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DEVELOPMENT AND ESTIMATION OF WEIGHTED XGAMMA EXPONENTIAL DISTRIBUTION WITH APPLICATIONS TO LIFETIME DATA

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SUMMARY

In this article, we introduce a weighted version of the Xgamma exponential distribution, extending its utility in modeling lifetime data. We derive several important distributional properties of the proposed model, including moments, residual life functions, generating functions, stochastic ordering, aging intensity, and entropy. These properties provide deeper insights into the behavior and structure of the proposed distribution. To estimate the model parameters, we discuss the maximum likelihood estimation approach, focusing on complete sample data. To demonstrate the practical applicability of the proposed distribution, we analyze two real-world lifetime data sets. The performance of the weighted Xgamma exponential distribution is compared with several well-established one- and two-parameter lifetime distributions, along with their weighted versions. Additionally, comparisons are made with length-biased and area-biased lifetime distributions to further assess the robustness of the proposed model. The results of these comparisons indicate that the proposed weighted distribution offers a superior fit, particularly for data sets exhibiting an increasing failure rate. The model's ability to outperform competing distributions highlights its potential as an effective alternative for analyzing lifetime data in reliability and survival studies.

Keywords: Xgamma exponential distribution; Weighted Xgamma; Characterizations; Point and interval estimation.

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1. Introduction

Introducing new probability distributions is a significant area of research in probability theory and statistical modeling, as it allows for creating models that can better describe and capture the complexities of real-world phenomena. Probability distributions are essential in characterizing the behavior of random variables, making them indispensable in fields like engineering, economics, biology, and social sciences. Developing new distributions often arises from the need to extend classical models or create new ones that address the limitations of existing distributions. Classical probability distributions such as the exponential, gamma, Weibull distributions, etc., have been widely used due to their simplicity and tractable mathematical properties. However, they may not always provide the best fit for real-world data. For example, in many applications, data may exhibit skewness, kurtosis, heavy tails, or multimodal behavior that traditional distributions fail to capture. In such cases, introducing new probability distributions becomes essential to provide more flexibility and accuracy in modeling. New lifetime distributions can be derived in several ways. One common method is through generalization or modification of existing distributions. For instance, a researcher may extend any baseline distribution to account for skewness or use transformations to handle heavy-tailed data. A common approach to introducing new lifetime distributions includes generalizing existing distributions, where classical models like the Weibull or exponential are extended by adding parameters or transformations to increase flexibility. Another approach is compounding or mixing distributions, where two or more distributions are combined to capture more complex data patterns. Parameterized extensions introduce additional shape, scale, or skewness parameters to enhance model flexibility. Generating families of distributions creates broad classes of models, such as exponentiated or Kumaraswamy families.

The motivation of our present study is to introduce a new-sprung weighted distribution and further explore various distributional characteristics. The need for the study of weighted distribution arises because when a sample is gathered by the experimenter following some standard probability model, the obtained sample may not represent the actual stochastic model considered due to accidental loss, destruction, or non-observance of some units until and unless each unit is selected with an equal chance of inclusion in the sample. Therefore, to quench the thirst for fitting our data in a better way, we resort to weighted distributions. Weighted distributions try to obtain the specified probabilities of the events/sample being observed and recorded by adjusting the probabilities of the actual occurrence of the events/sample. Mathematically, Consider a finite or infinite population of units carrying values of a non-negative random variable X having a distribution with probability density function (PDF) $g(x)$ where $x \geq 0$. Usually, we consider that the probability of selection of each unit is the same, regardless of the value of x it carries so that the PDF for the observation x is $g(x)$; but if the respective probability of observing x is $0 < w(x) < 1$, then the corresponding PDF of the recorded observation is

$$g_w(x) = \frac{g(x)w(x)}{\int_x g(x)w(x)dx}. \tag{1}$$

The above defined PDF emulates $g(x)$ only when $w(x)$ is constant.

The nascent idea of weighted distribution is attributed to Fisher (1934) who threw light on the influence of methods of ascertainment while approximating the frequencies. The idea was further strengthened by Rao (1965). He illustrated the practicality of the weighted distribution when $w(x) = x$ or $w(x) = x^a$. When the weight merely corresponds to the size of the respective sampling unit then the weighted scheme is said to be size-biased and is discussed by Patil and Ord (1976). Size-biased distribution is a celebrated version of weighted distribution and is further characterized as length-biased and area-biased distribution. Patil and Rao (1978) further discussed the concept of weighted distributions in the realm of human populations and wildlife management. Warren and Olsen (1964) coined the term 'line-intersect sampling' in the context of forestry literature. Grosenbaugh (1958) studied the concept and applicability of area sampling in the forestry regime. Lele and Keim (2006) used the theory of weighted distributions to estimate the resource selection probability function, which is defined by the probability of an individual or organism selecting a resource available to their surrounding environment. Larose and Dey (1998) modeled publication bias using weighted distributions under the Bayesian framework. Gupta and Keating (1986) established the relationships among various reliability measures of a lifetime distribution under length-bias sampling. Jain *et al.* (1989) generalized those results for a weighted lifetime distribution. Sunoj and Maya (2006) discussed some properties of weighted distributions in the context of repairable systems. It is to be noted that while using the weighted distribution for better fitting of recorded observations, appropriate choice of weight is immensely important.

The primary objective of this article is to introduce and thoroughly examine the various distributional properties of a newly proposed weighted distribution, along with its specific cases, namely the length-biased and area-biased distributions. In addition to studying these properties, a key aim is to develop and propose maximum likelihood (ML) estimation procedures for accurately estimating the unknown parameters and determining the survival/reliability characteristics of the proposed distribution. The research further aims to evaluate the performance and efficiency of these proposed estimators for different sample sizes and varying model parameters using Monte Carlo simulation techniques. Through these simulations, the robustness and accuracy of the estimation methods will be assessed, offering insight into how well the estimators perform under different conditions. Furthermore, the article intends to demonstrate the practical relevance and applicability of the proposed distribution by evaluating its fit with real-world data sets.

To attain our first objective, using Xgamma exponential distribution (XGED) as the baseline model, we study the size-biased (weighted) distribution with weight $w(x) = x^{\upsilon}$. For $\upsilon = 1$ and $\upsilon = 2$ we get length-biased and area-biased weighted distributions respectively. In survival/reliability analysis, many models are available to describe the lifetime data and to explore the inherent characteristics of the complete/ censored data. These lifetime models are classified based on their hazard rate. The one-parameter exponential distribution is a very celebrated survival/reliability model due to its constant hazard rate and lack of memory property. Since its application is restricted only to the constant hazard rate, therefore several generalizations based on the exponential model have been developed and characterized through non-constant hazard rates. XGED model is one of these which was introduced by Yadav *et al.* (2022) as an alternative to the one-parameter exponential distribution. The beauty of this distribution has been fully justified based on its monotone and bathtub-shaped hazard rate.

