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ON THE ASYMPTOTIC BEHAVIOR OF THE MAXIMUM AND RECORD VALUES OF MULTIVARIATE DATA USING THE R-ORDERING PRINCIPLE

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SUMMARY

By using a sup-norm, sufficient conditions for the convergence of multivariate extremes and the potential limit types were fully identified by Barakat *et al.* (2020a). In this paper, we prove an intriguing result that by using the sup-norm, the weak convergence of multivariate extremes to the Fréchet type implies the convergence of those multivariate extremes in an arbitrary D-norm to the same type-limit by using the same normalizing constants. As a result of this finding, the weak convergence to the Fréchet type takes place by employing any logistic norm. Moreover, the two other possible limit types (max-Weibull and Gumbel types) are discussed. Similar findings are also demonstrated for multivariate record values. Finally, we demonstrate in a real-world scenario how to model multivariate extreme data sets utilizing the R-ordering principle and different norms.

Keywords: Weak convergence; Multivariate extremes; Reduced ordering principle; Sup-norm; Logistic-norm; D-norm.

1. INTRODUCTION

In many applications, there is a need to order multivariate observations. Unlike univariate observations, multivariate observations lack any natural ordering properties. Barnet (1976) presented a fourfold classification of sub-ordering principles for multivariate random vectors (observations). These principles can be classified as follows: marginal ordering (M-ordering), reduced ordering (R-ordering), partial ordering (P-ordering), and conditional ordering (C-ordering). It worth to refere what are called depth functions,

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which are introduced by Tukey (1974) to provide an outward ordering in a multivariate sample. Statistical depth functions provide useful orderings of points in $\mathbb{R}^d$ and are becoming increasingly utilized in multivariate analysis, see, for example, Chebana and Ouarda (2011); He and Einmah (2017); Tat and Faridrohani (2021).

Here, our main interest is studying the ordered multivariate data based on the R-ordering principle. Bairamov and Gebizlioglu (1998) developed the R-ordering principle using the concept of norms. More specifically, let $||\cdot||$ be a norm defined in the real Euclidean space $\mathbb{R}^d$. Furthermore, let $F(x_1,\ldots,x_d)$ be a common d-dimensional distribution function (DF) of given random vectors $\underline{X}_1,\ldots,\underline{X}_n$, where $\underline{X}_i=(X_{i1},X_{i2},\ldots,X_{id})$, $i=1,2,\ldots,n$. Bairamov and Gebizlioglu (1998) relied on an elementary result in probability theory, which guarantees that $||\underline{X}_1||,||\underline{X}_2||,\ldots,||\underline{X}_n||$ are i.i.d random variables (RVs) with common DF $P(||\underline{X}_i||\leq x)=\mathscr{F}(x)$, $x\in\mathbb{R}$. Additionally, the RVs $||\underline{X}_1||,||\underline{X}_2||,\ldots,||\underline{X}_n||$ make up a series of univariate RVs if $F(x_1,\ldots,x_d)$ is continuous. As a result, these RVs ought to be arranged naturally and uniquely (i.e., without any tie among them). In this case, $\underline{X}_i$ is said to be less than $\underline{X}_j$ in a norm-sense, written $\underline{X}_i\prec\underline{X}_j$, if $||\underline{X}_i||<||\underline{X}_j||$, $i,j=1,2,\ldots,n$. Consequently, in this sense if $\underline{X}_{r:n}$ denotes the r-smallest of the set $\underline{X}_1,\underline{X}_2,\ldots,\underline{X}_n$, then we get the multivariate-order statistics (in the norm-sense) $\underline{X}_{1:n}\prec\underline{X}_{2:n}\prec\ldots\prec\underline{X}_{n:n}$. According to Bairamov (2006); Arnold *et al.* (2009), the R-ordering concept has undergone some significant developments and generalizations.

An important question that Bairamov and Gebizlioglu (1998) did not address is raised: Could we use any norm defined on $\mathbb{R}$ to order multivariate observations? First, it is not acceptable to use any non-monotone norm $||\cdot||$, for which $||\underline{X}_i||<||\underline{X}_j||$ (i.e., $\underline{X}_i\prec\underline{X}_j$) $i,j=1,2,\ldots,n$, $i\neq j$, while $\underline{0}\leq\underline{X}_j<\underline{X}_i$, where this inequality is taken componentwise. Also, the employed norm should be radially symmetric, i.e., changing the sign of arbitrary components does not alter the value of it. This means that the values of the employed norm in the R-ordering principle are completely determined by its values on the subset $\{\underline{x}\in\mathbb{R}^d:\underline{x}\geq\underline{0}\}$.

Example of the non-monotone and non-radially symmetric norm is the quadratic form norm $||\underline{X}_i||_A=\sqrt{(\underline{X}_i^TA\underline{X}_i)}$, $i=1,2,\ldots,n$, defined on $\mathbb{R}^2$ (say), where $A=A^T=(a_{ij})_{1\leq i,j\leq 2}$ is a positive definite 2×2 matrix, such that $a_{11}=a_{22}=1$ and $a_{12}=a_{21}=\delta$, $\delta\in(-1,0)$ (cf. Falk, 2019). Indeed, the norm

$$||\underline{x}||_A=\sqrt{\underline{x}^TA\,\underline{x}}=\sqrt{x_1^2+2\delta\,x_1x_2+x_2^2}$$

is not monotone, for example, set $\delta=-\frac{1}{2}$, $x_1=1$, $x_2=0$, $y_1=1$ and $y_2=\frac{1}{2}$. Then, $\underline{0}\leq\underline{x}\leq\underline{y}$, where the last inequality is taken componentwise, but $||\underline{x}||_A=1>||\underline{y}||_A=\frac{\sqrt{3}}{2}$. Also,

$$\begin{aligned}||(x_1,x_2)||_A &= \sqrt{(x_1,x_2)^TA\,(x_1,x_2)}=\sqrt{x_1^2+2\delta\,x_1x_2+x_2^2}\\ &\neq \sqrt{(x_1,-x_2)^TA\,(x_1,-x_2)}=\sqrt{x_1^2-2\delta\,x_1x_2+x_2^2}=||(x_1,-x_2)||_A.\end{aligned}$$

Therefore, the quadratic form norm is also not radially symmetric.

