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TRANSMUTED INVERSE XGAMMA DISTRIBUTION: STATISTICAL PROPERTIES, CLASSICAL ESTIMATION METHODS AND DATA MODELING

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SUMMARY

In this article, a new variant of Inverse Xgamma distribution is introduced by using the quadratic rank transmutation map (QRTM), named as Transmuted Inverse Xgamma (TIXG) distribution. The proposed model is positively skewed and has flexibility in the hazard rate function. A comprehensive account of mathematical and statistical properties of the newly obtained lifetime model are provided. Explicit expressions for moments, moment generating function, quantile functions, stochastic orderings, ageing intensity function and order statistics are formulated. We briefly discuss different classical estimations, including the maximum likelihood, maximum product spacings, least square, weighted least square and Cramèr-Von-Mises estimation methods. Monte Carlo simulation is carried out to compare the performance of the different estimation methods. Finally, to demonstrate the applicability of the model in real life, an illustrative example is performed by analyzing an environmental science dataset.

1. INTRODUCTION

The accuracy of any statistical research about real world events are determined by whether the assumed models are adequate for fitting that phenomena or not. In many applied fields, including engineering, health and economics etc. modeling and analyzing lifetime data are the prerequisite parts of research. Although, some well-known distribution families are frequently employed in modeling wide range of events, their modeling capabilities may not always meet expectations. Owing to this, considerable effort has been

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expended towards generating new families of distribution along with relevant statistical methodologies. Few examples of them are exponentiated Pareto distribution by Nassar *et al.* (2018), alpha power inverse Weibull distribution by Basheer (2019), three parameter Fréchet model by Al-Babtain *et al.* (2020), Marshall-Olkin Gompertz distribution by Eghwerido *et al.* (2021), new modified Lindley distribution by Chesneau *et al.* (2021), exponentiated XGamma distribution by Yadav *et al.* (2021), exponential transformed inverse Rayleigh distribution by Banerjee and Bhunia (2022), inverse A(α) distribution by Bhunia and Banerjee (2022), etc.

Several techniques for generating new distributions have been proposed in the literature. The quadratic rank transmutation map (QRTM) technique is one of them and that has been first proposed by Shaw and Buckley (2009). The construction of transmuted distribution is rather simple and defined as, a random variable X is said to follow a transmuted distribution if its cumulative distribution function satisfies the following relationships

$$F(x) = G(x)[(1+\lambda) - \lambda G(x)], \quad |\lambda| \leq 1, \tag{1}$$

which is on differentiation yields the corresponding probability density function as

$$f(x) = g(x)[1+\lambda - 2\lambda G(x)]. \tag{2}$$

Here, λ is an additional parameter known as transmutation parameter, $g(x)$ and $G(x)$ are the pdf and cdf of the baseline distribution respectively. Involvement of an extra parameter in the transmuted distribution generally brings more flexibility, in addition to possess the characteristics of baseline distribution. Recently, in two decades a lot of research works have been found on the basis of QRTM. For examples, Transmuted Extreme value distribution (Aryal and Tsokos, 2009), Transmuted Rayleigh distribution (Merovci, 2013), Transmuted Weibull distribution (Khan *et al.*, 2017), Transmuted Generalised Exponential distribution (Khan *et al.*, 2017), Transmuted Xgamma distribution (Biçer, 2019), etc.

In this era of generalizations, Sen *et al.* (2016) introduced Xgamma distribution with one shape parameter by using a special finite mixture of two well known lifetime distributions i.e. exponential and Gamma distribution. Yadav *et al.* (2019) used inverse transformation method of baseline variable to obtain the inverted form of Xgamma distribution. As there is only one parameter involved in the inverse Xgamma distribution, it did not give adequate flexibility for analyzing wide variety of lifetime data. Consequently, to expand the modeling abilities of the inverse Xgamma distribution, it is necessary to derive an alternative generalized form of this distribution. Therefore, the Transmuted Inverse Xgamma (TIXG) distribution, as its name suggests, is a transmuted variation of the Inverse Xgamma distribution, which is introduced in this article using the QRTM approach.