The cumulative distribution function (CDF) and corresponding PDF of XGED is given by

$$F_{\text{XGED}}(x;\zeta) = 1 - \frac{1}{2} e^{-\zeta x} \left[2 + \zeta x + \frac{1}{2} (\zeta x)^2 \right], \; x > 0, \zeta > 0 \tag{2}$$

and

$$f_{\text{XGED}}(x;\zeta) = \frac{1}{2} \zeta e^{-\zeta x} \left[1 + \frac{1}{2} (\zeta x)^2 \right], \; x > 0, \zeta > 0, \tag{3}$$

respectively. The hazard function for the above distribution is given as

$$h_{\text{XGED}}(x;\zeta) = \frac{\zeta \left[1 + \frac{1}{2} (\zeta x)^2 \right]}{2 + \zeta x + \frac{1}{2} (\zeta x)^2}. \tag{4}$$

This current study is motivated due to the variety of applications of the weighted distribution in lifetime data analysis. To the best of our knowledge, no attempt has been made in this direction, enhancing the uniqueness of the proposed study.

The article is structured as follows. Section 1 provides an overview of the relevant literature and outlines the motivation for the study. In Section 2, the newly proposed model is introduced, along with an in-depth examination of its various statistical properties. Section 3 details the point and interval estimation procedures for the model parameters and the survival/reliability characteristics, while Section 4 evaluates the performance of the classical estimators for distribution parameters and survival/reliability characteristics through a Monte Carlo simulation. In Section 5, the practical application of the proposed model is demonstrated using a real-world reliability and survival data set. Finally, Section 6 concludes the paper with a summary of key findings and remarks.

2. The proposed distribution and characterization

In this Section, we introduce the weighted Xgamma exponential distribution (WXGED) by considering the weight function as $w(x) = x^{\upsilon}$. The PDF and CDF with scale parameter $\zeta > 0$ and shape parameter $\upsilon > 0$ can be formally defined as

$$g_w(x;\upsilon,\zeta) = \frac{2\zeta^{\upsilon+1} x^{\upsilon} e^{-(\zeta x)} [1 + \frac{(\zeta x)^2}{2}]}{\upsilon!(\upsilon^2 + 3\upsilon + 4)}, \; x > 0 \tag{5}$$

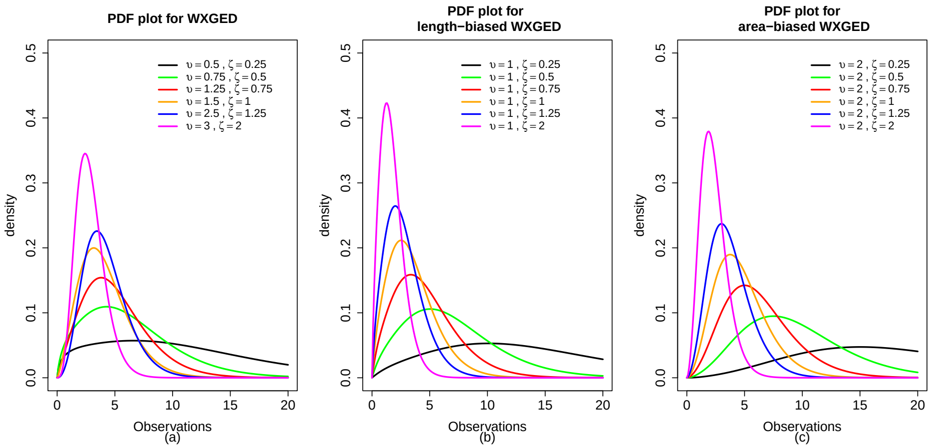

Figure 1 – PDF plot of WXGED for different υ and ζ values: (a) WXGED(υ,ζ) (b) WXGED$(\upsilon = 1,\zeta)$ and (c) WXGED$(\upsilon = 2,\zeta)$.

and

$$G_w(x;\upsilon,\zeta) = \frac{[2I_L(\upsilon+1,\zeta x)+I_L(\upsilon+3,\zeta x)]}{\upsilon!(\upsilon^2+3\upsilon+4)},\ x>0, \tag{6}$$

respectively, where I_L denotes the lower incomplete gamma integral[2]. The model proposed above produces length-biased and area-biased WXGED for $\upsilon = 1$ and $\upsilon = 2$ respectively. Putting $\upsilon = 1$ and $\upsilon = 2$ in Equations (5) and (6), we get PDF and CDF of length-biased and area-biased WXGED respectively. The shape of the PDF for different combinations of parameters (υ,ζ) is displayed in Figure 1. The PDF is also plotted in particular for $\upsilon = 1$ (length-biased), and $\upsilon = 2$ (area-biased). From those figures, it is observed that the PDF is positively skewed. The two most important survival characteristics are survival function and hazard rate, which, in the case of WXGED, are defined in the following equations:

$$S_w(t;\upsilon,\zeta) = \frac{[2I_U(\upsilon+1,\zeta t)+I_U(\upsilon+3,\zeta t)]}{\upsilon!(\upsilon^2+3\upsilon+4)},\ \ t>0 \tag{7}$$

and

$$h_w(t;\upsilon,\zeta) = \frac{2\zeta^{\upsilon+1}t^{\upsilon}e^{-\zeta t}\left[1+\frac{(\zeta t)^2}{2}\right]}{[2I_U(\upsilon+1,\zeta t)+I_U(\upsilon+3,\zeta t)]},\ \ t>0, \tag{8}$$

[2] The lower incomplete gamma integral $I_L(a,x)$ is defined as $\int_0^x t^{a-1}\exp(-t)dt$ and the upper incomplete gamma integral $I_U(a,x)$ is defined as $\int_x^\infty t^{a-1}\exp(-t)dt$, such that $I_L(a,x)+I_U(a,x) = \Gamma(a)$.

respectively, where I_U denotes upper incomplete gamma integral. Note that, from a theoretical point of view, the survival function is analogous to the reliability function and has similar interpretations.

Now, the mode of the distribution, which is nothing but the observation(s) with the highest frequency, can be obtained as a solution of $g_w'(x;\upsilon,\zeta)=0$, where

$$g_w'(x;\upsilon,\zeta)=g_w(x;\upsilon,\zeta)\left[\frac{d}{d(x)}\log g_w(x;\upsilon,\zeta)\right]=g_w(x;\upsilon,\zeta)\left[\frac{\upsilon}{x}-\zeta+\frac{\zeta^2 x}{1+\frac{(\zeta x)^2}{2}}\right]. \tag{9}$$

Since $g_w(x;\upsilon,\zeta)\neq 0$, therefore

$$\left[\frac{\upsilon}{x}-\zeta+\frac{\zeta^2 x}{1+\frac{(\zeta x)^2}{2}}\right]=0. \tag{10}$$

After simplification we get,

$$\zeta^3 x^3-\left[\zeta^2(\upsilon+2)\right]x^2+2\zeta x-2\upsilon=0. \tag{11}$$

The above equation can be easily solved by some numerical method which will finally provide us with the expression for mode.

2.1. *Shape of hazard rate*

The shape of the hazard rate of the proposed distribution is abstruse at first sight as it involves an incomplete gamma function in its expression. Therefore, we resort to the lemma given by Glaser (1980) which states the implication of monotonicity between $\eta(t)=-\frac{g'(t)}{g(t)}$ and $h(t)$. Using the result, the shape of the hazard rate can be deduced as follows.

