= \frac{1}{1-\gamma}\log\left\{(\alpha\beta)^{\gamma}\sum_{k=0}^{\infty}(-1)^k\binom{\beta\gamma-\gamma}{k}\int_0^1 u^{\alpha\gamma+\alpha k-1}(1-u)^{\gamma-1}du\right\}, \end{aligned}$$

by applying binomial expansion in $(1-u^{\alpha})^{\gamma(\beta-1)}$. Thus, we have

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\left\{(\alpha\beta)^{\gamma}\sum_{k=0}^{\infty}(-1)^k\binom{\beta\gamma-\gamma}{k}B(\alpha\gamma+\alpha k,\gamma)\right\},$$

which gives Eq. (32). □

LEMMA 8. *The survival function is given by*

$$S(x)=1-\left[1-\frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right]^{\beta},x\in R$$

.

PROOF. The proof follows directly from Eq. (6). □

LEMMA 9. *The hazard rate function is given by*

$$h(x)=\frac{\alpha\beta(1+e^{x})^{-\alpha}(1+e^{-x})^{-1}[1-(1+e^{x})^{-\alpha}]^{\beta-1}}{1-[1-(1+e^{x})^{-\alpha}]^{\beta}},x\in R.$$

PROOF. By definition

$$h(x)=\frac{f(x)}{1-F(x)}$$

Now the proof is straight forward in the light of Equations (6) and (7). □

## 3. LOCATION SCALE EXTENSION AND MAXIMUM LIKELIHOOD ESTIMATION

In this Section, we discuss an extended form of $ELD_{II}(\alpha,\beta)$ by introducing the location parameter $\mu$ and scale parameter $\sigma$, and discuss the maximum likelihood estimation of the parameters of extended form of $ELD_{II}(\alpha,\beta)$.

Let $Z$ follows the $ELD_{II}(\alpha,\beta)$ with PDF in Eq. (7). Then, $X=\mu+\sigma Z$ is said to have an extended $ELD_{II}$ with parameters $\mu,\sigma,\alpha$ and $\beta$, denoted by $EELD_{II}(\mu,\sigma;\alpha,\beta)$ with the following PDF,

$$f(x,\mu,\sigma,\alpha,\beta)=\frac{\alpha\beta e^{\frac{-\alpha(x-\mu)}{\sigma}}\left[1-\frac{e^{\frac{-\alpha(x-\mu)}{\sigma}}}{\left[1+e^{\frac{-(x-\mu)}{\sigma}}\right]^{\alpha}}\right]^{\beta-1}}{\sigma\left[1+e^{\frac{-(x-\mu)}{\sigma}}\right]^{\alpha+1}}, \tag{34}$$

in which $x\in R$, $\mu\in R$, $\alpha>0$, $\beta>0$.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population having $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$ with the PDF in Eq. (34). The log-likelihood function $l = \ln L(\mu,\sigma;\alpha,\beta)$ of the random sample is

$$\begin{aligned} l &= n\log\alpha + n\log\beta - n\log\sigma - \alpha\sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma} \\ &\quad +(\beta-1)\sum_{i=1}^{n}\log\left[1-\frac{e^{\frac{-\alpha(x_i-\mu)}{\sigma}}}{\left[1+e^{\frac{-(x_i-\mu)}{\sigma}}\right]^{\alpha}}\right]-(\alpha+1)\sum_{i=1}^{n}\log\left[1+e^{\frac{-(x_i-\mu)}{\sigma}}\right]. \end{aligned} \tag{35}$$

On differentiating Eq. (35) with respect to parameters $\mu$, $\sigma$, $\alpha$, $\beta$ and equating to zero, we obtain the following likelihood equations, in which $z_i = \frac{x_i-\mu}{\sigma}$ and $\Omega_{ji} = \left(\frac{e^{-z_i}}{1+e^{-z_i}}\right)^{j}$, for $j = 1, \alpha$

$$\frac{n\alpha}{\sigma} = \frac{(\alpha+1)}{\sigma}\sum_{i=1}^{n}\Omega_{1i} + \frac{\alpha(\beta-1)}{\sigma}\sum_{i=1}^{n}\frac{\Omega_{\alpha i}(1-\Omega_{1i})}{(1-\Omega_{\alpha i})}, \tag{36}$$

$$\frac{n}{\sigma} = \frac{\alpha}{\sigma}\sum_{i=1}^{n}z_i - \frac{(\alpha+1)}{\sigma}\sum_{i=1}^{n}z_i\Omega_{1i} - \frac{\alpha(\beta-1)}{\sigma}\sum_{i=1}^{n}z_i\frac{\Omega_{\alpha i}(1-\Omega_{1i})}{(1-\Omega_{\alpha i})}, \tag{37}$$

$$\frac{n}{\alpha} = \sum_{i=1}^{n}z_i + (\beta-1)\sum_{i=1}^{n}\frac{\Omega_{\alpha i}}{1-\Omega_{\alpha i}}\log(\Omega_{1i}) - \sum_{i=1}^{n}\log(1-\Omega_{1i}), \tag{38}$$

$$0 = \frac{n}{\beta} + \sum_{i=1}^{n}\log(1-\Omega_{\alpha i}). \tag{39}$$

On solving the system of Equations (36) - (39) with the help of some mathematical software such as MATLAB, MATHCAD, MATHEMATICA, R etc. one can obtain the maximum likelihood estimators (MLEs), $\hat{\theta} = \left(\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\beta}\right)$ of the parameters of $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$. Second order partial derivatives of Eq. (35) with respect to the parameters are obtained and find that the equations give negative values for $\alpha > 0, \beta > 0, \sigma > 0, \mu \in R$. These equations are given in Appendix A. Further this is verified with the help of MATHEMATICA Software. For interval estimation of $\theta = (\alpha,\beta,\mu,\sigma)$ and test of hypothesis, one requires the Fisher Information matrix, $I(\theta)$. The elements of this matrix can be worked out as follows:

$$
\begin{aligned}
I_{11} &= E\left(-\frac{\partial^2 l}{\partial \alpha^2}\right) = \frac{n}{\alpha^2} + (\beta - 1)J_1 \\
I_{12} &= E\left(-\frac{\partial^2 l}{\partial \alpha \partial \beta}\right) = J_2 \\
I_{13} &= E\left(-\frac{\partial^2 l}{\partial \alpha \partial \mu}\right) = -\frac{n}{\sigma} + \frac{(\beta-1)}{\sigma}J_3 + \frac{1}{\sigma}J_4 \\
I_{14} &= E\left(-\frac{\partial^2 l}{\partial \alpha \partial \sigma}\right) = \frac{1}{\sigma}J_5 + \frac{(\beta-1)}{\sigma}J_6 \\
I_{22} &= E\left(-\frac{\partial^2 l}{\partial \beta^2}\right) = \frac{n}{\beta^2} \\
I_{23} &= E\left(-\frac{\partial^2 l}{\partial \beta \partial \mu}\right) = \frac{\alpha}{\sigma}J_7 \\
I_{24} &= E\left(-\frac{\partial^2 l}{\partial \beta \partial \sigma}\right) = \frac{\alpha}{\sigma}J_8 \\
I_{33} &= E\left(-\frac{\partial^2 l}{\partial \mu^2}\right) = \frac{\alpha(\beta-1)}{\sigma^2}J_9 + \alpha J_{10} + \frac{(\alpha+1)}{\sigma^2}J_{11} \\
I_{34} &= E\left(-\frac{\partial^2 l}{\partial \mu \partial \sigma}\right) = \frac{n\alpha}{\sigma^2} + \frac{-\alpha(\beta-1)}{\sigma^2}J_{12} + (\alpha+1)J_{13} + \alpha J_{14} + (\alpha+1)J_{15} + \frac{(\alpha+1)}{\sigma^2}J_{16} \\
I_{44} &= E\left(-\frac{\partial^2 l}{\partial \sigma^2}\right) = -\frac{n}{\sigma^2} + \frac{\alpha}{\sigma^2}J_{17} + \frac{\alpha(\beta-1)}{\sigma^2}J_{18} + \frac{(\alpha+1)}{\sigma}J_{19},
\end{aligned}
$$