The family of D-norms is an example of a family of monotone and radially symmetric norms (cf. Falk, 2019). The D-norms are closely associated with the asymptotic behavior of the multivariate extreme theory (see Hofmann, 2009; Falk and Wisheckel, 2018). Moreover, the family of the D-norms includes numerous other subfamilies of norms, including the logistic family. Therefore, if we choose to order data by the R-ordering principle in the sense of norms, it is preferable to take the family of D-norms defined on $\mathbb{R}^d$ (or any of its subfamilies, like sup-norm), see Barakat *et al.* (2020a,b); Falk (2019). The notion of the D-norm was first proposed by Falk *et al.* (2004) and subsequently expanded upon by Falk *et al.* (2011). However, it was only much later that D-norms offered the framework for multivariate extreme value theory, as well as a mathematical topic that can be studied in isolation (e.g., see Fuller, 2016). The definition of D-norm is given below.

DEFINITION 1. *(cf. Falk, 2019) Let $\underline{x} = (x_1, x_2, \ldots, x_d)$ be a d-dimensional scaler vector in $\mathbb{R}^d$. Furthermore, let $\underline{Z} = (Z_1, Z_2, \ldots, Z_d)$ be a random vector, whose components satisfy $Z_i \geq 0$, $\mathrm{E}(Z_i) = 1, i = 1, 2, \ldots, d$. Then,*

$$||\underline{x}||_D := \mathrm{E}\left(\max_{1\leq i\leq d}(|x_i|Z_i)\right)$$

defines a norm, called D-norm, and the random vector $\underline{Z} = (Z_1, Z_2, \ldots, Z_d)$ is called the generator of this D-norm.

For employing the D-norm in the R-ordering principle, we have to adapt Definition 1 by replacing the vector $\underline{x}$ with the random vector $\underline{X}$ emphasizing that the expectation in the definition is taken only concerning $\underline{Z}$. The independence assumption between $\underline{X}$ and $\underline{Z}$ and taking the expectation only concerning $\underline{Z}$ were always used in Falk (2019) when one uses a random vector $\underline{X}$ instead of a scalar vector $\underline{x}$. This is the only way to ignore the randomness of $\underline{X}$ when taking the expectation. In this way, the generator selection and D-norm building should be carried out independently of $\underline{X}$. Therefore, from now on we deal with the RVs $||\underline{X}_1||_D, ||\underline{X}_2||_D, \ldots, ||\underline{X}_n||_D$, where

$$||\underline{X}_i||_D := \mathrm{E}_Z\left(\max_{1\leq t\leq d}|X_{it}|Z_t\right), \; i = 1, 2, \ldots, n \tag{1}$$

and

$$\mathrm{E}_Z(Z_t) = \int \ldots \int z_t \, dF_{Z_1,\ldots,Z_t,\ldots,Z_d}(z_1, \ldots, z_t, \ldots, z_d) = 1.$$

Below, we list some important properties of the D-norm (cf. Falk, 2019).

1. If we choose the constant generator $\underline{Z} = (1, 1, \ldots, 1)$, then $||\underline{X}_i||_D = \max_{1\leq t\leq d}|X_{it}| = ||\underline{X}_i||_\infty$, i.e., the sup-norm is a D-norm.

2. Let $W \geq 0$ be an RV with $E(W) = 1$ and put $\underline{Z} = (W, W, \dots, W)$. Then, $\underline{Z}$ also is a generator of the sup-norm, i.e., the generator of a D-norm is in general not uniquely determined.

3. Let $\underline{Z}$ be a random permutation of $(d, 0, \dots, 0) \in \mathbb{R}^d$ with equal probability $\frac{1}{d}$, i.e.,

$$Z_i = \begin{cases} d, \text{ with probability } \frac{1}{d}, \\ 0, \text{ with probability } 1 - \frac{1}{d}, \end{cases} \quad 1 \leq i \leq d,$$

and $Z_1 + \ldots + Z_d = d$. Then, the random vector $\underline{Z}$ is the generator of a D-norm $||\underline{X}_i||_D = \sum_{t=1}^{d} |X_{it}| = ||\underline{X}_i||_1$, $i = 1, 2, \dots, n$, i.e., $||\cdot||_1$ is a D-norm.