PROPOSITION 1. *The hazard function of the WXGED(υ,ζ) is an increasing function of t for given $\zeta>0$ and $\upsilon>0$.*

PROOF. for $T\sim$ WXGED(υ,ζ), $\eta(t)$ is given by

$$\eta(t)=\zeta-\frac{\upsilon}{t}-\frac{2\zeta^2 t}{2+(\zeta t)^2}. \tag{12}$$

Consequently,

$$\eta'(t)=\frac{4(\upsilon-1)\zeta^2 t^2+\zeta^4 t^4(\upsilon+2)+4\upsilon}{t^2(2+\zeta^2 t^2)^2}. \tag{13}$$

Since ζ,υ,t all are greater than zero, therefore $\eta'(t)>0,\ \forall\ t>0$. □

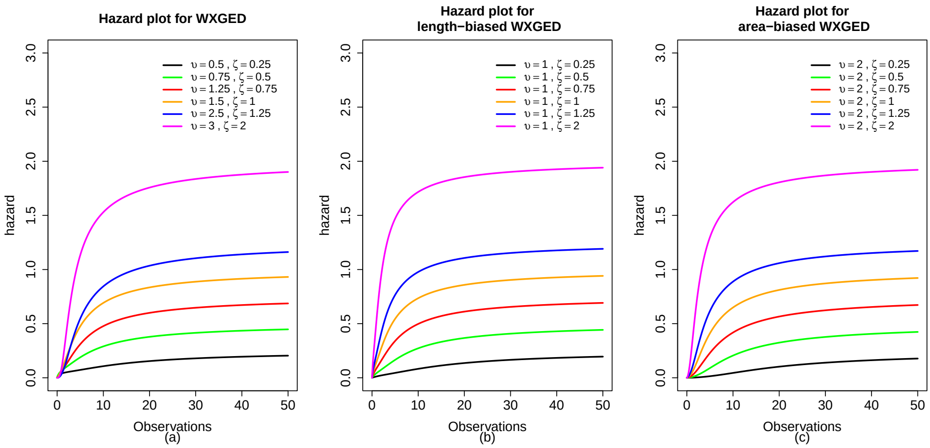

Figure 2 – Hazard plot of WXGED for different υ and ζ values: (a) WXGED(υ,ζ) (b) WXGED($\upsilon = 1,\zeta$) and (c) WXGED($\upsilon = 2,\zeta$).

Hence it is deduced that the hazard function of WXGED is an increasing function of t for given υ and ζ. The shape of the hazard rate for different combinations of parameters (υ,ζ) is displayed in Figure 2. The hazard rate function is also plotted in particular for $\upsilon = 1$ (length-biased), $\upsilon = 2$ (area-biased). From those figures, it is observed that the shape of the failure rate is indeed monotonically non-decreasing. Interestingly, the increasing hazard rate exhibited by the proposed distribution is highly spotted in the arena of clinical trials, inventory control, and many other fields in the industrial sector.

2.2. *Moments*

Moments constitute an important inferential aspect of any model. They form the basis to comment on measures like the central tendency of the distribution, scatteredness, asymmetricity, and peakedness. Next, we derive the general expression for the raw moments of WXGED in the following manner:

$$\mu'_r(x) = E(X^r) = \int_0^\infty x^r g_w(x;\upsilon,\zeta)dx. \tag{14}$$

After simplification we get,

$$\mu'_r(x) = \frac{2(\upsilon + r)!}{\zeta^r \upsilon!(\upsilon^2 + 3\upsilon + 4)}\left[1 + \frac{(\upsilon + r + 1)(\upsilon + r + 2)}{2}\right]; \quad r \geq 1. \tag{15}$$

Putting $\upsilon = 1$ and $\upsilon = 2$, we get the general expression for raw moments in the case of length-biased and area-biased distribution, respectively.

Particularly, mean and variance of WXGED(υ, ζ) are given by

$$\mu_1' = \frac{(\upsilon+1)(\upsilon^2+5\upsilon+8)}{\zeta(\upsilon^2+3\upsilon+4)} \tag{16}$$

and

$$\mu_2 = \frac{\upsilon+1}{\zeta^2(\upsilon^2+3\upsilon+4)}\left[(\upsilon+2)(\upsilon^2+7\upsilon+14) - \frac{(\upsilon+1)(\upsilon^2+5\upsilon+8)^2}{(\upsilon^2+3\upsilon+4)}\right], \tag{17}$$

respectively. By substituting $\upsilon = 1$ and $\upsilon = 2$ in the above equations, we get a similar expression for length-biased and area-biased distribution, respectively.

As mean and variance respectively give the measures of central tendency and dispersion of a distribution, measures of skewness and kurtosis provide the necessary means for understanding the shape of a particular distribution in terms of asymmetricity and peakedness, respectively. To obtain the coefficient of skewness and kurtosis, one needs to derive the second, third, and fourth central moments. These moments can be calculated using the expressions for the raw moments given above.

Evaluated expressions of coefficient of skewness and kurtosis for WXGED are

$$\gamma_1 = \left(\frac{\upsilon^2+3\upsilon+4}{\upsilon+1}\right)^{\frac{1}{2}} \frac{\left[(\upsilon+2)(\upsilon+3)(\upsilon^2+9\upsilon+22) - \frac{3(\upsilon+1)(\upsilon+2)(\upsilon^2+5\upsilon+8)(\upsilon^2+7\upsilon+14)}{(\upsilon^2+3\upsilon+4)} + \frac{2(\upsilon+1)^2(\upsilon^2+5\upsilon+8)^3}{(\upsilon^2+3\upsilon+4)^2}\right]}{\left[(\upsilon+2)(\upsilon^2+7\upsilon+14) - \frac{(\upsilon+1)(\upsilon^2+5\upsilon+8)^2}{(\upsilon^2+3\upsilon+4)}\right]^{\frac{3}{2}}} \tag{18}$$

and

$$\gamma_2 = \left(\frac{\upsilon^2+3\upsilon+4}{\upsilon+1}\right) \frac{[a-b+c-d]}{\left[(\upsilon+2)(\upsilon^2+7\upsilon+14) - \frac{(\upsilon+1)(\upsilon^2+5\upsilon+8)^2}{(\upsilon^2+3\upsilon+4)}\right]^2}, \tag{19}$$

respectively, where

$$a = (\upsilon+2)(\upsilon+3)(\upsilon+4)(\upsilon^2+11\upsilon+32),\ b = \frac{4(\upsilon+1)(\upsilon+2)(\upsilon+3)(\upsilon^2+9\upsilon+22)(\upsilon^2+5\upsilon+8)}{(\upsilon^2+3\upsilon+4)},$$

$$c = \frac{6(\upsilon+2)(\upsilon+1)^2(\upsilon^2+7\upsilon+14)(\upsilon^2+5\upsilon+8)^2}{(\upsilon^2+3\upsilon+4)^2},\ d = \frac{3(\upsilon+1)^3(\upsilon^2+5\upsilon+8)^4}{(\upsilon^2+3\upsilon+4)^3}.$$

It can be noted that the coefficient of skewness and kurtosis of WXGED are functions of υ. The coefficient of skewness and kurtosis plots for increasing values of υ are given in Figure 3. From those plots, it can be seen that the WXGED is positively skewed and leptokurtic. Although, for increasing values of υ, the coefficients of skewness and kurtosis decrease gradually.