where

$$z = \frac{x-\mu}{\sigma},$$

and

$$
\begin{aligned}
J_1 &= E\left(\frac{\left(\ln\frac{e^{-z}}{1+e^{-z}}\right)^2\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\left[1-\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}+\frac{e^{-z}}{1+e^{-z}}\right]}{\left[1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right]^2}\right) \\
J_2 &= E\left(\frac{\left(\ln\frac{e^{-z}}{1+e^{-z}}\right)\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}}{\left[1-\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right]}\right) \\
J_3 &= E\left(\frac{\left(1+\alpha\ln\frac{e^{-z}}{1+e^{-z}}\right)\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}}{(1+e^{-z})\left[1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right]^2}\right) \\
J_4 &= E\left(\frac{e^{-z}}{1+e^{-z}}\right) \\
J_5 &= E\left(\frac{ze^{-z}}{1+e^{-z}}\right) \\
J_6 &= E\left(\frac{ze^{-\alpha z}\left\{(1-\alpha z)\left[1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right]-\alpha\left[\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\log\left(\frac{e^{-z}}{1+e^{-z}}\right)+\log(1+e^{-z})\left[1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right]\right]\right\}}{\left[1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right]^2(1+e^{-z})^{\alpha+1}}\right)
\end{aligned}
$$

$$J_7 = E\left(\frac{\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}}{(1+e^{-z})\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)}\right)$$
$$J_8 = E\left(\frac{ze^{-\alpha z}}{(1+e^{-z})^{\alpha+1}\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)}\right)$$
$$J_9 = E\left(\frac{\left(\alpha+e^{-2z}\right)\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha-1}}{\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)^2}\right)$$
$$J_{10} = E\left(\frac{\left(\frac{e^{-z}}{1+e^{-z}}\right)^{2\alpha-1}}{\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)^2(1+e^{-z})^3}\right)$$
$$J_{11} = E\left(\frac{1}{(1+e^{-z})^2}\right)$$
$$J_{12} = E\left(\frac{(1+\alpha z)e^{-\alpha z}}{\left[\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)(1+e^{-z})^{\alpha+1}\right]}\right)$$
$$J_{13} = E\left(\frac{z(1+e^{-z})^{\alpha}e^{-(\alpha+1)z}}{\left[\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)(1+e^{-z})^{\alpha+1}\right]^2}\right)$$
$$J_{14} = E\left(\frac{ze^{-2\alpha z}}{\left[\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)(1+e^{-z})^{\alpha+1}\right]^2}\right)$$
$$J_{15} = E\left(\frac{ze^{-(2\alpha+1)z}}{\left[\left(1+\left(\frac{e^{-z}}{1+e^{-z}}\right)^{\alpha}\right)(1+e^{-z})^{\alpha+1}\right]^2}\right)$$
$$J_{16} = E\left(\frac{e^{-z}(1+e^{-z}+z)}{(1+e^{-z})^2}\right).$$

One can evaluate these expectations numerically using mathematical software like MATHEMATICA, MATLAB, PYTHON, R, etc., through numerical methods. Asymptotic normality of the MLEs of $\mathrm{EELD_{II}}(\mu,\sigma;\alpha,\beta)$ follows from standard theory for MLEs:

THEOREM 10. *Under the standard regularity conditions for asymptotic normality of MLEs, the MLEs* $\hat{\theta} = \left(\hat{\mu},\hat{\sigma},\hat{\alpha},\hat{\beta}\right)$ *of the parameter vector* $\theta = (\mu,\sigma,\alpha,\beta)$ *of* $EELD_{II}(\mu,\sigma;\alpha,\beta)$ *are asymptotically normally distributed in the sense that* $\sqrt{n}\left(\hat{\theta}-\theta\right) \to N\left(0, I(\theta)^{-1}\right)$ *as* $n\to\infty$*, where* $I(\theta)$ *is the Fisher information matrix.*

Regarding the regularity conditions, assume the true parameter $\theta_0$ lie in the interior of the parameter space. From the above it is evident that the log likelihood function is differentiable twice with respect to the parameters. Also we verified the Fisher information matrix as positive definite. Applying Taylor series expansion on the log likelihood function:

$$l(\hat{\theta}) = l(\theta_0) + (\hat{\theta}-\theta_0)^T S(\theta_0) + \frac{1}{2}(\hat{\theta}-\theta_0)^T \nabla^2 l\left(\tilde{\theta}\right)\left(\hat{\theta}-\theta_0\right), \quad (40)$$

where $S(\theta_0) = \frac{\partial l(\theta)}{\partial \theta}|\theta=\theta_0$ is the score function and $\nabla^2 l\left(\tilde{\theta}\right)$ is the Hessian matrix of the log-likelihood function evaluated at some intermediate point $\tilde{\theta}$ between $\hat{\theta}$ and $\theta_0$.

The maximum likelihood estimator $\hat{\theta}$ maximises the log-likelihood function. That is, $S\left(\hat{\theta}\right)=0$. By substituting this first-order condition in Eq. (40), we obtain

$$0=S(\theta_0)+\nabla^2 l\left(\hat{\theta}\right)\left(\hat{\theta}-\theta_0\right),$$

which implies

$$\sqrt{n}\left(\hat{\theta}-\theta_0\right)\approx\left(\frac{1}{n}\nabla^2 l\left(\tilde{\theta}\right)\right)^{-1}\frac{1}{\sqrt{n}}S(\theta_0).$$

By law of large numbers, the Hessian matrix of the log-likelihood function converges in probability to the negative of Fisher information matrix Van der Vaart (2000). Thus, as $n\to\infty$, $\frac{-1}{n}\nabla^2 l(\theta_0) \xrightarrow{P} I(\theta_0)$. Now, as $n\to\infty$, in the light of central limit theorem, we have $\frac{1}{\sqrt{n}}S(\theta_0) \xrightarrow{D} N(0,I(\theta_0))$. Hence, as $n\to\infty$, by Slutsky's lemma $\sqrt{n}\left(\hat{\theta}-\theta_0\right) \xrightarrow{D} N\left(0,I(\theta_0)^{-1}\right)$.

## 4. Application

For numerical illustration we consider the following datasets.

**Dataset 1** Prostrate cancer dataset available in "https://www.umass.edu//statdata". This dataset is also used by Hosmer Jr *et al.* (2013). These data are copyrighted by John Wiley and Sons Inc. We are considering the continuous variable Prostatic Specimen Antigen Value (PSA) in mg/ml of 380 patients.
Mean=15.41, variance=399.90, skewness=3.28, kurtosis=13.18.

**Dataset 2** Myopia Data dataset available in "https://www.umass.edu//statdata". These data are copyrighted by John Wiley and Sons Inc. We are considering the continuous variable vitreous chamber depth(VCD) in mm of 618 patients. Mean=15.41, variance=0.441, skewness=0.108, kurtosis=0.221.

We obtain the maximum likelihood estimates (MLEs) of the parameters of the $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$ by using the *nlm()* package in R software. The Akaike information criterion (AIC), Bayesian information criterion (BIC), Consistent Akaike information criterion (CAIC), Hannan Quinn information criterion (HQIC) and Kolmogorov-Smirnov Statistic (KSS) values are computed for comparing the model $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$ with the existing models - $\text{LD}(\mu,\sigma)$, $\text{LD}_{\text{I}}(\mu,\sigma,\alpha,\beta)$, $\text{LD}_{\text{II}}(\mu,\sigma,\alpha)$ are given in Table 2.

From Table 2 it is seen that the AIC, BIC, CAIC and HQIC values are minimum for $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$ compared to other models. Figure 2, Figure 3, Figure 4 and Figure 5 also confirm this result.