The uniqueness of this study stems from the fact that, we provide an extensive explanation of mathematical and statistical features of TIXG distribution with the hopes of attracting further applications in biology, medical science, reliability, engineering and many others applied field of study. Also, from frequentist perspective, we study five different estimation methods and compare their performance for varying sample sizes with different combinations of parameter values through the Monte Carlo simulation technique. Furthermore, to the best of our knowledge, no attempt has been made to compare all of these TIXG estimators in terms of mathematical and statistical characteristics. Another appealing feature of this distribution is its flexibility in hazard rate function and practicability in modeling positively skewed data.

The remaining part of this study is organized as in the following sequence. In Section 2, TIXG distribution has been introduced and its survival properties are comprehensively discussed. Explicit expression of various mathematical and statistical properties are presented in Section 3. Different estimation methods of the unknown parameters of TIXG distribution have been considered in Section 4. Next in Section 5, numerical results of the simulation study are presented for verifying the performance of the estimation methods. In Section 6, an environmental data is considered to demonstrate the superiority of the TIXG lifetime distribution over some well known distributions. Finally, the conclusion of the study is addressed in Section 7.

2. Transmuted inverse xgamma distribution

In this Section, we introduce the Transmuted Inverse Xgamma distribution by considering the Inverse Xgamma as baseline distribution and utilizing it in the QRTM formula defined in Equations (1) and (2), respectively. Before going on to the subsequent derivations, firstly, we review the pdf and cdf of the Inverse Xgamma distribution and they are expressed as

$$g(x;\theta)=\frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}, \quad x>0 \tag{3}$$

and

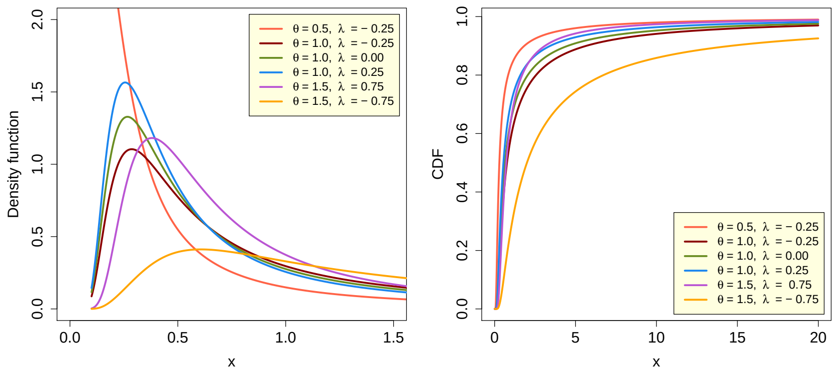
$$G(x;\theta)=\left[1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right]e^{-\frac{\theta}{x}}, \quad x>0, \tag{4}$$

respectively. $\theta>0$ is the shape parameter of the IXG distribution. Therefore, considering the pdf in Eq. (3) and cdf in Eq. (4) along with the above QRTM formula, the TIXG is defined as follows. A random variable X is said to follow TIXG distribution with parameters $\theta>0$ and $-1\leq\lambda\leq 1$ if it has the following cdf

$$F(x;\theta,\lambda)=(1+\lambda)\left[\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right]-\lambda\left[\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right]^2 \tag{5}$$

and the corresponding pdf of TIXG distribution is expressed as

$$f(x;\theta,\lambda)=\frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\right]. \tag{6}$$



Probability density function Cumulative distribution function

Figure 1 – The pdf and cdf plots of TIXG distribution for different parameter choices.

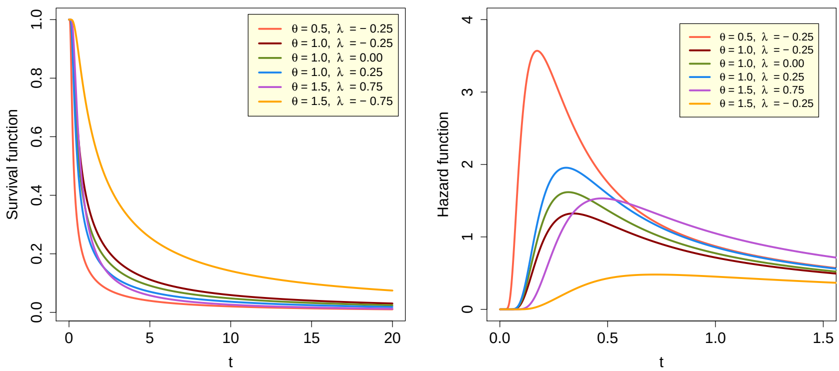
The form of the newly obtained distribution is controlled by the parameter $\theta > 0$, much like in the IXG distribution. Also the transmuted parameter λ, determines the behaviour of the distribution and gives more flexibility compared to the baseline distribution. The structure of the proposed distribution indicates that it is a more advanced model for analyzing complicated datasets. When $\lambda = 0$, the TIXG distribution reduces to the IXG distribution. In the rest of the paper, we will use the notation $X \sim \text{TIXG}(\theta, \lambda)$ to indicate a random variable X with positive values belongs to the TIXG distribution with parameters θ and λ. Figure 1 illustrates some possible shapes of the pdf and cdf of this distribution for the chosen values of the parameter. Clearly, the proposed model is positively skewed and unimodal.