4. In general, each logistic norm $||\underline{X}_i||_p = \left(\sum_{t=1}^{d} |X_{it}|^p\right)^{\frac{1}{p}}$, $1 \leq p < \infty$, is a D-norm. For $1 < p < \infty$ a generator is given by

$$\underline{Z}^{(p)} = (Z_1^{(p)}, \dots, Z_d^{(p)}) = \left(\frac{W_1^{1/p}}{\Gamma(1-p^{-1})}, \dots, \frac{W_d^{1/p}}{\Gamma(1-p^{-1})}\right),$$

where $W_1, \dots, W_d$ are i.i.d standard Fréchet-distributed RV, i.e.,

$$P(W_i \leq w) = e^{-w^{-1}}, \; w \geq 0, \; i = 1, \dots, d, \; \text{ with } \; E(W_i^{1/p}) = \Gamma(1-p^{-1}),$$

where $1 \leq i \leq d$ (it is known that $\lim_{p\to\infty} ||\underline{X}_i||_p = ||\underline{X}_i||_\infty$ and $||\underline{X}_i||_p \leq ||\underline{X}_i||_q$, whenever $q \leq p$).

5. In complete accordance with the univariate case we call a DF G on $\mathbb{R}^d$ max-stable, if for every $n \in \mathbb{N}$ there exist normalizing constants $a_{t,n} > 0$ and $b_{t,n}$, $t = 1, 2, \dots, d$, such that $G^n(a_{1,n}x_1 + b_{1,n}, \dots, a_{d,n}x_d + b_{d,n}) = G(x_1, \dots, x_d)$. The theory of D-norms allows a mathematically elegant characterization of the family of standard max-stable DF. Namely, any DF G on $\mathbb{R}^d$ is a standard max-stable DF if and only if there exists a D-norm on $\mathbb{R}^d$ such that $G(\underline{x}) = \exp(-||\underline{x}||_D)$, $\underline{x} \leq \underline{0} \in \mathbb{R}^d$ (see Falk, 2019, Theorem 2.3.3).

We end this introductory Section by re-stating the Extremal Types Theorem, which serves as the main subject of this paper's investigation.

THEOREM 2. *(Extremal Types Theorem, see Leadbetter et al., 1983) Let $X_{n:n} = \max(X_1, X_2, \dots, X_n)$, where $X_1, X_2, \dots, X_n$ are i.i.d RVs with univariate DF $F(x)$. Then, by using the elementary relation $\lim\limits_{x\to 1} \frac{\log x}{x-1} = 1$, we get*

$$F_{n:n}(a_n x + b_n) = P(X_{n:n} \leq a_n x + b_n) \xrightarrow[n]{w} H(x), \tag{2}$$

where $a_n > 0$, $b_n \in \mathbb{R}$ *are some suitable normalizing constants and* $H(\cdot)$ *is a non-degenerate DF, if and only if*

$$n(1-F(a_nx+b_n)) \underset{n}{\longrightarrow} -\log H(x),$$

where the symbols "$\underset{n}{\longrightarrow}$*" and "*$\overset{w}{\underset{n}{\longrightarrow}}$*" stand for convergence and weak convergence, as* $n \to \infty$*, respectively. Moreover, the limit function* $H(\cdot)$ *must have one of the three possible types* $H_{i,\beta}(x) = \exp(-u_{i,\beta}(x))$, $i = 1,2,3$, $\beta > 0$, *where* $H_{3,\beta}(x) = H_3(x) = \exp(-u_3(x))$, *and*

$$\left.\begin{array}{lll} \textit{Type I (Fréchet type):} & u_{1,\beta}(x) = \begin{cases} x^{-\beta}, & x > 0, \\ \infty, & x \leq 0, \end{cases} \\ \textit{Type II (max-Weibull type):} & u_{2,\beta}(x) = \begin{cases} (-x)^{\beta}, & x \leq 0, \\ 0, & x > 0, \end{cases} \\ \textit{Type III (Gumbel type):} & u_{3,\beta}(x) = u_3(x) = e^{-x}, \ -\infty < x < \infty. \end{array}\right\} \qquad (3)$$

Conversely, any such DF $\exp(-u_{i,\beta}(x))$, $i \in \{1,2,3\}$, *appears as a limit in Eq. (2) and in fact does so when* $\exp(-u_{i,\beta}(x))$ *is itself the DF of each* X_i*. In this case, we write* $F \in \mathscr{D}(H_{i,\beta})$ *and call* $H_{i,\beta}(\cdot), i = 1,2,3$, *max-stable DFs.*

In this paper, we prove an interesting result that the weak convergence of multivariate extremes by using the sup-norm (defined by $||\underline{X}_i||_\infty = \max\limits_{1\leq t\leq d}|X_{it}|$, for more details see Section 2) to the Fréchet type implies the convergence of those multivariate extremes in an arbitrary D-norm to the same type-limit by using the same normalizing constants. As a result of this finding, any logistic norm (defined by $||\underline{X}_i||_p = \left(\sum_{t=1}^{d}|X_{it}|^p\right)^{\frac{1}{p}}$, $1 \leq p < \infty$, for more details see Section 2) may be used to achieve the weak convergence to the Fréchet type in this situation. Throughout this paper, we use $\mathscr{F}(\cdot)$ and $\mathscr{F}_D(\cdot)$ to denote the DFs of the RVs $||\underline{X}||_\infty$ and $||\underline{X}||_D$, respectively, i.e., $\mathscr{F}(x) = P(||\underline{X}||_\infty \leq x)$ and $\mathscr{F}_D(x) = P(||\underline{X}||_D \leq x)$.