*TABLE 2*
*Estimated values and standard errors of the parameters with the corresponding KSS, AIC , BIC, CAIC, HQIC values.*

| Dataset | | Distribution LD $(\mu,\sigma)$ | $LD_I$, $(\mu,\sigma,\alpha,\beta)$, | $LD_{II}$ $(\mu,\sigma,\alpha)$ | $EELD_{II}$ $(\mu,\sigma,\alpha,\beta)$ |
|---|---|---|---|---|---|
| Dataset 1 | $\hat{\mu}$ | 11.478 | -70.041 | 0.872 | 4.820 |
| | | (0.234) | (0.452) | (0.662) | (0.032) |
| | $\hat{\sigma}$ | 7.919 | 433.953 | 0.293 | 0.739 |
| | | (0.404) | (0.370) | (0.267) | (0.234) |
| | $\hat{\alpha}$ | – | 6315.143 | 0.020 | 0.033 |
| | | | (0.832) | (0.051) | (0.044) |
| | $\hat{\beta}$ | – | 48.255 | – | 0.389 |
| | | | (0.757) | | (0.155) |
| | KSS | 0.045 | 0.039 | 0.035 | 0.031 |
| | p-value | 0.561 | 0.695 | 0.862 | 0.901 |
| | AIC | 3177.298 | 3005.468 | 2824.581 | 2808.022 |
| | BIC | 3178.458 | 3007.787 | 2826.319 | 2810.339 |
| | CAIC | 3180.458 | 3011.787 | 2829.319 | 2814.339 |
| | HQIC | 3174.944 | 3000.761 | 2821.049 | 2803.313 |
| Dataset 2 | $\hat{\mu}$ | 2.581 | 0.474 | 2.008 | 6.277 |
| | | (0.231) | (0.627) | (0.113) | (0.563) |
| | $\hat{\sigma}$ | 0.490 | 3.720 | 0.311 | 6.007 |
| | | (0.482) | (0.237) | (0.083) | (0.337) |
| | $\hat{\alpha}$ | – | 13.495 | 0.397 | 24.160 |
| | | | (0.570) | (0.185) | (0.226) |
| | $\hat{\beta}$ | – | 5.391 | – | 2.984 |
| | | | (0.251) | | (0.149) |
| | KSS | 0.155 | 0.088 | 0.053 | 0.038 |
| | p-value | 0.477 | 0.593 | 0.682 | 0.896 |
| | AIC | 1675.865 | 1623.548 | 1643.798 | 1619.694 |
| | BIC | 1677.496 | 1626.810 | 1646.245 | 1622.957 |
| | CAIC | 1679.496 | 1630.810 | 1649.245 | 1626.957 |
| | HQIC | 1673.663 | 1619.144 | 1640.496 | 1615.291 |

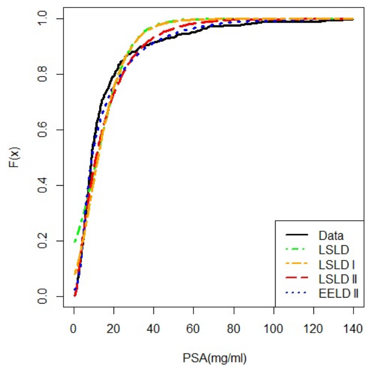



*Figure 2* – Empirical distribution of the Dataset 1 along with the fitted CDFs.

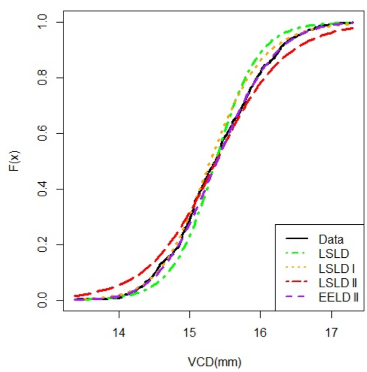



*Figure 3* – Empirical distribution of the Dataset 2 along with the fitted CDFs.

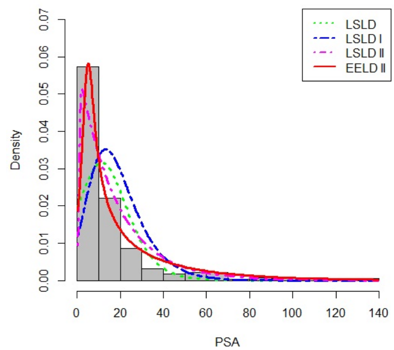



*Figure 4* – Fitted density functions to the histogram of Dataset 1.

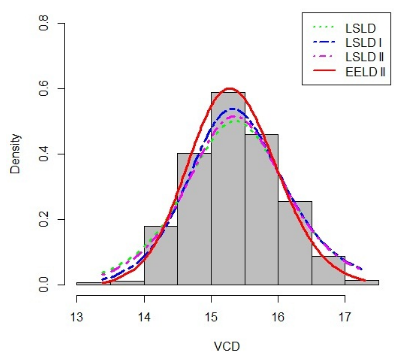



*Figure 5* – Fitted density functions to the histogram of Dataset 2.

## 5. Testing of hypothesis

In this Section, the generalized likelihood ratio test procedure is used for testing the parameters of the $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$. Here, we consider the following tests:

**Test 1** $H_{01}:\alpha=1$ against $H_{11}:\alpha\neq 1$.

**Test 2** $H_{02}:\beta=1$ against $H_{12}:\beta\neq 1$.

**Test 3** $H_{03}:\alpha=1,\beta=1$ against $H_{13}:\alpha\neq 1,\beta\neq 1$.

In this case, the test statistic is,

$$\lambda=-2\log\Lambda=2\left[\log L\left(\hat{\theta};y|x\right)-\log L\left(\hat{\theta}^{*};y|x\right)\right], \tag{41}$$

where $\hat{\theta}$ is the maximum likelihood estimator of $\theta=(\mu,\sigma;\alpha,\beta)$ with no restriction, and $\hat{\theta}^{*}$ is the maximum likelihood estimator of $\theta$ when $\alpha=1$ in case of Test 1 and $\beta=1$ in case of Test 2 and $\alpha=1$, $\beta=1$ in case of Test 3 respectively. Using Taylor series expansion of the log likelihood function around $\theta^{*}$

$$\lambda\approx\left(\hat{\theta}^{*}-\hat{\theta}\right)^{T}\nabla_{\theta}^{2}l\left(\theta^{*}\right)\left(\hat{\theta}^{*}-\hat{\theta}\right).$$

By the properties MLEs, the distribution of the likelihood ratio test statistic can be approximated in the following form

$$\lambda\approx\sqrt{n}\left(\hat{\theta}^{*}-\hat{\theta}\right)^{T}I\left(\theta^{*}\right)\sqrt{n}\left(\hat{\theta}^{*}-\hat{\theta}\right),$$

where, as $n\to\infty$ $\sqrt{n}\left(\hat{\theta}^{*}-\hat{\theta}\right)\xrightarrow{D}N\left(0,I\left(\theta_{0}\right)^{-1}\right)$ in the light of Theorem 10, this implies $\lambda$ follows an asymptotic chi-square distribution with degrees of freedom equal to the difference in the number of parameters estimated under $H_0$ and $H_1$. Hence, the test statistic $\lambda$ given in Eq. (41) is asymptotically distributed as $\chi^2$ with one degree of freedom in case of Test 1 and Test 2 and with two degrees of freedom in case of Test 3. The computed values of $\log L\left(\hat{\theta};y|x\right)$, $\log L\left(\hat{\theta}^{*};y|x\right)$ and test statistic in case of the two datasets are listed in Table 3. Since the critical value at the significance level 0.05 and degree of freedom one for two tailed test is 5.023 and for degrees of freedom two is 7.327, the null hypothesis is rejected in all cases, which shows the appropriateness of the $\text{EELD}_{\text{II}}$ to the datasets.

## 6. Simulation

A simulation study is carried out to evaluate the performance of the parameters of the $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$ for varying degrees of skewness. We applied the inverse transform

*TABLE 3*
*Calculated values of the test statistic.*

| Dataset | Hypothesis | $\log L\left(\overset{\wedge}{\theta};y\|x\right)$ | $\log L\left(\overset{\wedge *}{\theta};y\|x\right)$ | Test statistic |
|---|---|---|---|---|
| Dataset 1 | $H_{01}:\alpha=1$ | -1400.005 | -1498.745 | 197.480 |
| | $H_{02}:\beta=1$ | -1400.005 | -1409.030 | 18.051 |
| | $H_{03}:\alpha=1,\beta=1$ | -1400.005 | -1586.649 | 373.288 |
| . | | | | |
| Dataset 2 | $H_{01}:\alpha=1$ | -805.847 | -807.974 | 4.253 |
| | $H_{02}:\beta=1$ | -805.847 | -818.899 | 26.104 |
| | $H_{03}:\alpha=1,\beta=1$ | -805.847 | -531.587 | 60.171 |

method of Ross (2022) for generating random variables from the $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$ distribution. Let $X$ be a random variable with CDF in Eq. (6). Simulate $X$ according to $X=F^{-1}(U)$, $U\sim U[0,1]$.

Here, we are considering the following four parameter sets with varying degrees of skewness, and sample sizes $n$ =15, 25, 50, 100, 200, 300 and 500.