2.1. *Survival properties*

The basic elements of failure time for a given system or any biological phenomenon include the survival function, hazard rate function and reversed hazard rate function, which are discussed below.

- The survival function $S(t)$, is defined as the probability that an individual or any item is survived at least t, (with $t \geq 0$) unit of time and denoted as $S(t) = P(X \geq t) = 1 - F(t)$. Thus, the survival function of the TIXG distribution is expressed as

$$S(t) = \lambda\left\{\left(1 + \frac{\theta x + \frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}e^2 - (1+\lambda)\left\{\left(1 + \frac{\theta x + \frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\} + 1.$$



Survival function Hazard rate function

Figure 2 – The survival function and hazard rate function plots of TIXG distribution for different parameter choices.

- Another important characteristic of interest for quantifying the real life phenomena of a lifetime distribution is the hazard rate function or failure rate function. It can be interpreted as the conditional probability of failure, given it has survived up to at least the time t ($t \geq 0$) and is defined as $h(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{S(t)}$, where $f(t)$ and $S(t)$ are the probability density and survival functions of the corresponding distribution. Therefore, the hazard function for the TIXG distribution is given as follows

$$h(t) = \frac{\frac{\theta^2}{(1+\theta)t^2}\left(1+\frac{\theta}{2t^2}\right)e^{-\frac{\theta}{t}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}\right]}{\lambda\left\{\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}^2-(1+\lambda)\left\{\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}+1}.$$

- Reversed (or proportional) hazard rate function of a random life phenomenon is defined as the ratio between the lifetime probability density and its distribution function. For the TIXG distribution it is expressed as

$$H(t) = \frac{f(t)}{F(t)} = \frac{\frac{\theta^2}{(1+\theta)t^2}\left(1+\frac{\theta}{2t^2}\right)e^{-\frac{\theta}{t}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}\right]}{(1+\lambda)\left[\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right]-\lambda\left[\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right]^2}.$$

Figure 2 reveals the possible shapes of the survival function and hazard rate function with different combinations of parameters θ and λ. The survival graph tends to decline

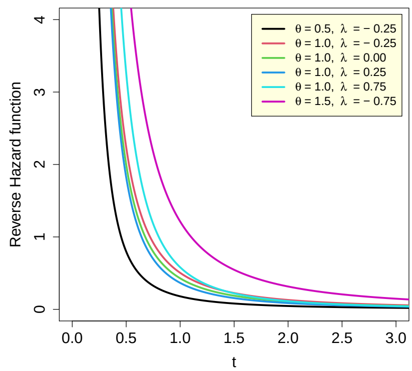


Figure 3 – The reverse hazard rate function plot of TIXG distribution for different parameter choices.

as time increases, therefore this distribution can be utilized in lifetime studies. Also, we see that the shape of the hazard rate function initially increases and then starts to decline and over the time it converges with some constant value. The reversed hazard rate function in Figure 3 also decreases with a reversed J-shape pattern. In real phenomenon, this type of model is very useful. For example, the infant mortality rate is very high at initial stages but gradually declines over the time when the immunity grows in their body and with the improvement of medical facilities.

3. Statistical properties

In this Section, we discuss some important mathematical and statistical properties of the TIXG distribution such as moments, moment generating functions, mode, quantile functions, stochastic orderings, ageing intensity function and order statistics, etc.