2. The main result

Barakat *et al.* (2020a) considered a sequence of bivariate random vectors $\underline{X}_i = (X_{i1}, X_{i2})$, $i = 1,2,\ldots,n$, distributed according as a common bivariate DF $F(x_1,x_2)$. They proved Theorem 3.1, which reveals the asymptotic behavior of the DF

$$\mathscr{F}_{n:n}(x) = P\left(\max(||\underline{X}_1||_\infty, \ldots, ||\underline{X}_n||_\infty) \leq x\right) = \mathscr{F}^n(x),$$

in terms of the asymptotic behavior of the marginal maxima concerning the marginals DFs F_1 and F_2 of F, under the condition that the two marginal maxima are asymptotically independent. Moreover, Barakat *et al.* (2020a) demonstrated how to eliminate the restrictive requirement $X_{it} > 0, t = 1,2, i = 1,2,\ldots,n$, and extend Theorem 3.1 to multivariate extremes (cf. Remark 3.2). Ultimately, they proved Theorem 3.2 extending the result of Theorem 3.1 into the case of asymptotically dependent between marginal maxima.

2.1. *An intriguing and unique limit feature of the Fréchet type*

In this sub-Section, we show that when the weak convergence of $\mathscr{F}_{n:n}$ occurs to the Fréchet type, we prove an interesting result that Theorem 3.1 and its extensions given in Barakat *et al.* (2020a) are true for the D-norm. Consequently, they are also true for any arbitrary logistic norm. In the next theorem, we will keep the notations used in Theorem 3.1 in Barakat *et al.* (2020a) for the employed normalizing constants. Namely, we define the normalizing constants $(a_{j,n}, b_{j,n})$ such that the subscript j is equal 1, or 2, or 3, according to the weak convergence taken place to the type $H_{1,\beta}(x)$, or $H_{2,\beta}(x)$, or $H_3(x)$, respectively. The Extremal Types Theorem (Theorem 2) entails that any DF belonging to the domain of attraction of the Fréchet type (i.e., the relation in Eq. (2) holds for $H_{1,\beta}$ for some $\beta > 0$) should have the infinite right endpoint $x_{1,0}$ and in this case we can take $b_{1n} = 0$. Bearing in mind this fact, we have the following theorem:

THEOREM 3. *Let* $\mathscr{F}(a_{1,n}x) \in \mathscr{D}(H_{1,\beta}(x))$, $\beta > 0$. *Then,* $\mathscr{F}_D(a_{1,n}x) \in \mathscr{D}(H_{1,\beta}(Ax))$, *for some constant* $0 < A \leq 1$.

PROOF. In view of Eq. (1), we get the D-norms (arbitrary D-norms)

$$||\underline{X}_i||_D := \mathrm{E}_Z\left(\max_{1\leq t\leq d}|X_{it}|Z_t\right),\ i = 1,2,\ldots,n,$$

where $\underline{Z} = (Z_1, Z_2, \ldots, Z_d)$ is any generator. Now, by using the inequality $||\underline{X}_i||_\infty \leq ||\underline{X}_i||_D$ (cf. Falk, 2019, Page 4), besides applying Jensen's inequality, the inequality $\max_{1\leq t\leq d}\omega_t\omega_t^\star \leq \max_{1\leq t\leq d}\omega_t \max_{1\leq t\leq d}\omega_t^\star$, for any sequences $\{\omega_t\}$ and $\{\omega_t^\star\}$, $\omega_t^\star, \omega_t > 0$, and using the relation $\mathrm{E}_Z(Z_t) = 1$, $t = 1,2,\ldots,d$, we get

$$\begin{aligned} ||\underline{X}_i||_\infty \leq ||\underline{X}_i||_D &= \mathrm{E}_Z\left[\max_{1\leq t\leq d}|X_{it}|Z_t\right] \\ &\leq \left[\max_{1\leq t\leq d}|X_{it}|\right]\mathrm{E}_Z\left[\max_{1\leq t\leq d}Z_t\right] = C\ ||\underline{X}_i||_\infty, \quad i = 1,2,\ldots,n, \end{aligned} \tag{4}$$

where $C = \mathrm{E}_Z\left[\max_{1\leq t\leq d}Z_t\right] \geq 1$ (by using again Jensen's inequality). Moreover, due to the Eq. (4), we have

$$P\left(C||\underline{X}_i||_\infty \leq x\right) \leq P\left(||\underline{X}_i||_D \leq x\right) \leq P\left(||\underline{X}_i||_\infty \leq x\right),$$

or equivalently

$$P\left(||\underline{X}_i||_\infty \leq a_{1,n}(C^{-1}x)\right) \leq P\left(||\underline{X}_i||_D \leq a_{1,n}x\right) \leq P\left(||\underline{X}_i||_\infty \leq a_{1,n}x\right),$$

which can be written as

$$\mathscr{F}\left(a_{1,n}\,(C^{-1}x)\right) \leq \mathscr{F}_D(a_{1,n}x) \leq \mathscr{F}(a_{1,n}x).$$

Consequently,

$$\mathscr{F}^n\left(a_{1,n}\left(C^{-1}x\right)\right) \leq \mathscr{F}_D^n(a_{1,n}x) \leq \mathscr{F}^n(a_{1,n}x). \quad (5)$$