**(i)** $\alpha=1.501,\ \beta=0.058,\ \mu=0.395,\ \sigma=4.432$ (Galton's percentile measure of skewness=1.7).

**(ii)** $\alpha=0.184,\ \beta=0.355,\ \mu=0.701,\ \sigma=0.111$ (Galton's percentile measure of skewness=0.7).

**(iii)** $\alpha=0.05,\ \beta=0.4,\ \mu=1.2,\ \sigma=0.5$ (Galton's percentile measure of skewness=2.8).

**(iv)** $\alpha=1.5,\ \beta=0.08,\ \mu=5,\ \sigma=1.2$ (Galton's percentile measure of skewness=0.5).

We computed the mean, standard deviation, skewness, excess kurtosis, absolute bias and Mean Square Error (MSE) of the parameters of $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$ and presented in Table 4 to Table 7 . From the tables it can be seen that both the absolute bias and MSEs in respect of each parameters of the $\text{EELD}_{\text{II}}(\mu,\sigma;\alpha,\beta)$ are in decreasing order as the sample size increases.

*TABLE 4*
*Mean, standard deviation, skewness, kurtosis, absolute bias and MSE of the estimated parameters of parameter set (i).*

| n | Parameter | Mean | SD | Skewness | Kurtosis | Bias | MSE |
|---|---|---|---|---|---|---|---|
| 10 | $\alpha$ | 5.06E-01 | 1.27E-00 | 1.84E-00 | -8.20E-01 | 9.95E-01 | 8.25E-02 |
| | $\beta$ | 8.23E-01 | 1.23E-00 | 9.13E-01 | -7.74E-01 | 7.70E-01 | 4.94E-02 |
| | $\mu$ | 1.06E-00 | 2.33E-00 | 1.44E-00 | -8.53E-01 | 6.71E-01 | 3.48E-02 |
| | $\sigma$ | 2.82E+00 | 8.43E-01 | 1.66E+00 | -4.76E-01 | 1.61E+00 | 2.59E-01 |
| 25 | $\alpha$ | 7.75E-01 | 2.13E-01 | 5.81E-01 | -7.32E-01 | 7.26E-01 | 4.25E-02 |
| | $\beta$ | 1.60E-01 | 9.87E-01 | 5.33E-01 | -5.12E-01 | 1.02E-01 | 8.42E-03 |
| | $\mu$ | 8.26E-01 | 3.54E-01 | 2.54E-01 | -7.12E-01 | 3.01E-01 | 2.07E-02 |
| | $\sigma$ | 3.12E+00 | 5.23E-01 | 1.19E+00 | -4.14E-01 | 1.31E+00 | 2.24E-01 |
| 50 | $\alpha$ | 9.37E-01 | 2.08E-01 | 4.77E-01 | -5.02E-01 | 5.64E-01 | 3.84E-03 |
| | $\beta$ | 8.40E-02 | 5.25E-01 | 4.84E-01 | -3.97E-01 | 2.60E-02 | 6.21E-03 |
| | $\mu$ | 7.27E-01 | 2.29E-01 | 1.33E-01 | -4.32E-01 | 2.65E-01 | 1.97E-03 |
| | $\sigma$ | 3.60E+00 | 3.14E-01 | 8.85E-01 | -3.95E-01 | 8.31E-01 | 1.83E-01 |
| 100 | $\alpha$ | 1.46E+00 | 1.92E-01 | 4.63E-01 | -4.85E-01 | 3.64E-02 | 2.95E-03 |
| | $\beta$ | 7.03E-02 | 1.13E-01 | 4.71E-01 | -3.86E-01 | 2.34E-03 | 7.43E-06 |
| | $\mu$ | 4.74E-01 | 1.84E-01 | 9.88E-02 | -4.01E-01 | 2.11E-02 | 5.67E-04 |
| | $\sigma$ | 3.72E+00 | 2.67E-01 | 7.81E-01 | -3.45E-01 | 1.16E-01 | 1.64E-01 |
| 200 | $\alpha$ | 1.48E+00 | 1.90E-01 | 3.46E-01 | -4.11E-01 | 1.82E-02 | 1.96E-03 |
| | $\beta$ | 6.67E-02 | 8.74E-02 | 4.60E-01 | -2.85E-01 | 1.26E-03 | 6.61E-06 |
| | $\mu$ | 4.31E-01 | 1.62E-01 | 8.40E-02 | -3.81E-01 | 3.69E-03 | 4.78E-04 |
| | $\sigma$ | 4.06E+00 | 2.26E-01 | 5.79E-01 | -2.26E-01 | 7.26E-02 | 7.81E-02 |
| 300 | $\alpha$ | 1.49E+00 | 1.88E-01 | 2.49E-01 | -1.95E-01 | 9.15E-03 | 7.03E-04 |
| | $\beta$ | 6.15E-02 | 8.04E-02 | 4.12E-01 | -2.31E-01 | 1.03E-04 | 1.19E-07 |
| | $\mu$ | 4.11E-01 | 9.50E-02 | 7.11E-02 | -2.54E-01 | 2.05E-03 | 3.14E-04 |
| | $\sigma$ | 4.12E+00 | 2.07E-01 | 3.79E-01 | 1.15E-01 | 2.05E-02 | 2.46E-03 |
| 500 | $\alpha$ | 1.50E+00 | 1.64E-01 | 2.33E-01 | 0.37E-01 | 1.92E-03 | 1.69E-04 |
| | $\beta$ | 6.03E-02 | 7.58E-02 | 3.84E-01 | -1.04E-01 | 9.32E-05 | 2.35E-08 |
| | $\mu$ | 4.06E-01 | 5.50E-02 | 5.80E-02 | -1.06E-01 | 1.04E-03 | 3.41E-05 |
| | $\sigma$ | 4.24E+00 | 1.21E-01 | 1.28E-01 | 1.08E-01 | 8.16E-03 | 1.06E-04 |

*TABLE 5*
*Mean, standard deviation, skewness, kurtosis, absolute bias and MSE of the estimated parameters of parameter set (ii).*

| n | Parameter | Mean | SD | Skewness | Kurtosis | Bias | MSE |
|---|---|---|---|---|---|---|---|
| 10 | $\alpha$ | 8.86E-01 | 7.86E-01 | -1.92E+00 | -5.35E-01 | 7.02E-01 | 5.03E-01 |
| | $\beta$ | 8.04E-01 | 7.14E-01 | -7.83E-01 | -4.47E-01 | 4.49E-01 | 2.02E-02 |
| | $\mu$ | 1.61E+00 | 2.56E-01 | -9.19E-01 | -1.74E-01 | 9.09E-01 | 8.10E-02 |
| | $\sigma$ | 5.11E-01 | 5.61E-01 | -7.05E-01 | -1.47E-01 | 4.00E-01 | 1.60E-01 |
| 25 | $\alpha$ | 7.80E-01 | 7.01E-01 | -1.68E+00 | -4.86E-01 | 5.96E-01 | 3.68E-01 |
| | $\beta$ | 7.22E-01 | 6.37E-01 | -7.11E-01 | -3.38E-01 | 3.67E-01 | 1.89E-02 |
| | $\mu$ | 1.02E+00 | 2.11E-01 | -8.94E-01 | -1.51E-01 | 2.98E-01 | 6.50E-02 |
| | $\sigma$ | 4.23E-01 | 3.84E-01 | -5.63E-01 | -1.02E-01 | 3.12E-01 | 1.57E-01 |
| 50 | $\alpha$ | 6.35E-01 | 5.38E-01 | -1.55E+00 | -4.52E-01 | 4.51E-01 | 2.89E-01 |
| | $\beta$ | 5.81E-01 | 5.77E-01 | -6.44E-01 | -1.46E-01 | 2.26E-01 | 1.30E-02 |
| | $\mu$ | 8.88E-01 | 1.45E-01 | -6.22E-01 | -4.32E-01 | 1.87E-01 | 1.53E-02 |
| | $\sigma$ | 2.78E-01 | 3.08E-01 | -5.16E-01 | -9.84E-01 | 1.67E-01 | 8.51E-02 |
| 100 | $\alpha$ | 4.85E-01 | 3.42E-01 | -7.03E-01 | -4.41E-01 | 3.01E-01 | 1.36E-01 |
| | $\beta$ | 4.02E-01 | 2.33E-01 | -4.74E-01 | -7.48E-02 | 4.67E-02 | 3.26E-03 |
| | $\mu$ | 7.21E-01 | 1.32E-01 | -5.75E-01 | -4.68E-01 | 1.98E-02 | 5.88E-04 |
| | $\sigma$ | 1.42E-01 | 2.99E-01 | -3.83E-01 | -8.54E-01 | 3.12E-02 | 1.46E-03 |
| 200 | $\alpha$ | 2.83E-01 | 2.75E-01 | -5.01E-01 | -4.26E-01 | 9.94E-02 | 1.48E-02 |
| | $\beta$ | 3.85E-01 | 2.24E-01 | -4.53E-01 | 7.26E-01 | 2.95E-02 | 1.30E-03 |
| | $\mu$ | 7.11E-01 | 1.16E-01 | -5.64E-01 | -3.88E-01 | 1.01E-02 | 1.53E-04 |
| | $\sigma$ | 1.26E-01 | 2.57E-01 | -3.72E-01 | 2.46E-01 | 1.53E-02 | 3.51E-04 |
| 300 | $\alpha$ | 2.15E-01 | 2.06E-01 | -4.31E-01 | -4.01E-01 | 3.07E-02 | 1.42E-03 |
| | $\beta$ | 3.67E-01 | 2.14E-01 | -3.46E-01 | 5.44E-01 | 1.23E-02 | 2.28E-04 |
| | $\mu$ | 7.09E-01 | 9.59E-02 | -2.17E-01 | -2.81E-01 | 8.02E-03 | 9.65E-05 |
| | $\sigma$ | 1.20E-01 | 2.37E-01 | -2.74E-01 | 1.84E-01 | 9.02E-03 | 1.22E-04 |
| 500 | $\alpha$ | 1.92E-01 | 1.77E-01 | -3.91E-01 | -3.37E-01 | 8.46E-03 | 1.07E-04 |
| | $\beta$ | 3.57E-01 | 1.87E-01 | -2.24E-01 | 1.16E-01 | 1.87E-03 | 5.25E-06 |
| | $\mu$ | 7.02E-01 | 6.58E-02 | -1.58E-01 | -2.06E-01 | 1.01E-03 | 1.53E-06 |
| | $\sigma$ | 1.12E-01 | 1.84E-01 | -9.80E-02 | 7.73E-02 | 1.13E-03 | 1.91E-06 |