The moment generating function (MGF) $M_x(t)$ is a powerful tool to generate moments but as it is demarcated by a limited range, we go for evaluating the expression for

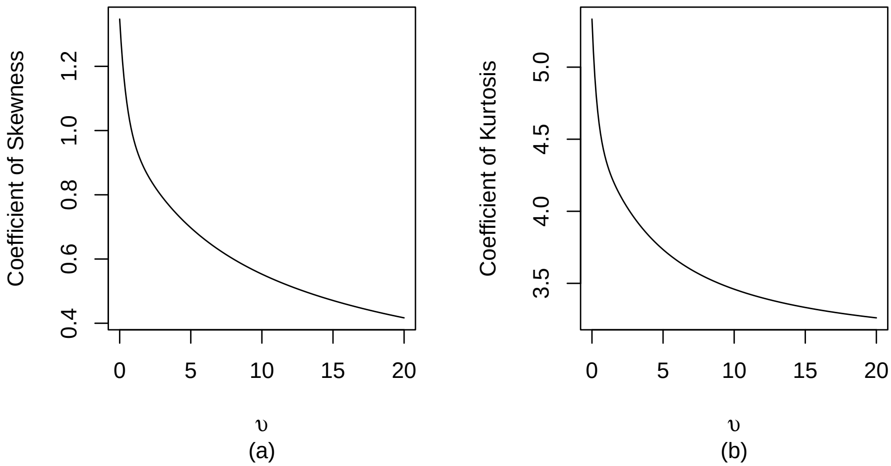

Figure 3 – (a) Coefficient of Skewness plot and (b) Coefficient of Kurtosis plot of WXGED for increasing υ.

the characteristic function $\Phi_x(t)$. The beauty of the latter is highlighted by the fact that it always exists while the former may or may not exist.

The expression of MGF after due simplification is given by

$$M_x(t) = \frac{1}{\upsilon^2 + 3\upsilon + 4}\left(\frac{\zeta}{\zeta - t}\right)^{\upsilon+1}\left[2 + (\upsilon+1)(\upsilon+2)\left(\frac{\zeta}{\zeta - t}\right)^2\right]. \quad (20)$$

Characteristic function $\Phi_x(t)$ can be obtained by putting it in $M_x(t)$ in place of t, where $i = \sqrt{-1}$. The respective expressions for length-biased and area-biased distribution can be obtained by putting $\upsilon = 1$ and $\upsilon = 2$. Further, if the life of the unit x under observation exceeds t'; one may seek the expressions for moments and moment-generating functions in that situation. Evaluated expressions are termed conditional moments and conditional MGFs, respectively. Mathematically the conditional moments and the conditional MGF are obtained as

$$E[X^r|X > t'] = \frac{\int_{t'}^{\infty} x^r g_w(x;\upsilon,\zeta)dx}{P(X > t')} \text{ and } E[e^{tX}|X > t'] = \frac{\int_{t'}^{\infty} e^{tx} g_w(x;\upsilon,\zeta)dx}{P(X > t')}, \quad (21)$$

respectively. After substituting the values from Equation (5) and Equation (7) in Equation (21), the expressions for conditional moments and MGF are given as

$$E[X^r|X>t'] = \frac{2I_U(\upsilon+r+1,\zeta t')+I_U(\upsilon+r+3,\zeta t')}{\zeta^r[2I_U(\upsilon+1,\zeta t')+I_U(\upsilon+3,\zeta t')]} \tag{22}$$

and

$$E[e^{tX}|X>t'] = \frac{\frac{2\zeta^{\upsilon+1}}{(\zeta-t)^{\upsilon+1}}I_U(\upsilon+1,(\zeta-t)t')+\frac{\zeta^{\upsilon+3}}{(\zeta-t)^{\upsilon+3}}I_U(\upsilon+3,(\zeta-t)t')}{2I_U(\upsilon+1,\zeta t')+I_U(\upsilon+3,\zeta t')}, \tag{23}$$

respectively. For the corresponding expressions in the case of length-biased and area-biased distribution put $\upsilon=1$ and $\upsilon=2$ in Equations (22) and (23), respectively.

REMARK 2. *With* $t'\to 0$; $E(X^r|X>t')\to E(X^r)$ *and* $E(e^{tX}|X>t')\to E(e^{tX})$, *respectively.*

2.3. *Mean residual lifetime*

The beauty of the mean residual lifetime (MRL) function lies in the fact that it summarizes the entire residual life distribution. It holds immense importance in the fields of demography, actuarial sciences, and different areas of the industrial sector. MRL function $e(x)$ of a non-negative random variable X with survivor function $S(x)$ is given as

$$\begin{aligned} e(x) &= E(X-t|X>t) \\ &= \frac{\int_t^\infty y g_w(y;\upsilon,\zeta)dy}{S_w(t;\upsilon,\zeta)}-t. \end{aligned} \tag{24}$$

After the required calculations we get

$$e(x) = \frac{2I_U(\upsilon+2,\zeta t)+I_U(\upsilon+4,\zeta t)}{2\zeta I_U(\upsilon+1,\zeta t)+\zeta I_U(\upsilon+3,\zeta t)}-t. \tag{25}$$

To obtain the due expressions for length-biased and area-biased distribution put $\upsilon=1$ and $\upsilon=2$ in the above equations, respectively.

2.4. *Residual lifetime distribution*

The residual lifetime $R(t)$ is the additional life of a unit with the assertion that it has survived up to time t, i.e. $R(t)=X-t|X>t$, $t>0$ whereas reverse residual lifetime $\bar{R}(t)$ is the time to observe the failure given that the unit has life less than or equal to t i.e; $\bar{R}(t)=t-X|X\le t$, $t>0$. The survival function for $R(t)$ is given by

$$S_{R(t)} = \frac{\bar{G}(x+t)}{\bar{G}(t)}, \tag{26}$$

where $\bar{G}(\cdot)$ is the survival function. Substituting values from Equation (7) in Equation (26) we get

$$S_{R(t)} = \frac{2I_U(\upsilon+1,\zeta(x+t)) + I_U(\upsilon+3,\zeta(t+x))}{2I_U(\upsilon+1,\zeta t) + I_U(\upsilon+3,\zeta t)}. \tag{27}$$

Now, the expressions for $\bar{R}(t)$, $t > 0$ are obtained in similar way. The corresponding PDF and the hazard function for $R(t)$ and $\bar{R}(t)$ can be derived by using the relation $f(t) = -S^{'}(t)$ and $h(t) = \frac{f(t)}{S(t)}$, respectively. By putting $\upsilon = 1$ and $\upsilon = 2$ in the above equation respective expressions for length-biased and area-biased distribution are obtained.

2.5. *Stochastic ordering*

Stochastic ordering has proved to be highly useful in many diverse areas of probability and statistics, especially in the financial sector. In a financial setting, stochastic orders help to decide on the maximum return subject to a given utility function. The detailed description of stochastic ordering can be found in Shaked and Shanthikumar (1994). The following theorem shows the stochastic ordering of two random variables X and Y in the context of likelihood ratio order.