By using the condition of Theorem 3.1 given in Barakat *et al.* (2020a), that $\mathscr{F}(a_{1,n}x) \in \mathscr{D}(H_{1,\beta}(x))$, the sequences of DFs on the LHS and RHS of Eq. (5) weakly converge to the non-degenerate DFs $H_{1,\beta}(\frac{x}{C})$ and $H_{1,\beta}(x)$, respectively. Therefore, $\lim_{n\to\infty} \mathscr{F}_D^n(a_{1,n}x) = 0$, for any $x \leq 0$. On the other side, $\lim_{n\to\infty} \mathscr{F}_D^n(a_{1,n}x) \to 1$, as $x \to \infty$, which means that the sequence $\mathscr{F}_D^n(a_{1,n}x)$ has non-defective limit DF. Finally, from the facts that $H_{1,\beta}(x)$ is strictly increasing in the interval $0 < x < \infty$, and $0 < H_{1,\beta}(\frac{x}{C}) < H_{1,\beta}(x) < 1$, $0 < x < \infty$, we can find $0 < x_0 < \infty$, such that $0 < \lim_{n\to\infty} \mathscr{F}_D^n(a_{1,n}x) < 1$ (actually there are infinite numbers of such value x_0). Therefore, the relation in Eq. (5) entails that the sequence of DFs $\mathscr{F}_D^n(a_{1,n}x)$ converges to a non-degenerate DF. On the other hand, in view of the Extremal Types Theorem (Theorem 2), this sequence should converge to a max-stable DF $H_{i,\beta^\star}(Ax)$, $i \in \{1,2,3\}$, say. Therefore, in view of Eq. (5) we should have

$$H_{1,\beta}(C^{-1}x) \leq H_{i,\beta^\star}(Ax) \leq H_{1,\beta}(x), \ \forall x \in \mathbb{R}, A > 0. \quad (6)$$

On taking a quick look at the functional shape of the three limit types $H_{1,\beta}$, $H_{2,\beta}$ and H_3, Eq. (6) is impossible satisfied unless $i = 1$. Moreover, a more deep look leads to the conclusion that $\beta^\star = \beta$ and $0 < A \leq 1$. Let us shed more light on the proof. In our case we have

$$e^{-\left(\frac{x}{C}\right)^{-\beta}} \leq e^{-(Ax)^{-\beta^\star}} \leq e^{-x^{-\beta}}, \ \forall x \geq 0, \ A, \ \beta, \ \beta^\star > 0,$$

which in turn implies

$$x^{-\beta} \leq (Ax)^{-\beta^\star} \leq \left(\frac{x}{C}\right)^{-\beta}, \ \forall x \geq 0. \quad (7)$$

Clearly, the inequalities in Eq. (7) cannot be held, $\forall x \geq 0$, unless $\beta^\star = \beta$. Thus, we get $C^\beta \geq A^{-\beta} \geq 1$, i.e., $1 \geq A$ and $AC \geq 1$. □

EXAMPLE 4. *Suppose that the components $X_{i1} > 0$ and $X_{i2} > 0$ of the bivariate random vectors $\underline{X}_i = (X_{i1}, X_{i2})$, $i = 1, 2, \ldots, n$, are distributed as F_1 and F_2, respectively. Furthermore, let $F_1 \in \mathscr{D}(H_{1,\alpha})$ and $F_2 \in \mathscr{D}(H_{1,\beta})$, $0 < \alpha < \beta$. Then, in view of the result of Barakat* et al. *(2020a), we get $\mathscr{F} \in \mathscr{D}(H_{1,\alpha})$. Consequently, Theorem 3.1 in Barakat* et al. *(2020a) entails that $\mathscr{F}_D \in \mathscr{D}(H_{1,\alpha})$. Moreover, as a consequence of Theorem 3 in the present paper, $\mathscr{F}_p \in \mathscr{D}(H_{1,\alpha})$, $0 \leq p < \infty$, where $\mathscr{F}_p(x) = P((X_{i1}^p + X_i^p)^{\frac{1}{p}} \leq x)$. Moreover, since the Logistic norm is a special example of D-norm, so Theorem 3 can be implemented to this norm. It is worth mentioning that the last result at $p = 1$ was proved by Sreehari* et al. *(2011).*

Under the conditions of Theorem 3 in this paper, Theorems 3.1 with its extensions given in Barakat *et al.* (2020a) can now be applied for arbitrary logistic norm $||\cdot||_p$, whenever we deal with the Fréchet type. Apart from the multivariate extreme value theorem, this result has wide applications in many aspects of applied sciences. For example, in many applications, the researchers' concern is focused on the convolution of two RVs rather than these RVs. In this context, by considering the norm $||\cdot||_1$ for positive RVs, Theorem 3 in this paper (whenever the convergence takes place to the Fréchet type) enables us to determine the asymptotic behavior of the maximum concerning the convolution in terms of the asymptotic behavior of the maxima pertained to its components.

EXAMPLE 5. *Let X and Y be independent Cauchy RVs. Thus,* $F_X(t)=F_Y(t)=\frac{1}{2}+\frac{1}{\pi}\arctan t, -\infty<t<\infty$. *It's known that* $F_X(a_{1,n}t+b_{1,n})=F_Y(a_{1,n}t+b_{1,n})\in\mathscr{D}(H_{1,1}(t))$ *(cf. Ahsanullah and Nevzorov, 2001), with normalizing constants* $a_{1,n}=\tan\left(\frac{\pi}{2}-\frac{\pi}{n}\right)\sim\frac{n}{\pi}$ *and* $b_{1,n}=0$. *In view of Theorem 3.2 Part (1)-(ii) in Barakat et al. (2024), we have* $F_{|X|}(a_{1,n}t+b_{1,n})=F_{|Y|}(a_{1,n}t+b_{1,n})\in\mathscr{D}(H_{1,1}(\frac{1}{2}t))$. *Therefore, an application of Theorem 3.1 Part (1) in Barakat et al. (2020a) yields* $\mathscr{F}(a_{1,n}t+b_{1,n})\in\mathscr{D}(H_{1,1}((\frac{1}{2}t)))$. *On the other hand, due to Example 4.7 in Barakat et al. (2024), we get* $F_{||(X,Y)||_p}(a_{1,n}t+b_{1,n})\in\mathscr{D}(H_{1,1}(t))$, $\forall p\in[1,\infty)$. *This means that* $\mathscr{F}(a^\star_{1,n}t+b_{1,n})\in\mathscr{D}(H_{1,1}((t)))$ *and* $F_{||(X,Y)||_p}(a^\star_{1,n}t+b_{1,n})\in\mathscr{D}(H_{1,1}(\frac{t}{2}))$, *where* $2a^\star_{1,n}=a_{1,n}$. *The last result endorses Theorem 3, with* $A=\frac{1}{2}$.