*TABLE 6*
*Mean, standard deviation, skewness, kurtosis, absolute bias and MSE of the estimated parameters of parameter set (iii).*

| n | Parameter | Mean | SD | Skewness | Kurtosis | Bias | MSE |
|---|---|---|---|---|---|---|---|
| 10 | $\alpha$ | 5.21E-01 | 8.81E-01 | 1.33E+00 | -5.81E-01 | 4.71E-01 | 2.22E-02 |
| | $\beta$ | 8.82E-01 | 1.80E+00 | 1.21E+00 | -2.95E-01 | 4.82E-01 | 2.32E-02 |
| | $\mu$ | 1.74E+00 | 6.85E-01 | 6.78E-01 | -8.56E-01 | 5.44E-01 | 2.96E-02 |
| | $\sigma$ | 1.21E+00 | 5.63E-01 | 8.93E-01 | -8.81E-01 | 7.13E-01 | 5.08E-02 |
| 25 | $\alpha$ | 4.56E-01 | 8.45E-01 | 1.28E+00 | -5.35E-01 | 4.06E-01 | 6.59E-03 |
| | $\beta$ | 8.41E-01 | 1.78E+00 | 8.95E-01 | -2.64E-01 | 4.41E-01 | 7.78E-03 |
| | $\mu$ | 1.62E+00 | 6.33E-01 | 6.56E-01 | -8.13E-01 | 4.15E-01 | 6.89E-03 |
| | $\sigma$ | 1.09E+00 | 5.78E-01 | 8.84E-01 | -7.28 E-01 | 5.88E-01 | 1.38E-02 |
| 50 | $\alpha$ | 2.55E-01 | 7.86E-01 | 9.77E-01 | -4.12E-01 | 2.05E-01 | 8.41E-04 |
| | $\beta$ | 6.73E-01 | 1.06E+00 | 7.49E-01 | -1.58E-01 | 2.73E-01 | 1.49E-03 |
| | $\mu$ | 1.57E+00 | 6.42E-01 | 6.12E-01 | -7.40E-01 | 3.66E-01 | 2.26E-03 |
| | $\sigma$ | 1.04E+00 | 5.12E-01 | 8.12E-01 | -6.67E-01 | 5.37E-01 | 5.77E-03 |
| 100 | $\alpha$ | 1.87E-01 | 7.71E-01 | 9.01E-01 | -4.07E-01 | 1.37E-01 | 1.88E-04 |
| | $\beta$ | 5.16E-01 | 9.76E-01 | 6.95E-01 | -2.47E-02 | 1.16E-01 | 1.35E-04 |
| | $\mu$ | 1.39E+00 | 6.13E-01 | 5.99E-01 | -6.81E-01 | 1.85E-01 | 3.42E-04 |
| | $\sigma$ | 8.75E-01 | 4.69E-01 | 7.43E-01 | -6.12E-01 | 3.75E-01 | 1.41E-03 |
| 200 | $\alpha$ | 1.16E-01 | 7.13E-01 | 7.93E-01 | 3.87E-01 | 6.60E-02 | 2.18E-05 |
| | $\beta$ | 4.82E-01 | 7.44E-01 | 6.64E-01 | 6.12E-01 | 8.20E-02 | 3.36E-05 |
| | $\mu$ | 1.31E+00 | 5.86E-01 | 5.47E-01 | -5.88E-01 | 5.99E-02 | 1.79E-05 |
| | $\sigma$ | 7.87E-01 | 4.52E-01 | 7.06E-01 | -5.47E-01 | 2.87E-01 | 4.12E-04 |
| 300 | $\alpha$ | 8.70E-02 | 6.54E-01 | 5.46E-01 | 3.04E-01 | 3.70E-02 | 4.56E-06 |
| | $\beta$ | 4.67E-01 | 5.39E-01 | 5.86E-01 | 2.07E-01 | 6.70E-02 | 1.50E-05 |
| | $\mu$ | 1.28E+00 | 5.34E-01 | 4.09E-01 | -3.27E-01 | 3.02E-02 | 3.04E-06 |
| | $\sigma$ | 6.34E-01 | 4.09E-01 | 5.13E-01 | -4.88E-01 | 1.34E-01 | 5.99E-05 |
| 500 | $\alpha$ | 6.40E-02 | 6.18E-01 | 5.21E-01 | 2.98E-01 | 1.40E-02 | 3.92E-07 |
| | $\beta$ | 4.42E-01 | 5.01E-01 | 5.69E-01 | 1.97E-02 | 4.20E-02 | 3.53E-06 |
| | $\mu$ | 1.24E+00 | 5.16E-01 | 3.79E-01 | -3.06E-01 | 2.61E-02 | 1.36E-06 |
| | $\sigma$ | 5.53E-01 | 3.98E-01 | 5.07E-01 | -4.62E-01 | 5.25E-02 | 5.55E-06 |

*TABLE 7*
*Mean, standard deviation, skewness, kurtosis, absolute bias and MSE of the estimated parameters of parameter set (iv).*