THEOREM 3. *Let X and Y be two independent random variables that follow WXGED with shape parameters υ_1 and υ_2 and scale parameters ζ_1 and ζ_2 respectively. If $\upsilon_1 > \upsilon_2$ and $\zeta_2 > \zeta_1$, then X is stochastically greater than Y in likelihood ratio order, i.e. $(Y \leq_{lr} X)$ for all x.*

PROOF. For given $X \sim$ WXGED(υ_1, ζ_1) and $Y \sim$ WXGED(υ_2,ζ_2), we have

$$\begin{aligned} \psi &= \frac{g_w(x;\upsilon_1,\zeta_1)}{g_w(x;\upsilon_2,\zeta_2)}, \\ \psi &= \left(\frac{\zeta_1}{\zeta_2}\right)^{\upsilon_1-\upsilon_2} x^{\upsilon_1-\upsilon_2} e^{-x\left(\zeta_1+\frac{\zeta_1^3 x^2}{2}-\zeta_2-\frac{\zeta_2^3 x^2}{2}\right)} \frac{\upsilon_2!(\upsilon_2^2+3\upsilon_2+4)}{\upsilon_1!(\upsilon_1^2+3\upsilon_1+4)} \end{aligned} \tag{28}$$

and

$$\frac{d}{dx}\psi = \psi\left[\left(\frac{\upsilon_1-\upsilon_2}{x}\right) - (\zeta_1-\zeta_2) - \frac{3x^2}{2}\left(\zeta_1^3-\zeta_2^3\right)\right]. \tag{29}$$

The above equation increases in x for all $\upsilon_1 > \upsilon_2$ and $\zeta_2 > \zeta_1$. □

COROLLARY 4. *Let $X \sim$ WXGED(υ_1, ζ_1) and $Y \sim$ WXGED(υ_2,ζ_2). If $\upsilon_1 > \upsilon_2$ and $\zeta_2 > \zeta_1$ then X is stochastically greater than Y in likelihood ratio order and the result also holds in hazard rate order and mean residual life order.*

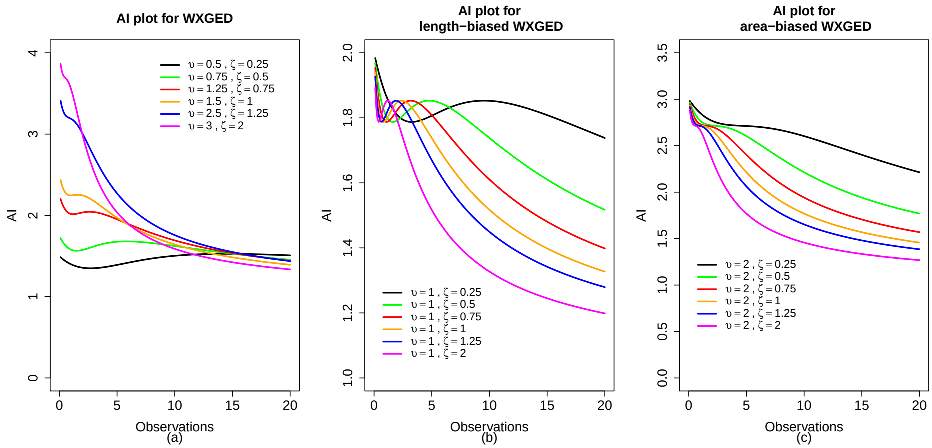

Figure 4 – AI plot of WXGED for different υ and ζ values: (a) WXGED(υ,ζ) (b) WXGED$(\upsilon = 1,\zeta)$and (c) WXGED$(\upsilon = 2,\zeta)$.

COROLLARY 5. *It can also be noted that, as shown in Jain* et al. *(1989), if we choose our weight function $w(x)$ as a monotonically increasing function of x, then the weighted version of the distribution dominates the original distribution in likelihood ratio order. Since our weight function x^{υ} is indeed a monotonically increasing function of x, random variables from WXGED are stochastically greater than random variables from XGED in likelihood ratio order.*

2.6. *Aging intensity*

Aging is very crucial and an inherent property of a system (may be living or non-living) and is obtained using a hazard rate. Generally, it is defined as a ratio of the failure rate to the average hazard rate. For a given failure rate, there exists a unique aging intensity (AI) function but not the other way around. Further, it can be noted that the monotonic behavior of the failure rate function does not ensure the same monotonic behavior of the AI function. For more details about AI, see Nanda *et al.* (2007). AI is expressed in the following formula:

$$L_x(t) = \frac{-t\,g(t)}{S(t)\log S(t)}; \quad t > 0. \tag{30}$$

In the case of the proposed model, the expression for AI is given by

$$L_x(t)=\frac{2t^{\upsilon+1}\zeta^{\upsilon+1}e^{-\zeta t}\left(1+\frac{(\zeta t)^2}{2}\right)}{\mathscr{S}(t;\upsilon,\zeta)}, \tag{31}$$

where

$$\mathscr{S}(t;\upsilon,\zeta)=[2I_U(\upsilon+1,\zeta t)+I_U(\upsilon+3,\zeta t)][\log\upsilon!+\log(\upsilon^2+3\upsilon+4) \\ -\log(2I_U(\upsilon+1,\zeta t)+I_U(\upsilon+3,\zeta t))].$$

The shape of the AI function for different combinations of parameters (υ,ζ) is displayed in Figure 4. The AI function is also plotted in particular for $\upsilon=1$ (length-biased), $\upsilon=2$ (area-biased). From those figures, it is observed that the shape of the AI function increases for some observations, but at the beginning and the end of the curve, it decreases.

2.7. *Measure of uncertainty*

Renyi entropy is immensely useful in the fields of statistical inference, econometrics, and pattern recognition in computer science for measuring the uncertainty associated with the phenomena. For WXGED, the expression for the Renyi entropy function is given by

$$\mathrm{RE}(\nu)=\frac{1}{1-\nu}\log\int_0^\infty g_w(x;\upsilon,\zeta)^\nu=\frac{1}{1-\nu}\log\int_0^\infty\left(\frac{2\zeta^{\upsilon+1}x^\upsilon e^{-(\zeta x)}[1+\frac{(\zeta x)^2}{2}]}{\upsilon!(\upsilon^2+3\upsilon+4)}\right)^\nu dx.$$

After simplification, the final expression of Renyi entropy is obtained as

$$\mathrm{RE}(\nu)=\frac{1}{1-\nu}\log\left[\left(\frac{2\zeta^{\upsilon+1}}{\upsilon!(\upsilon^2+3\upsilon+4)}\right)^\nu\sum_{i=0}^{\nu}\binom{\nu}{i}\left(\frac{\zeta^2}{2}\right)^i\frac{(\upsilon\nu+2i)!}{(\zeta\nu)^{\upsilon\nu+2i+1}}\right]. \tag{32}$$

2.8. *Bonferroni and Lorenz curve*

Bonferroni curve and Lorenz curve are two popular indices used in the financial arena, particularly in the context of income inequality. These two measures have some applications in reliability and life-testing experiments as well. The Bonferroni curve is given by the following expression:

$$B_G(p)=\frac{G_1(x)}{G(x)},$$

where

$$G_1(x)=\tfrac{1}{\mu}\int_0^x t\,g(t)dt,\ \mu=E(X).$$

Putting $p = G(x)$ and $q = G^{-1}(p)$ we get

$$B_G(p) = \frac{1}{p\mu}\int_0^q t\, g(t) dt, \;\; p \in (0,1].$$

The Lorenz curve is simply given by

$$L_G(p) = pB_G(p).$$

The evaluated expressions for these two curves in the case of WXGED are as follows:

$$\begin{aligned} B_G(p) &= \frac{2}{p\zeta\mu\upsilon!(\upsilon^2+3\upsilon+4)}\left[I_L(\upsilon+2,\zeta q)+\frac{1}{2}I_L(\upsilon+4,\zeta q)\right], \\ L_G(p) &= \frac{2}{\mu\zeta\upsilon!(\upsilon^2+3\upsilon+4)}\left[I_L(\upsilon+2,\zeta q)+\frac{1}{2}I_L(\upsilon+4,\zeta q)\right]. \end{aligned} \tag{33}$$

3. Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n coming from WXGED with PDF and CDF defined in Equations (5) and (6), respectively. The log-likelihood (LL) function for the proposed distribution is given by

$$\begin{aligned} \log L(\zeta,\upsilon|\underset{\sim}{x}) = n\log(2) - n\log(\upsilon!) - n\log(\upsilon^2+3\upsilon+4) + n(\upsilon+1)\log\zeta - \zeta\sum_{i=1}^{n} x_i \\ + \upsilon\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log(1+\frac{\zeta^2 x_i^2}{2}). \end{aligned} \tag{34}$$

Differentiating $\log L(\zeta,\upsilon|\underset{\sim}{x})$ with respect to υ, ζ and equating differentials to zero the following score equations are obtained

$$-n\left[\left(-\gamma+\sum_{k=1}^{\upsilon}\frac{1}{k}\right)\right] - \frac{n(2\upsilon+3)}{(\upsilon^2+3\upsilon+4)} + n\log(\zeta) + \sum_{i=i}^{n}\log x_i = 0 \tag{35}$$

and

$$\frac{n(\upsilon+1)}{\zeta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{\zeta x_i^2}{(1+\frac{\zeta^2 x_i^2}{2})} = 0, \tag{36}$$

where γ = Euler-Mascheroni constant.

Solving Equations (35) and (36) for υ and ζ, respectively, we get the ML estimators of the parameters. The solutions for the above score equations can not be obtained analytically because of their complicated expressions. Therefore, numerical procedures like the Newton-Raphson algorithm or fixed-point iterations are used to find the solutions to these equations.

Further, one may be interested in estimating the survival and hazard functions defined in Equations (7) and (8), as it conveys meaningful information about the survival/reliability properties of the lifetime distribution. The invariance property of the ML estimator allows us to directly obtain the ML estimators of the survival and hazard function by simply plugging in the ML estimators of the distribution parameters. Hence, if $\hat{\upsilon}$ and $\hat{\zeta}$ denote the ML estimators of υ and ζ, respectively, then the ML estimators of the survival and hazard function of WXGED is defined as $\hat{S}(t) = S_w(t;\hat{\upsilon},\hat{\zeta})$ and $\hat{h}(t) = h_w(t;\hat{\upsilon},\hat{\zeta})$, respectively.

The $100(1-\alpha)\%$ asymptotic confidence intervals (ACI) for the distribution parameters υ and ζ can be constructed using the asymptotic normality property of the ML estimators, which are well-discussed in the literature. The ACIs for the survival and hazard functions are also available through the delta method, which was beyond the scope of this study.

4. SIMULATION STUDY

In this Section, a Monte Carlo simulation study has been conducted to evaluate the performance of the ML estimators for the parameters and survival/reliability characteristics of the proposed distribution. The study is carried out based on different combinations of distributional parameters, specifically $(1,1.75),(2,2.5)$ and $(2.75,3)$. To provide a comprehensive comparison among the estimators, the first two parameter combinations where $\nu = 1,2$ correspond to the length-biased and area-biased distributions, respectively, and the last parameter combination indicates a size-biased (weighted) distribution, allowing us to explore the effect of different biasing schemes on the estimators' performance. Further, the ML estimators of the survival and the hazard function are evaluated at an arbitrary point $t = 4$. To assess the performance of the estimators under different conditions, five sample sizes are chosen: $n = 10,20,30,50$, and 100. The accuracy of the estimators is measured in terms of bias and mean squared error (MSE). Once the sequence of samples is generated, the average estimate (AE), bias, and MSE are calculated for each parameter combination, sample size, and biasing scheme. The AE, bias, and MSEs for the considered parameters and survival/reliability characteristics are obtained by calculating using the following formula based on $N = 10000$ replications:

$$\text{AE} = \frac{1}{N}\sum_{i=1}^{N}\hat{\Theta}_i\,,\ \text{Bias} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta}_i - \Theta) \text{ and } \text{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta}_i - \Theta)^2,$$

where $\Theta = (\upsilon,\zeta,S_w(t;\upsilon,\zeta),h_w(t;\upsilon,\zeta))$.

In addition to point estimation, ACIs for the parameters υ and ζ are constructed for the same variations in distributional parameters and sample sizes considered in the study. The interval estimators are evaluated in terms of coverage probability (CP) and average width (AW), where

$$\mathrm{CP} = \frac{1}{N}\sum_{i=1}^{N} I(L_i \leq \Theta' \leq U_i) \text{ and } \mathrm{AW} = \frac{1}{N}\sum_{i=1}^{N}(U_i - L_i),$$

where $\Theta' = (\upsilon, \zeta)$, $I(\cdot)$ is an indicator function, L_i is the lower bound and U_i is the upper bound of the ACIs, respectively.

A lower MSE indicates a better-performing estimator. The results of both the point estimation and the interval estimation for the different parameter variations and sample sizes are presented in Table 1. The simulation study revealed several important trends. First, as the sample size increases, both the MSE and the bias of each estimator decrease consistently across all parameter combinations. This indicates that larger sample sizes lead to more accurate and reliable estimates, as the estimators tend to converge to the true parameter values. In terms of interval estimation, the CP of the ML estimator approaches the nominal confidence level in all of the cases considered. This means that the proportion of intervals that contain the true parameter value becomes closer to the desired confidence level as the sample size grows. Furthermore, the AW of the ACIs decreases as the sample size increases. This reduction in interval width signifies greater precision in the interval estimates, as larger sample sizes provide more information about the parameter, resulting in tighter confidence intervals. Overall, the results show that as the sample size grows, the estimators become more accurate (with smaller bias and MSE), and the confidence intervals become both narrower and more reliable.

5. Real data applications

In this Section, two data sets have been taken to show the application of the proposed model. Data set 1 has been taken from Nichols and Padgett (2006) which represents the breaking stress of carbon fibers (in GBA). Data set 2 represents the data of lifetime relating to relief times (in minutes) of 20 patients receiving an analgesic. This data was originally reported by Gross and Clark (1975) and it was used by Shukla (2019) in a comparative study on one parameter lifetime distributions. The descriptive statistics of the two data sets along with the box plot and histogram are given in Table 2, Figure 5, and Figure 7, respectively. A model-fitting summary of both data sets has been shown in Tables 3 and 4, respectively.

For Data set 1, we have compared our proposed WXGED model with different distributions namely weighted exponential distribution (WED), generalized Lindley distribution (GLD), generalized Exponential distribution (GED), Rayleigh distribution (RD), XGED, Lindley distribution (LD), exponential distribution (ED), and Xgamma distribution (XGD). total time on test (TTT) plot for the Data set 1 and empirical CDF (eCDF) vs Theoretical CDF plots are also given in Figure 6.

Since our proposed model has increasing failure rate properties, only the distributions with increasing hazard rate properties are considered for comparison, except for ED, which has a constant failure rate. The motivation for taking ED as an alternative

TABLE 1
AE, Bias and MSE of different pairs of υ, ζ, corresponding $S(t)$ and $h(t)$ and 95% CP and AW for ACIs for different combinations of n.