2.2. *Discussion of the max-Weibull and Gumbel limit types*

The Extremal Types Theorem entails that any DF belonging to the domain of attraction of the max-Weibull type (i.e., the relation in Eq. (2) holds for $H_{2,\beta}$ for some $\beta>0$) should have the finite right endpoint $x_{2,0}$ and in this case the Extremal Types Theorem also allows us to take $b_{2,n}=x_{2,0}$, and $a_{2,n}\xrightarrow[n]{}0$. On the other hand, when $\mathscr{F}(a_{3,n}x+b_{3,n})\in\mathscr{D}(H_3(x))$, the Extremal Types Theorem entails that $\frac{b_{3,n}}{a_{3,n}}\xrightarrow[n]{}\infty$. Thus, because of Khinchin's type theorem (cf. Barakat *et al.*, 2019), $\mathscr{F}^n\left(C^{-1}a_{i,n}x+C^{-1}b_{i,n}\right)$, $i=2,3; b_{2,n}=x_{2,0}\neq 0$, cannot in general weakly converge to any non-degenerate limit DF. Therefore, the proof method of Theorem 3 in this paper cannot be applied in these cases. The following example endorses this by showing that the weak convergence to the Gumbel type under the sup-norm does not imply convergence under any other D-norm if we used the same normalizing constants.

EXAMPLE 6. *Suppose that the components* X_{i1} *and* X_{i2} *of the bivariate random vectors* $\underline{X}_i=(X_{i1},X_{i2})$, $i=1,2,\ldots,n$, *are independent and standard exponential RVs. Then, one* *obtain, with the normalizing constants* $a_{3,n} = 1$ *and* $b_{3,n} = \log n$

$$\begin{aligned} P\left(\max_{1\le i\le n} \|\underline{X}_i\|_\infty \le x + \log n\right) &= P\left(X_{11} \le x + \log n, X_{12} \le x + \log n\right)^n \\ &= P\left(X_{11} \le x + \log n\right)^n P\left(X_{12} \le x + \log n\right)^n \\ &= (1 - \exp(-(x + \log n)))^{2n} \\ &= \left(1 - \frac{\exp(-x)}{n}\right)^{2n} \\ &\xrightarrow[n]{w} \exp(-2\exp(-x)) = H_3(x - \log 2). \end{aligned}$$

On the other hand, for every $i = 1, 2, \ldots, n$, *the RV* $X_{i1}+X_{i2}$ *follows the gamma distribution with parameter 2, i.e., its DF is*

$$P(X_{i1} + X_{i2} \le x) = 1 - (1 + x)e^{-x},\ x \ge 0.$$

As a consequence, using the same normalizing constants $a_n = 1$ *and* $b_n = \log n$, *we get*

$$\begin{aligned} P\left(\max_{1\le i\le n} \|\underline{X}_i\|_1 \le x + \log n\right) &= P^n\left(X_{11} + X_{12} \le x + \log n\right) \\ &= (1 - (1 + x + \log n)\exp(-x - \log n))^n \\ &= \left(1 - (1 + x + \log n)\frac{\exp(-x)}{n}\right)^n \xrightarrow[n]{} 0, \end{aligned}$$

where the last limit convergence relation to zero can be checked by taking the logarithm and using the limit relation

$$\lim_{\varepsilon\to 0} \frac{\log(1+\varepsilon)}{\varepsilon} = 1.$$

Example 4 shows that for the Gumbel type, the same normalizing constants as the sup-norm cannot be used to prove the convergence for any other D-norm. In this case, a natural question arises: are there no other normalizing constants that may be used? In answer to this question, we can use $a_n = 1$ *and* $b_n = \log n + \log\log n$, *to get (cf. Ahsanullah and Nevzorov, 2001, Page 91)*

$$P\left(\max_{1\le i\le n} \|\underline{X}_i\|_1 \le x + \log n + \log\log n\right) \xrightarrow[n]{w} H_3(x).$$

Example 4 shows that Theorem 3 does not apply to the Gumbel type whenever we use the same normalizing constants for both norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Moreover, the above discussion concerning the max-Weibull type shows that the method of Theorem 3 does not apply whenever $x_{2,0} \neq 0$. The following proposition shows that Theorem 3 does not apply to the Gumbel and max-Weibull types in general.