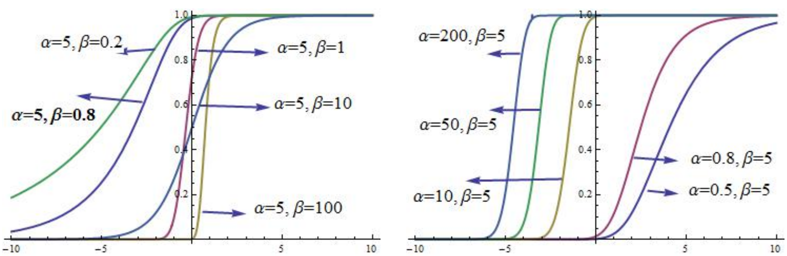
| n | Parameter | Mean | SD | Skewness | Kurtosis | Bias | MSE |
|---|---|---|---|---|---|---|---|
| 10 | $\alpha$ | 1.02E+00 | 1.25E+00 | -1.55E+00 | -1.21E+00 | 4.80E-01 | 2.30E-02 |
| | $\beta$ | 2.88E-01 | 1.51E+00 | -1.29E+00 | -6.47E-01 | 2.08E-01 | 4.33E-03 |
| | $\mu$ | 6.12E+00 | 1.05E+00 | -8.79E-01 | -7.48E-01 | 1.12E+00 | 1.25E-01 |
| | $\sigma$ | 2.57E+00 | 8.69E-01 | -7.48E-01 | -6.79E-01 | 1.37E+00 | 1.88E-01 |
| 25 | $\alpha$ | 1.14E+00 | 1.10E-01 | -1.35E+00 | -1.03E+00 | 3.60E-01 | 5.18E-03 |
| | $\beta$ | 3.12E-01 | 9.85E-01 | -8.75E-01 | -4.47E-01 | 2.32E-01 | 2.15E-03 |
| | $\mu$ | 5.94E+00 | 8.82E-01 | -7.45E-01 | -5.46E-01 | 9.40E-01 | 3.53E-02 |
| | $\sigma$ | 2.33E+00 | 6.47E-01 | -6.43E-01 | -4.48E-01 | 1.13E+00 | 5.11E-02 |
| 50 | $\alpha$ | 1.22E+00 | 9.81E-01 | -1.05E+00 | -7.12E-01 | 2.80E-01 | 1.57E-03 |
| | $\beta$ | 4.29E-02 | 8.46E-01 | -8.47E-01 | -4.15E-01 | 3.71E-02 | 2.75E-05 |
| | $\mu$ | 5.89E+00 | 8.01E-01 | -6.44E-01 | -2.14E-01 | 8.90E-01 | 1.58E-02 |
| | $\sigma$ | 2.04E+00 | 6.33E-01 | -5.85E-01 | -4.13E-01 | 8.40E-01 | 1.41E-02 |
| 100 | $\alpha$ | 1.38E+00 | 9.47E-01 | -8.42E-01 | -6.60E-01 | 1.20E-01 | 1.44E-04 |
| | $\beta$ | 5.13E-02 | 7.14E-01 | -7.97E-01 | -4.06E-01 | 2.87E-02 | 8.24E-06 |
| | $\mu$ | 5.64E+00 | 7.09E-01 | -6.01E-01 | -4.92E-02 | 6.40E-01 | 4.10E-03 |
| | $\sigma$ | 1.85E+00 | 5.83E-01 | -5.67E-01 | -3.99E-01 | 6.50E-01 | 4.23E-03 |
| 200 | $\alpha$ | 1.40E+00 | 8.02E-01 | -8.11E-01 | -6.20E-01 | 1.00E-01 | 5.00E-05 |
| | $\beta$ | 5.82E-02 | 6.82E-01 | -7.61E-01 | -3.77E-01 | 2.18E-02 | 2.38E-06 |
| | $\mu$ | 5.58E+00 | 6.39E-01 | -5.83E-01 | 8.66E-01 | 5.80E-01 | 1.68E-03 |
| | $\sigma$ | 1.57E+00 | 5.54E-01 | -5.47E-01 | -3.63E-01 | 3.70E-01 | 6.85E-04 |
| 300 | $\alpha$ | 1.42E+00 | 7.83E-01 | -7.46E-01 | -5.43E-01 | 8.00E-02 | 2.13E-05 |
| | $\beta$ | 6.60E-02 | 6.17E-01 | -7.22E-01 | -2.96E-01 | 1.40E-02 | 6.53E-07 |
| | $\mu$ | 5.31E+00 | 5.42E-01 | -4.39E-01 | 3.94E-01 | 3.10E-01 | 3.20E-04 |
| | $\sigma$ | 1.42E+00 | 4.73E-01 | -5.18E-01 | -2.87E-01 | 2.20E-01 | 1.61E-04 |
| 500 | $\alpha$ | 1.47E+00 | 7.09E-01 | -6.18E-01 | -4.79E-01 | 3.00E-02 | 1.80E-06 |
| | $\beta$ | 7.10E-02 | 5.42E-01 | -6.43E-01 | -1.75E-01 | 9.00E-03 | 1.62E-07 |
| | $\mu$ | 5.29E+00 | 4.96E-01 | -3.77E-01 | 3.04E-01 | 2.90E-01 | 1.68E-04 |
| | $\sigma$ | 1.31E+00 | 3.97E-01 | -4.69E-01 | -2.25E-01 | 1.10E-01 | 2.42E-05 |

## 7. Exponentiated Logistic Regression Model

Binary regression models are used for regressing certain independent variables on a dichotomous dependent variable. In this Section, we consider a regression model based on $ELD_{II}$ and compare it with the other logit models. A random variable $X$ is said to follow exponentiated logistic regression model of type II ($ELRM_{II}$) if it has the following representation

$$p = \left[1 - \frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}\right]^{\beta}, \tag{42}$$

where $x \in R$ , $\alpha > 0$ and $\beta > 0$. A graphical representation of $ELRM_{II}$ for particular values of $\alpha$ and $\beta$ are given in Figure 6.




(a) For fixed $\alpha$ and varying $\beta$ (b) For fixed $\alpha$ and varying $\beta$

*Figure 6* – Plots of $ELRM_{II}$ for varying values of $\alpha$ and $\beta$.

In order to obtain the maximum likelihood estimators of the parameters of $ELRM_{II}$, let us have a sample of n independent observations of the pairs $(x_i, y_i)$, i=1,2,...,n, where $x_i$ is the value of the independent variable and $y_i$ denotes the value of the dichotomous outcome variable for the $i^{\text{th}}$ subject. Let $p_i = P(Y_i = 1|X_i)$, so that $P(Y_i = 0|X_i) = 1 - p_i$. The probability of observing the outcome $Y_i$ whether it is 0 or 1 is given by $P(Y_i|X_i) = p_i^{y_i}(1-p_i)^{1-y_i}$. If there are n sets of values of $X_i$, say $\boldsymbol{X}$, the probability of observing a particular sample of n values of Y, say $\boldsymbol{Y}$ is given by the product of n probabilities, since the observations are independent. That is

$$P(\mathbf{Y}|\mathbf{X}) = \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i}. \tag{43}$$

Let $z = a + \sum_{r=1}^{s} b_r X_r$ and $\Theta = (\alpha, \beta, a, b_1, b_2, \ldots, b_s)$ be the vector of parameters of the $ELRM_{II}$ and let $\widehat{\Theta} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{a}, \widehat{b_1}, \widehat{b_2}, \ldots, \widehat{b_s}\right)$ be the maximum likelihood estimator (MLE) of Θ. The log-likelihood function of $\text{ELRM}_{\text{II}}$ is given by

$$l = \log\, L(y|z,\theta) = \sum_{i=1}^{n} y_i \log p_i + \sum_{i=1}^{n} (1-y_i)\log(1-p_i). \tag{44}$$

The MLEs of the parameters are obtained by solving the following set of likelihood equations, in which

$$\delta_j = \left(\frac{e^{-z}}{1+e^{-z}}\right)^j, j = 1,\ \alpha,$$

$$\frac{\partial l}{\partial \alpha} = 0,$$

or, equivalently,

$$\sum_{i=1}^{n} y_i \frac{\delta_\alpha \log(\delta_1)}{(1-\delta_\alpha)} - \sum_{i=1}^{n} (1-y_i)\frac{(1-\delta_\alpha)^{\beta-1}\delta_\alpha \log(\delta_1)}{1-(1-\delta_\alpha)^\beta} = 0, \tag{45}$$

$$\frac{\partial l}{\partial \beta} = 0$$

or, equivalently,

$$\sum_{i=1}^{n} y_i \log(1-\delta_\alpha) - \sum_{i=1}^{n} (1-y_i)\frac{(1-\delta_\alpha)^{\beta}\log(1-\delta_\alpha)}{1-(1-\delta_\alpha)^\beta} = 0, \tag{46}$$

$$\frac{\partial l}{\partial a} = 0$$

or, equivalently,

$$\sum_{i=1}^{n} y_i \frac{\delta_\alpha (1-\delta_1)}{(1-\delta_\alpha)} - \sum_{i=1}^{n} (1-y_i)\frac{(1-\delta_\alpha)^{\beta-1}\delta_\alpha (1-\delta_1)}{1-(1-\delta_\alpha)^\beta} = 0 \tag{47}$$

and

$$\frac{\partial l}{\partial b_j} = 0,$$

or, equivalently,

$$x_j \sum_{i=1}^{n} y_i \frac{\delta_\alpha (1-\delta_1)}{(1-\delta_\alpha)} - x_j \sum_{i=1}^{n} (1-y_i)\frac{(1-\delta_\alpha)^{\beta-1}\delta_\alpha (1-\delta_1)}{1-(1-\delta_\alpha)^\beta} = 0 \quad for\ j = 1,2,\ldots,s. \tag{48}$$

Second order partial derivatives of Eq. (44) with respect to the parameters are observed with the help of `MATHEMATICA` software and found that the equation gives negative values, for all $\alpha > 0$, $\beta > 0$, $a, b \in R$. Now we can obtain the $\widehat{\Theta}$ by solving the likelihood Equations (45) to (48) with the help of mathematical software packages such as `MATHCAD`, `MATHEMATICA`, `R` etc.