3.1. *Moments and inverse moments*

The r^{th} order raw moment of a random variable X having the TIXG(θ, λ) distribution is obtained as follows

$$\begin{aligned}
\mu'_r =& E(X^r) \\
=& \int_0^\infty x^r \frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\right]\mathrm{d}x \\
=& \Gamma(1-r)\frac{(1+\lambda)\theta^{(r+1)}}{(1+\theta)}+\Gamma(3-r)\frac{(1+\lambda)\theta^r}{2(1+\theta)}-\Gamma(1-r)\frac{2^r\lambda\theta^{(r+1)}}{(1+\theta)}-\Gamma(3-r)
\end{aligned}$$

$$\frac{2^{(r-3)}\lambda\theta^{r}}{(1+\theta)}-\Gamma(2-r)\frac{2^{(r-1)}\lambda\theta^{r+1}}{(1+\theta)^{2}}-\Gamma(4-r)\frac{2^{(r-4)}\lambda\theta^{r}}{(1+\theta)^{2}}-\frac{2^{(r-3)}\lambda\theta^{(r+1)}}{(1+\theta)^{2}}$$
$$\Gamma(3-r)-\Gamma(5-r)\frac{2^{(r-6)}\lambda\theta^{r}}{(1+\theta)^{2}}. \tag{7}$$

The moment of the proposed distribution is expressed in Eq. (7) and we see that there does not exist any classical moment for the TIXG distribution. Hence, our interest has shifted to derive the inverse moments and the related measures from that expression. The r^{th} order inverse moment about the origin of this distribution is given as follows

$$\begin{aligned}\mu'_{r-1}=&E(X^{-r})\\=&\int_0^\infty \frac{1}{x^r}\frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\right]\mathrm{d}x\\=&\Gamma(r+1)\frac{(1+\lambda)\theta^{1-r}}{(1+\theta)}+\Gamma(r+3)\frac{(1+\lambda)\theta^{-r}}{2(1+\theta)}-\Gamma(r+1)\frac{2^{-r}\lambda\theta^{(1-r)}}{(1+\theta)}-\Gamma(r+3)\\&\frac{\lambda\theta^{-r}}{2^{(r+3)}(1+\theta)}-\Gamma(r+2)\frac{\lambda\theta^{1-r}}{2^{(r+1)}(1+\theta)^2}-\Gamma(r+4)\frac{\lambda\theta^{-r}}{2^{(r+4)}(1+\theta)^2}-\Gamma(r+3)\\&\frac{\lambda\theta^{1-r}}{2^{(r+3)}(1+\theta)^2}-\Gamma(r+5)\frac{\lambda\theta^{-r}}{2^{(r+6)}(1+\theta)^2}.\end{aligned} \tag{8}$$

The harmonic mean for the random variable X is to be obtained by putting $r=1$ in the above Eq. (8)

$$E\left(\frac{1}{X}\right)=\int_0^\infty \frac{1}{x}f(x)dx=\frac{1}{1+\theta}\left[1+\frac{\lambda}{2}+\frac{3}{\theta}+\frac{21\lambda}{8\theta}-\frac{7\lambda}{8(1+\theta)}-\frac{27\lambda}{16\theta(1+\theta)}\right].$$

3.2. *Moment generating function and characteristic function*

The moment generating function (mgf) of a random variable X having the TIXG(θ,λ) distribution is derived by taking expectation of e^{tX} as follows

$$\begin{aligned}M_X(t)=&E(e^{tX})\\=&\frac{\theta^2(1+\lambda)}{(1+\theta)}\int_0^\infty\frac{e^{tx}}{x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\mathrm{d}x-\frac{2\lambda\theta^2}{(1+\theta)}\int_0^\infty\frac{e^{tx}}{x^2}\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{2\theta}{x}}\left(1+\frac{\theta}{2x^2}\right)\mathrm{d}x\\=&\sum_{m=0}^\infty\frac{t^m}{m!}\left[\frac{(1+\lambda)\theta^{(m+1)}}{(1+\theta)}\Gamma(1-m)+\frac{(1+\lambda)\theta^m}{2(1+\theta)}\Gamma(3-m)-\frac{2^m\lambda\theta^{(m+1)}}{(1+\theta)}\Gamma(1-m)\right.\\&-\frac{2^{(m-3)}\lambda\theta^m}{(1+\theta)}\Gamma(3-m)-\frac{2^{(m-1)}\lambda\theta^{(m+1)}}{(1+\theta)^2}\Gamma(2-m)-\frac{2^{(m-4)}\lambda\theta^m}{(1+\theta)^2}\Gamma(4-m)\\&\left.-\frac{2^{(m-3)}\lambda\theta^{(m+1)}}{(1+\theta)^2}\Gamma(3-m)-\frac{2^{(m-6)}\lambda\theta^m}{(1+\theta)^2}\Gamma(5-m)\right].\end{aligned} \tag{9}$$

From the Eq. (9), it is clearly seen that the mgf of the proposed distribution does not exist.