PROPOSITION 7. *Theorem 3 does not apply to the max-Weibull and Gumbel types in general.*

PROOF. Suppose that the components X_{i1} and X_{i2} of the bivariate random vectors $\underline{X}_i = (X_{i1}, X_{i2})$, $i = 1, 2, \ldots, n$, are independent, where X_{i1} and X_{i2} have DFs F_1 and F_2. Furthermore, let $F_1 \star F_2$ be the convolution of F_1 and F_2, i.e., $F_1 \star F_2$ is the DF of $||\underline{X}_i||_1$. First, we prove the theorem for the max-Weibull type. Let $F_1(a_{2,n}x+b_{2,n}) \in \mathscr{D}(H_{2,\beta_1}(x))$ and $F_2(c_{2,n}x+d_{2,n}) \in \mathscr{D}(H_{2,\beta_2}(x))$. Then, in view of Theorem 3.1 Part (2) in Barakat *et al.* (2020a), if $x_1^0 < x_2^0$, we have $\mathscr{F}(c_{2,n}x + d_{2,n}) \in \mathscr{D}(H_{2,\beta_2}(x))$, while, if $x_2^0 < x_1^0$, we have $\mathscr{F}(a_{2,n}x + b_{2,n}) \in \mathscr{D}(H_{2,\beta_1})$. On the other hand, regardless of the interrelation between x_1^0 and x_2^0, because of Theorem 3.5 in Sreehari *et al.* (2011), we have $\mathscr{F}_{||\cdot||_1} = F_1 \star F_2 \in \mathscr{D}(H_{2,\beta_1+\beta_1})$. Since, H_{2,β_1}, H_{2,β_2}, and $H_{2,\beta_1+\beta_2}$ are of a different type, this proves that Theorem 3 does not apply to the max-Weibull type.

Turning now to prove the proposition for Gumbel type. Sreehari *et al.* (2011) revealed an interesting fact in Remark 3.9, Part (2) that if X and Y are independent RVs with $X + Y$ belonging to the max domain of attraction $\mathscr{D}(H_3)$, then it is not true that both X and Y should belong to the max domain of attraction $\mathscr{D}(H_3)$. On the other hand, because of Theorem 2.1, in Barakat *et al.* (2020a), in this case, it is impossible that $\mathscr{F} \in \mathscr{D}(H_3)$. This proves that Theorem 3 does not apply to the Gumbel type. □

2.3. *Similar limit results for the multivariate records*

It is known that the record values are rarely observed with large n. Therefore, an applied scientist would not find much use for an asymptotic theory of these statistics. Nonetheless, a thorough explanation of this theory is warranted due to its mathematical relevance (cf. Galambos, 1987). Recently, Barakat and Harpy (2021) investigated the asymptotic behavior of the multivariate record values using the R-ordering principle. They determined the necessary and sufficient conditions for weak convergence of the multivariate record values based on sup-norm.

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of mutually independent RVs with a common DF $F(x)$, we say X_j is an upper record value. if $X_j > X_i$ for every $j > i$. By convention X_1 is an upper record value. Clearly, the upper record values in the sequence $\{X_n,\ n \geq 1\}$ are the successive maxima. Let us assume that X_j is observed at time j. Then, the record time sequence $\{T_n, n > 1\}$ is defined by $T_1 = 1$ with probability 1 and, for $n > 1, T_n = \min\{j : X_j > X_{T_{n-1}}\}$. The record value sequence $\{R_n\}$ is then defined by $R_n = X_{T_n},\ n = 1, 2, \ldots$. The DF of the upper record value is given by (cf. Arnold *et al.*, 1998) $P(R_n \leq x) = \Gamma_n(U_F(x)),\ n > 1$, where $U_F(x) = -\log(1 - F(x))$ and $\Gamma_n(x) = \frac{1}{\Gamma(n)} \int\limits_0^x t^{n-1} e^{-t} dt$ is the incomplete gamma function. A DF F is said to belong to the domain of upper record value attraction of a non-degenerate DF Ψ, written $F \in \mathscr{D}_{urec}(\Psi)$, if there exist normalizing constants $a_n > 0$ and b_n such that, $P(R_n \leq a_n x + b_n) = F_{R_n}(a_n x + b_n) \xrightarrow[n]{w} \Psi(x)$. The class limit laws for record values consists of $\Psi_{i,\beta} = \Phi(-\log(-\log H_{i,\beta}(x))), i = 1, 2, 3$ (cf. Arnold *et al.*, 1998), where $\Phi(\cdot)$ is the standard normal distribution. For more details about the lower and upper record values and their asymptotic behavior, with applications, see Barakat (2012); Barakat and Elgawad (2017).

Now, in the same way, as in multivariate extremes, we can define the upper record-value sequence $\{R_n^*\}$ associated with the sequence $\{\underline{X}_i\}$, in the norm-sense, as the successive maxima in the sequence $\{\underline{X}_i\}$ in the norm sense, i.e., we say $\underline{X}_j$ is an upper record vector in the norm-sense if $\underline{X}_i \prec \underline{X}_j$ for every $i < j$.

The Duality Theorem for record values (cf. Arnold *et al.*, 1998) entails that any DF belonging to the domain of attraction of the log-normal type $\Psi_{1,\beta}$ for some $\beta > 0$ should have the infinite right endpoint $x_{1,0}$ and in this case we can take $b_{1n} = 0$. Bearing in mind this fact, we have the following result:

THEOREM 8. *Let* $\mathscr{F}(a_{1,n}x) \in \mathscr{D}_{urec}(\Psi_{1,\beta}(x))$. *Then,* $\mathscr{F}_D(a_{1,n}x) \in \mathscr{D}_{urec}(\Psi_{1,\beta}(Ax))$, *for some constant* $0 < A \leq 1$.

PROOF. The proof of this theorem follows exactly as the proof of Theorem 3 in the present paper, by noting that $\Gamma_n(U_F(x))$ is an increasing function of F and the functional shape of the limit $\Psi_{1,\beta}$, leads to the same results as those of the limit $H_{1,\beta}$. □

3. MODELING MULTIVARIATE EXTREMES OF A REAL DATA SET

In this Section, we provide an illustrative example of how to model multivariate extreme data sets using the R-ordering principle via alternative norms. This topic has not been addressed in the literature to our knowledge. Moreover, using three different logistic norms, the example shows that the extreme data sets under consideration were fitted by the max-Weibull type, but with different shape parameters.