### *7.1. Applications*

For numerical illustration, we consider the following two datasets.

**Dataset 3** Prostrate cancer dataset available in "https://www.umass.edu//statdata". These data are copyrighted by John Wiley and Sons Inc. The variable *capsule* denotes the status of the tumor, whether it is penetrated or not, which we consider as the dichotomous dependent variable ($Y$) and *Prostatic Specimen Antigen Value (PSA)* in mg/ml of 380 patients as the explanatory ($X$) variable.

**Dataset 4** Shock dataset obtained from Afifi and Azen (2014) (see https://www.umass.edu/statdata/statdata/stat-logistic.html). These data were collected at the Shock Research Unit at the University of Southern California, Los Angeles, California. Data were collected on 113 critically ill patients. Here, we consider the explanatory variable as the urine output (ml/hr) at the time of admission and the dependent variable $Y$ as whether the person survived or not.

Here we consider the simplest model, $Z = a + bX$. We obtained the MLEs of the parameters $\alpha, \beta$, a and b with the help of `R` software 3.0.3 by using the *nlm* package. The estimated values of the parameters of the logistic regression model (LRM), type I logistic regression model ($\text{LRM}_\text{I}$), type II logistic regression model ($\text{LRM}_\text{II}$) and $\text{ELRM}_\text{II}$, along with the computed values of the AIC, BIC, CAIC and HQC and pseudo $R^2$ values McFadden (1973) are given in Tables 8 and 10. The values of AIC, BIC, CAIC and HQC are relatively less in case of the $\text{ELRM}_\text{II}$ compared to other existing models while the values of the of pseudo $R^2$ (such as McFadden's $R^2$, McFadden's Adj $R^2$) are more in case of $\text{ELRM}_\text{II}$. We have obtained plots of cumulative distribution functions of the fitted regression models along with the empirical distribution function in the case of Dataset 3 in Figure 7 and that in the case of Dataset 4 in Figure 8. Further, we obtained the weighted residual plots in each of the cases and presented in Figure 9 and Figure 10. The plots in Figure 7 to Figure 10 also support the suitability of the $\text{ELRM}_\text{II}$ models to both the datasets considered here. Hence, based on the above observations we can conclude that the $\text{ELRM}_\text{II}$ gives better fit to the given datasets compared to other existing models.

*TABLE 8*
*Estimated values and standard errors of the parameters with the corresponding Pseudo $R^2$ values and Information criteria values - Dataset 3.*

| | Distribution | | | |
|---|---|---|---|---|
| | LRM $(a,b)$ | $LRM_I$ $(\alpha,\beta,a,b)$ | $LRM_{II}$ $(\alpha,a,b)$ | $ELRM_{II}$ $(\alpha,\beta,a,b)$ |
| $\hat{\alpha}$ | – | 184.237 | 0.028 | 0.159 |
| | | (0.783) | (0.013) | (0.239) |
| $\hat{\beta}$ | – | 3.467 | – | 3.830 |
| | | (0.847) | – | (0.587) |
| $\hat{a}$ | -1.113 | 1.379 | 5.994 | 6.731 |
| | (0.432) | (0.347) | (0.778) | (0.886) |
| $\hat{b}$ | 0.050 | 0.011 | 1.025 | 0.229 |
| | (0.014) | (0.122) | (0.068) | (0.069) |
| AIC | 467.702 | 468.758 | 464.367 | 462.163 |
| BIC | 468.861 | 471.077 | 466.106 | 464.482 |
| CAIC | 470.861 | 475.077 | 469.106 | 468.482 |
| HQC | 465.348 | 464.051 | 460.836 | 457.456 |
| McFadden's $R^2$ | 0.094 | 0.101 | 0.105 | 0.113 |
| McFadden's Adj $R^2$ | 0.087 | 0.093 | 0.089 | 0.098 |
| Cox Snell $R^2$ | 0.120 | 0.127 | 0.132 | 0.142 |
| Cragg-Uhler(Nagelkerke)$R^2$ | 0.162 | 0.171 | 0.179 | 0.192 |

*TABLE 9*
*Estimated values and standard errors of the parameters with the corresponding Pseudo $R^2$ values and Information criteria values - Dataset 4.*

| | Distribution | | | |
|---|---|---|---|---|
| | LRM $(a,b)$ | $\text{LRM}_\text{I}$ $(\alpha,\beta,a,b)$ | $\text{LRM}_\text{II}$ $(\alpha,a,b)$ | $\text{ELRM}_\text{II}$ $(\alpha,\beta,a,b)$ |
| $\hat{\alpha}$ | – | 12.406 | 0.003 | 0.003 |
| | | (0.435) | (0.028) | (0.011) |
| $\hat{\beta}$ | – | 0.073 | – | 1.116 |
| | | (0.066) | – | (0.148) |
| $\hat{a}$ | -4.679 | 5.241 | 4.725 | -58.378 |
| | (0.801) | (0.749) | (0.342) | (0.648) |
| $\hat{b}$ | 0.854 | 5.854 | 12.130 | 39.432 |
| | (0.344) | (0.686) | (0.414) | (0.525) |
| AIC | 391.372 | 390.494 | 405.743 | 384.773 |
| BIC | 393.003 | 393.756 | 408.190 | 388.036 |
| CAIC | 395.003 | 397.756 | 411.190 | 392.036 |
| HQC | 389.169 | 386.091 | 402.441 | 380.370 |
| McFadden's $R^2$ | 0.085 | 0.097 | 0.056 | 0.110 |
| McFadden's Adj $R^2$ | 0.076 | 0.087 | 0.037 | 0.091 |
| Cox Snell $R^2$ | 0.091 | 0.102 | 0.060 | 0.116 |
| Cragg-Uhler(Nagelkerke)$R^2$ | 0.135 | 0.152 | 0.090 | 0.172 |

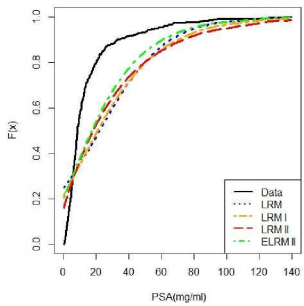



*Figure 7* – Empirical distribution of the Dataset 3 along with the fitted CDFs.

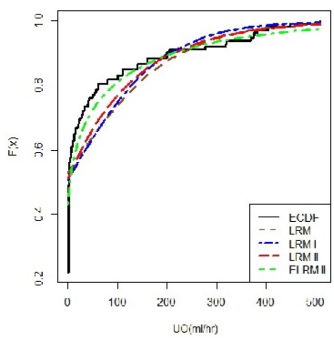



*Figure 8* – Empirical distribution of the Dataset 4 along with the fitted CDFs.

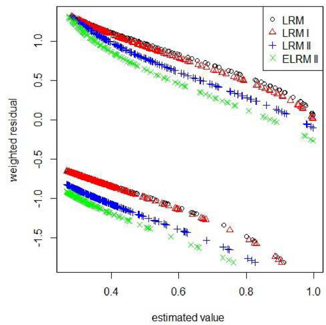



*Figure 9* – Weighted residual plots of fitted regression models for Dataset 3.

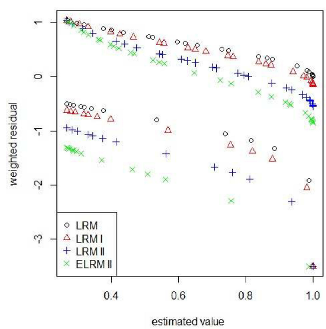



*Figure 10* – Weighted residual plots of fitted regression models for Dataset 4.