							$S(t)\|_{t=4}$			$h(t)\|_{t=4}$		
n	(υ, ζ)	AE	Bias	MSE	CP	AW	AE	Bias	MSE	AE	Bias	MSE
10	(1, 1.75)	1.87107 2.27944	0.87107 0.52944	4.39602 1.53162	0.92270 0.95530	4.17087 3.10467	0.06236	-0.00079	0.00284	1.44647	0.31877	0.60573
	(2, 2.5)	3.58645 3.42748	1.58645 0.92748	14.37851 4.79753	0.87140 0.90490	6.64027 4.50047	0.02752	0.00205	0.00097	2.16374	0.54037	1.72842
	(2.75, 3)	4.85724 4.21863	2.10724 1.21863	24.45878 7.89245	0.84850 0.84450	8.06914 5.27907	0.01697	0.00226	0.00048	2.63955	0.70197	2.71868
20	(1, 1.75)	1.31947 1.95477	0.31947 0.20477	0.78403 0.32489	0.97330 0.95950	2.62904 1.87546	0.06312	-0.00003	0.00155	1.25534	0.12763	0.14592
	(2, 2.5)	2.63090 2.87259	0.63090 0.37259	2.62429 0.92408	0.94170 0.94050	4.60986 2.91038	0.02626	0.00079	0.00046	1.84269	0.21931	0.35499
	(2.75, 3)	3.59500 3.49303	0.84500 0.49303	4.50198 1.56442	0.90170 0.92670	5.79289 3.55085	0.01582	0.00111	0.00022	2.22421	0.28663	0.57853
30	(1, 1.75)	1.19989 1.87683	0.19989 0.12683	0.40216 0.17318	0.97330 0.94730	2.07053 1.44828	0.06331	0.00016	0.00104	1.20643	0.07873	0.07980
	(2, 2.5)	2.38689 2.72878	0.38689 0.22878	1.25257 0.46220	0.95120 0.93880	3.61597 2.26773	0.02628	0.00081	0.00032	1.75840	0.13503	0.18619
	(2.75, 3)	3.25413 3.29491	0.50413 0.29491	2.13414 0.75746	0.93090 0.93750	4.60782 2.81262	0.01572	0.00101	0.00015	2.10958	0.17200	0.28903
50	(1, 1.75)	1.11465 1.82481	0.11465 0.07481	0.17786 0.08342	0.95620 0.96060	1.55516 1.07959	0.06302	-0.00013	0.00063	1.17484	0.04713	0.04051
	(2, 2.5)	2.20892 2.62590	0.20892 0.12590	0.53682 0.20986	0.95720 0.96520	2.64870 1.67577	0.02597	0.00050	0.00019	1.69865	0.07528	0.08869
	(2.75, 3)	3.04545 3.17595	0.29545 0.17595	0.95233 0.35139	0.94500 0.94290	3.47825 2.12751	0.01512	0.00041	0.00009	2.04146	0.10388	0.13885
100	(1, 1.75)	1.05786 1.78736	0.05786 0.03736	0.07614 0.03713	0.94250 0.96820	1.07507 0.74201	0.06308	-0.00007	0.00032	1.15112	0.02341	0.01832
	(2, 2.5)	2.08837 2.55273	0.08837 0.05273	0.21655 0.08664	0.96210 0.96840	1.76962 1.12975	0.02591	0.00044	0.00009	1.65477	0.03139	0.03758
	(2.75, 3)	2.88369 3.07958	0.13369 0.07958	0.39611 0.14831	0.95400 0.94680	2.36377 1.45343	0.01500	0.00028	0.00004	1.98458	0.04700	0.05960

TABLE 2
Descriptive statistics for the considered data sets.

Data	Size	Mean	Median	Mode	Skewness	Kurtosis	variance	IQR
Data set 1	100	2.6215	2.7000	2.7500	0.3681	0.1055	1.0278	1.3800
Data set 2	20	1.9000	1.7000	1.7500	1.7198	2.9241	0.4958	0.5750

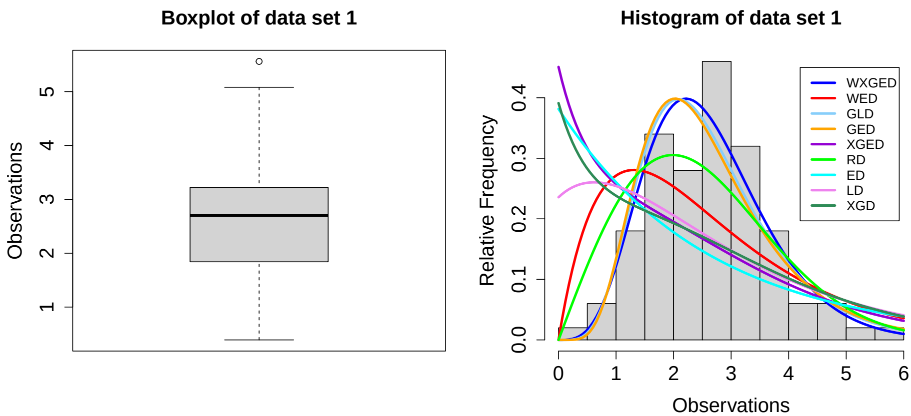

Figure 5 – Boxplot and Histogram with theoretical density for Data set 1.

model is that the generative distribution of our proposed model is Exponential. The comparison among the distributions is done using the Akaike information criterion (AIC), Bayesian information criterion (BIC), and Kolmogorov–Smirnov (K-S) distance values. The distributions are assigned a rank based on the smallest to largest K-S values. From Table 3, we can see that our proposed model provides the best fit among all the other distributions. Similarly, for Data set 2, we have compared the WXGED model

TABLE 3
Model fitting summary of Data set 1.

Model	Estimates	-LL	AIC	BIC	K-S	Rank
WXGED	$\upsilon = 3.43523, \zeta = 2.39482$	142.65490	289.30990	294.52020	0.08965	1st
WED	$\alpha_1 = 0.00774, \lambda = 0.75912$	166.37800	336.75590	341.96630	0.20661	5th
GLD	$\alpha_2 = 1.24695, \theta_1 = 5.87928$	144.92840	293.85680	299.06710	0.10118	2nd
GED	$\alpha_3 = 7.78938, \beta = 1.01317$	146.18060	296.36120	301.57160	0.10768	3rd
RD	$\theta_2 = 1.98617$	149.50040	301.00090	303.60600	0.13832	4th
XGED	$\theta_3 = 0.90186$	186.89680	375.79370	378.39880	0.33383	9th
LD	$\theta_4 = 0.61732$	181.75620	365.51240	368.11750	0.26326	6th
ED	$\theta_5 = 0.38146$	196.37470	394.74930	397.35452	0.32058	8th
XGD	$\theta_6 = 0.85092$	184.65540	373.31080	378.52110	0.29058	7th

with WED, weighted Maxwell distribution (WMD), RD, XGED, LD, ED, and XGD. The comparisons are made based on AIC, BIC, and K-S values. From Table 4, we can see that our proposed model provides the best fit among all the other distributions. TTT plot for the Data set 2 and eCDF vs theoretical CDF plots are given in Figure 8. Table 5 contains the ML estimators of the parameters, survival, and the hazard function