We consider the extremes of real bivariate data for air pollution from the London Air Quality Network (LAQN). Namely, data was taken from the site Barking Dagenham at Rush Green square, which monitors sulfur dioxide (SO_2) (stands for the marginal X_{i1} with DF F_1) and Nitrogen oxides (NO) (stands for the marginal X_{i2} with DF F_2) every hour in the period from 1-1-2010 to 31-12-2015. These data sets were treated in Barakat *et al.* (2020a) for the sup-norm. Here we add the logistic norms $||\cdot||_1$ and $||\cdot||_2$, for comparison purposes. The daily maximum observations of these data sets (exactly 1949 daily maximum observations for each pollutant) are used to apply the block maxima method on the general extreme value DF (GEV) (see Barakat *et al.*, 2019)

$$G_\gamma(x;\mu,\sigma) = \exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\gamma}}\right\}, \tag{8}$$

defined on $\{x : 1+\gamma(x-\mu)/\sigma > 0\}$, where with $\gamma = 0$, $\gamma = \frac{1}{\beta} > 0$, and $\gamma = -\frac{1}{\beta} < 0$, the GEV $G_\gamma(x;\mu,\sigma)$ corresponds to the Gumbel, max-Weibull, and Fréchet types, respectively (defined in 3). On the other hand, the same method is applied to the norms

$||\underline{X}_i||_\ell = ||(X_{i1}, X_{i2})||_\ell$, $\ell = 1, 2, \infty$. The maximum likelihood estimates (MLEs) and the 95% asymptotic confidence intervals (95% CI) are obtained for the parameters μ, σ, and γ, by using the `MATLAB` Version 7.11.0.584 (R2010b).

Tables 1 and 2 give the result of this study, which reveals that the limit DFs of both $\max\limits_{1\le i\le n}\{X_{i1}\}$ and $\max\limits_{1\le i\le n}\{X_{i2}\}$ are max-Weibull with $\gamma = 0.281$ and $\gamma = 0.978$, respectively. On the other hand, the limit DF of $\max\limits_{1\le i\le n}||\underline{X}_i||_\infty$, $\max\limits_{1\le i\le n}||\underline{X}_i||_1$, and $\max\limits_{1\le i\le n}||\underline{X}_i||_2$ are also a max-Weibull DF with $\gamma = 0.952$, $\gamma = 0.239$, and $\gamma = 0.322$, respectively. This means that the limit DFs of $\max\limits_{1\le i\le n}||\underline{X}_i||_\infty$, $\max\limits_{1\le i\le n}||\underline{X}_i||_1$ and $\max\limits_{1\le i\le n}||\underline{X}_i||_2$ are all the max-Weibull type, but with different shape parameters.

TABLE 1
The MLEs of the parameters of $G_\gamma(x;\mu,\sigma)$ for SO_2 and NO.

	SO_2			NO		
	γ	μ	σ	γ	μ	σ
MLE's	0.281	3.36	1.655	0.978	6.98	8.119
95% CI	(0.21,0.31)	(3.21,3.51)	(1.54,1.78)	(0.97,1.08)	(6.59,7.38)	(7.59,8.68)

TABLE 2
The MLEs of the parameters of $G_\gamma(x;\mu,\sigma)$ for $||\cdot||_\ell, \ell = \infty, 1, 2$.

	$\|\|\cdot\|\|_\infty$			$\|\|\cdot\|\|_1$		
	γ	μ	σ	γ	μ	σ
MLE's	0.952	8.02	7.97	0.239	9.975	8.80
95% CI	(0.91,0.98)	(7.81,8.87)	(7.78,8.57)	(0.22,0.3)	(9.56,10.39)	(8.31,9.33)
	$\|\|\cdot\|\|_2$					
	γ	μ	σ			
MLE's	0.322	8.81	8.466			
95% CI	(0.31,0.40)	(8.41,9.21)	(7.96,9.00)			

In addition to the main goal of this Section which is to show how to model multivariate extreme data sets using the R-ordering principle. The finding of this Section endorses the result of Section 2, especially Proposition 7, as well as some recent limit results concerning the multivariate extreme value theorem.

4. Discussion

1. Bearing in mind the relation between the parameter β in (3) and γ in (8), the finding of this Section reveals that $F_1 \in \mathscr{D}(H_{2,3.559})$ ($\beta_1 = 3.559$), $F_2 \in \mathscr{D}(H_{2,1.023})$ ($\beta_2 = 1.023$), and $\mathscr{F} \in \mathscr{D}(H_{2,\beta})$, where $\beta = 1.05 \approx 1.023 = \beta_2$. This means that $\mathscr{F} \in \mathscr{D}(H_{2,\beta_1})$ and in view of Theorem 3.1 Part (2) in Barakat *et al.* (2020a) $x_1^0 > x_2^0$.

2. The finding of this Section reveals that $F_1 \in \mathscr{D}(H_{2,3.559})$, $F_2 \in \mathscr{D}(H_{2,1.023})$, and $\mathscr{F}_{||\cdot||_1} = F_1 \star F_2 \in \mathscr{D}(H_{2,4.184})$, where $4.184 \approx \beta_1 + \beta_1 = 4.582$. This result endorses Proposition 7.

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