### *7.2. Testing of hypothesis*

Next we discuss the generalised likelihood ratio test procedures for testing the parameters of the $\mathrm{ELRM_{II}}(\alpha,\beta,a,b)$ and attempt a brief simulation study. Here, we consider the following tests:

**Test 1** $H_{01}:\alpha=1$ vs $H_{11}:\alpha\neq 1$;

**Test 2** $H_{02}:\beta=1$ vs $H_{12}:\beta\neq 1$;

**Test 3** $H_{03}:a=0$ vs $H_{13}:a\neq 0$;

**Test 4** $H_{04}:b=0$ vs $H_{14}:b\neq 0$;

**Test 5** $H_{05}:\alpha=1,\ a=0$ vs $H_{13}:\alpha\neq 1,\ a\neq 0$;

**Test 6** $H_{06}:\ \beta=1,\ a=0$ vs $H_{15}:\beta\neq 1,\ a\neq 0$;

**Test 7** $H_{07}:\alpha=1,\ \beta=1,\ a=0$ vs $H_{17}:\alpha\neq 1,\ \beta\neq 1,\ a\neq 0$.

In this case, the test statistic is

$$-2\log\Lambda = 2\Big[\ln L(\hat{\Omega};y|x) - \ln L(\hat{\Omega}^*;y|x)\Big], \tag{49}$$

where $\hat{\Omega}$ is the maximum likelihood estimator of $\Omega=(\alpha,\beta,a,b)$ with no restriction, and $\hat{\Omega}^*$ is the maximum likelihood estimator of $\Omega$ when $\alpha=1$ in case of Test 1, $\beta=1$ in case of Test 2, $a=0$ in case of Test 3, $b=0$ in case of Test 4, $\alpha=1, a=0$ in case of Test 5, $\beta=1, a=0$ in case of Test 6, $\alpha=1,\beta=1,a=0$ in case of Test 7. The test statistic $-2\log\Lambda$ given in Eq. (49) is asymptotically distributed as $\chi^2$ with one degree of freedom freedom in Test 1, Test 2, Test 3, Test 4 and 2 degree of freedom in Test 5, Test 6 and three degree of freedom in Test 7. The computed values of $\ln L(\hat{\Omega};y|x)$, $\ln L(\hat{\Omega}^*;y|x)$ and test statistic in case of both the two datasets are listed in Table 10. Since the critical value at the significance level 0.05 and degree of freedom one for two tailed test is 5.023 and for degrees of freedom two is 7.327 and for degrees of freedom three is 9.348, the null hypothesis is rejected in all cases, which shows the appropriateness of the $\mathrm{ELRM_{II}}$ to the datasets.

*TABLE 10*
*Calculated values of the test statistics in case of ELRM$_{II}$.*

| | Test | $\ln L\left(\hat{\Omega}; y\|x\right)$ | $\ln L\left(\hat{\Omega}^*; y\|x\right)$ | Test statistic |
|---|---|---|---|---|
| Dataset 3 | Test 1 | -227.0811 | -230.4308 | 6.6994 |
| | Test 2 | -227.0811 | -229.8361 | 5.5100 |
| | Test 3 | -227.0811 | -232.1870 | 10.2118 |
| | Test 4 | -227.0811 | -256.1444 | 58.1266 |
| | Test 5 | -227.0811 | -232.9611 | 11.7602 |
| | Test 6 | -227.0811 | -234.0933 | 14.0244 |
| | Test 7 | -227.0811 | -259.6120 | 65.0618 |
| Dataset 4 | Test 1 | -68.1231 | -70.9346 | 5.7864 |
| | Test 2 | -68.1231 | -72.8451 | 9.6074 |
| | Test 3 | -68.1231 | -72.7806 | 9.4784 |
| | Test 4 | -68.1231 | -75.0685 | 14.0542 |
| | Test 5 | -68.1231 | -74.1214 | 11.9966 |
| | Test 6 | -68.1231 | -75.2569 | 14.2696 |
| | Test 7 | -68.1231 | -76.3648 | 16.4834 |

### *7.3. Simulation*

Next, we conduct a simulation study to assess the performance of the MLEs of the parameters of the ELRM$_{\text{II}}$. We consider the following two sets of parameters.

1. $\alpha = 0.159$, $\beta = 3.830$, $a = 6.731$, $b = 0.229$ (positively skewed),

2. $\alpha = 2.5$, $\beta = 1.116$, $a = -58.378$, $b = 39.432$ (negatively skewed).

The computed values of the bias and MSE corresponding to sample sizes 100, 200, 300 and 500 respectively are given in Table 11. From the Table, it can be seen that both the absolute bias and MSEs in respect of each parameter of the ELRM$_{\text{II}}$ are in decreasing order as the sample size increases.

*TABLE 11*
*Bias and MSE within brackets of the simulated datasets in case of $ELRM_{II}$.*

| Parameter Set | sample size: | $\alpha$ | $\beta$ | $a$ | $b$ |
|---|---|---|---|---|---|
| | 100 | 1.64E-01 | 6.14E-02 | 3.05E-01 | 2.86E-01 |
| | | (8.07E-02) | (4.53E-03) | (7.10E-03) | (1.33E-02) |
| | 200 | 8.32E-02 | 3.67E-02 | 1.19E-01 | 9.82E-02 |
| | | (2.64E-02) | (2.34E-03) | (2.78E-03) | (5.81E-03) |
| $\alpha = 0.159$, $\beta = 3.830$, | 300 | 5.85E-02 | 7.73E-03 | 8.85E-02 | 7.55E-02 |
| a= 6.731, b=0.229 | | (6.73E-03) | (1.99E-03) | (1.74E-03) | (2.46E-03) |
| | 500 | 1.02E-02 | -2.32E-03 | 3.64E-02 | -4.41E-02 |
| | | (3.69E-03) | (8.47E-04) | (9.41E-04) | (1.06E-04) |
| | 100 | 5.71E-01 | 6.77E-01 | -6.08E-02 | 2.12E-01 |
| | | (2.46E-01) | (9.26E-02) | (7.68E-02) | (4.56E-02) |
| | 200 | 2.54E-01 | 3.57E-01 | -4.81E-02 | 9.93E-02 |
| | | (1.28E-02) | (6.30E-03) | (5.59E-03) | (1.51E-02) |
| $\alpha = 2.5$, $\beta = 1.116$, | 300 | 6.70E-02 | 1.03E-01 | 8.02E-03 | 7.02E-02 |
| a= -58.378, b=39.432 | | (7.82E-03) | (1.28E-03) | (3.65E-04) | (8.22E-03) |
| | 500 | 2.76E-03 | 8.87E-02 | 2.12E-03 | 5.13E-02 |
| | | (2.97E-04) | (6.65E-04) | (9.93E-05) | (5.97E-04) |

## 8. Conclusion

Exponentiated version of $LD_{II}$ was studied by Manju (2016). Also the distribution with the same PDF has been considered by Sapkota (2020) and called it as the Exponentiated-Exponential Logistic Distribution. Through the present paper we considered a detailed study of the Exponentiated-Exponential logistic distribution. We have investigated its properties, discussed parameter estimation, and demonstrated its usefulness in analysing real-life medical data along with its corresponding regression model. The developed model provides researchers with valuable tools for accurately modelling and analysing medical phenomena, thereby contributing to advancements in healthcare research and decision-making.

# Appendix

## A. Equations

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \alpha^2} &= \frac{-n}{\alpha^2} - (\beta-1)\sum_{i=1}^{n} \frac{\ln(\Omega_{1i})^2\,\Omega_{\alpha i}\,(1-\Omega_{\alpha i}+\Omega_{1i})}{(1+\Omega_{\alpha i})^2},\\
\frac{\partial^2 l}{\partial \beta^2} &= \frac{-n}{\beta^2},\\
\frac{\partial^2 \alpha}{\partial \mu^2} &= \frac{-\alpha(\beta-1)}{\sigma^2}\left[\sum_{i=1}^{n} \frac{(1-\Omega_{1i})^3\left[(1+\Omega_{\alpha i})\Omega_{\alpha-1i}(\alpha+e^{-2z_i})-\alpha\Omega_{2\alpha-1i}\right.}{(1+\Omega_{\alpha i})^2}\right]\\
&\quad -\frac{(\alpha+1)}{\sigma}\sum_{i=1}^{n}(1-\Omega_{1i})^2,\\
\frac{\partial^2 l}{\partial \sigma^2} &= \frac{n}{\sigma^2} - \frac{\alpha}{\sigma^2}\sum_{i=1}^{n} z_i - \frac{\alpha(\beta-1)}{\sigma^2(1+\Omega_{\alpha i})}\left[e^{-\alpha z_i}(\sigma(\alpha z_i-1)-1)\right.\\
&\quad \left.+\left[(\alpha+1)z_i e^{(\alpha+1)z_i}-\alpha z_i e^{-\alpha z_i}\right](1-\Omega_{1i})^{\alpha+2}\right]\\
&\quad -\frac{(\alpha+1)}{\sigma}\sum_{i=1}^{n} z_i(z_i-1)\Omega_{1i}(1-\Omega_{1i}).
\end{aligned}
$$

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