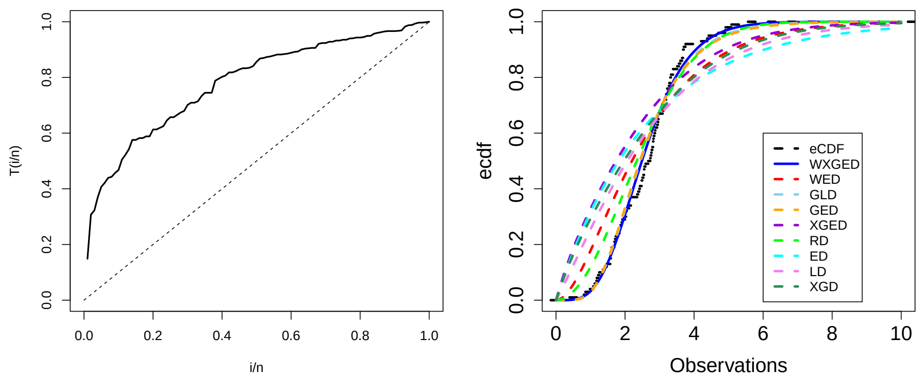

Figure 6 – TTT plot and eCDF vs Theoretical CDF plot for Data set 1.

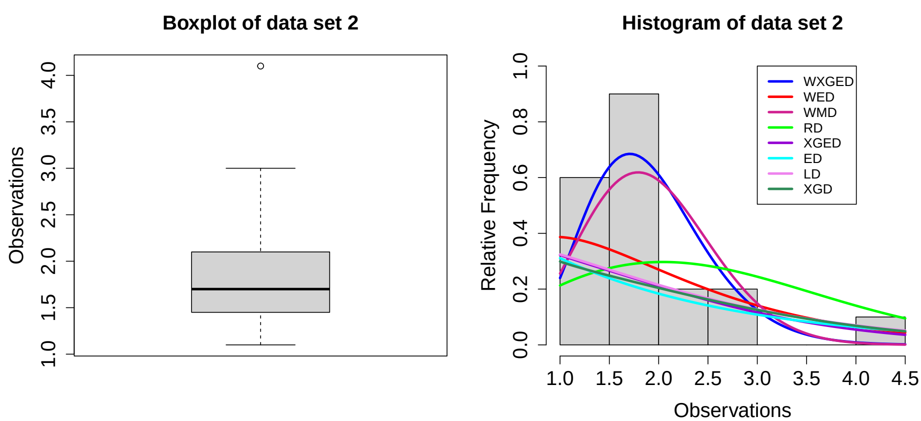

Figure 7 – Boxplot and Histogram with theoretical density for Data set 2.

TABLE 4
Model fitting summary of Data set 2.

Model	Estimates	-LL	AIC	BIC	K-S	Rank
WXGED	$\upsilon = 6.91655, \zeta = 5.19197$	17.85488	39.70976	41.70122	0.17416	1st
WED	$\alpha_1 = 0.01439, \lambda_1 = 1.04551$	26.16318	56.32636	58.31782	0.32212	4th
WMD	$\alpha_2 = 1.15002, \lambda_2 = 1.69344$	19.17005	42.34010	44.33156	0.19672	2nd
RD	$\lambda_4 = 1.42846$	22.47881	46.95762	47.95335	0.25658	3rd
XGED	$\lambda_5 = 1.27147$	30.90790	63.81580	64.81153	0.45961	8th
LD	$\lambda_6 = 0.81612$	30.24955	62.49910	63.49483	0.39108	5th
ED	$\lambda_7 = 0.52632$	32.83708	67.67416	68.66989	0.43951	6th
XGD	$\lambda_8 = 1.10748$	31.50824	67.01649	69.00795	0.42915	7th

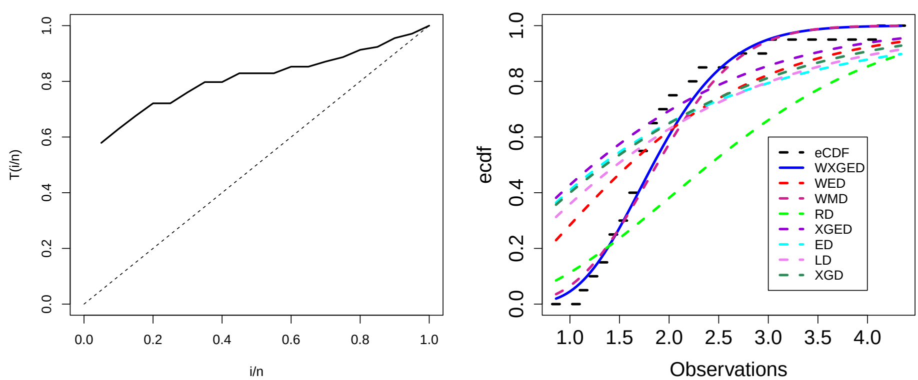

Figure 8 – TTT plot and eCDF vs Theoretical CDF plot for Data set 2.

(evaluated at the median of the corresponding data sets) and respective 95% ACI of distribution parameters υ and ζ. Taking one data set with a larger sample size and the other one with a smaller sample size is motivated by the desire to see how the proposed estimators for WXGED perform on both smaller and larger data sets. The goodness-of-fit of the distributions is solely assessed through the alignments of the theoretical CDF and the eCDF plots here, as the goodness-of-fit tests might perform poorly on small data sets.

TABLE 5
ML estimates of parameters, survival/reliability characteristics, and 95% ACIs of parameters.

	MLE		95% ACI					
			υ		ζ			
	$\hat{\upsilon}$	$\hat{\zeta}$	Lower	Upper	Lower	Upper	$\hat{S}(t)\|_{t=\text{median}}$	$\hat{h}(t)\|_{t=\text{median}}$
Data set 1	3.43523	2.39482	1.65924	5.21123	1.67101	3.11863	0.42100	0.85031
Data set 2	6.91655	5.19197	3.14734	10.68577	3.05386	7.33009	0.59241	1.15648

6. Concluding remarks

In conclusion, this study presents a newly developed weighted version of the XGED, motivated by the broad range of applications that weighted distributions offer across various fields. The newly introduced weighted distribution has been thoroughly investigated in terms of its statistical properties, with a specific focus on its length-biased and area-biased forms. One of the key findings is that the proposed distribution exhibits an increasing failure rate property, which makes it particularly well-suited for modeling survival and reliability data in practical applications. The unknown parameters of the distribution, along with its survival/reliability characteristics, have been estimated using the ML estimation method. Additionally, ACIs based on the ML estimation method have been proposed for different model parameter variations, and their corresponding CP and AW have been reported. A Monte Carlo simulation study has been conducted to evaluate the performance of the estimators, comparing them in terms of their MSE. To validate the practical applicability of the proposed weighted distribution, two real-world data sets have been analyzed. The findings demonstrate that the proposed distribution provides a superior fit compared to several existing weighted distributions and other commonly used distributions with increasing failure rates. This highlights the potential of the newly introduced distribution in real-life data modeling, particularly in fields where reliability and survival data are of interest. All computational work for this study has been performed using R software (version 4.2.1). Overall, the study not only introduces a versatile new weighted distribution but also establishes its practical utility and improved performance through a comprehensive analysis of theoretical properties, simulation studies, and real-world applications.

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