As the characteristic function (cf) of any real-valued random variable always exist, we obtain the cf of the TIXG distribution as follows

$$\begin{aligned}\phi_X(t) &= E(e^{itX})\\ &= \sum_{m=0}^{\infty}\frac{(it)^m}{m!}\Bigg[\Gamma(1-m)\frac{(1+\lambda)\theta^{(m+1)}}{(1+\theta)}+\Gamma(3-m)\frac{(1+\lambda)\theta^m}{2(1+\theta)}-\Gamma(1-m)\frac{2^m\lambda\theta^{(m+1)}}{(1+\theta)}\\ &-\Gamma(3-m)\frac{2^{(m-3)}\lambda\theta^m}{(1+\theta)}-\Gamma(2-m)\frac{2^{(m-1)}\lambda\theta^{(m+1)}}{(1+\theta)^2}-\Gamma(4-m)\frac{2^{(m-4)}\lambda\theta^m}{(1+\theta)^2}\\ &-\Gamma(3-m)\frac{2^{(m-3)}\lambda\theta^{(m+1)}}{(1+\theta)^2}-\Gamma(5-m)\frac{2^{(m-6)}\lambda\theta^m}{(1+\theta)^2}\Bigg],\end{aligned}$$

where, $i=\sqrt{-1}$ denotes an imaginary unit.

3.3. *Mode*

Mode of the TIXG(θ,λ) distribution can be found as a solution of the following equation,

$$\frac{\partial}{\partial x}\log f(x)=0$$

and by substituting the pdf in Eq. (6), we get

$$\frac{\partial}{\partial x}\log\left\{\frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\right]\right\}=0.$$

After simplifying the above differentiation, finally the expression becomes

$$\frac{\lambda e^{-\frac{\theta}{x}}\left(\frac{2\theta^2x^2+\theta^3}{(1+\theta)x^4}\right)}{1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}}-\frac{\theta}{x^2}+\frac{2}{x}+\frac{\frac{\theta}{x^3}}{1+\frac{\theta}{2x^2}}=0. \tag{10}$$

By solving Eq. (10) numerically, we will be able to obtain the mode of the proposed distribution.

3.4. *Quantile function*

Let $Q(p)$ be the quantile function of order p of TIXG distribution where $0<p<1$. Then, the quantile function will be obtained by solving the following equation,

$$F(Q(p)) = P[X\le Q(p)] = p$$

$$\Rightarrow(1+\lambda)\left\{\left(1+\frac{\theta Q(p)+\frac{\theta^2}{2}}{(1+\theta)Q^2(p)}\right)e^{\left(-\frac{\theta}{Q(p)}\right)}\right\}-\lambda\left\{\left(1+\frac{\theta Q(p)+\frac{\theta^2}{2}}{(1+\theta)Q^2(p)}\right)e^{\left(-\frac{\theta}{Q(p)}\right)}\right\}^2=p. \tag{11}$$

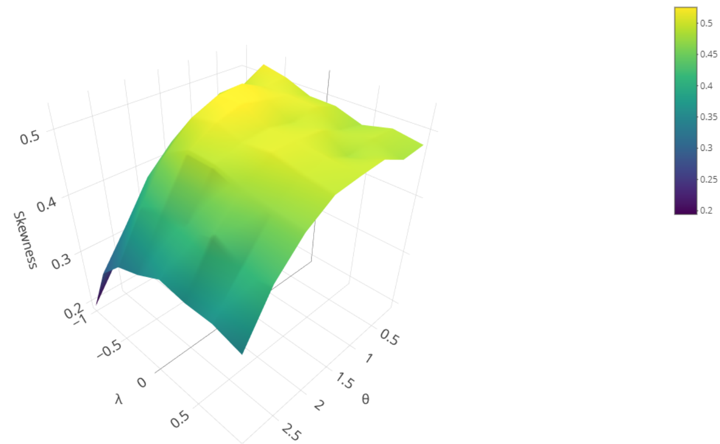


Figure 4 – Plot of skewness of TIXG distribution for different parameter choices.

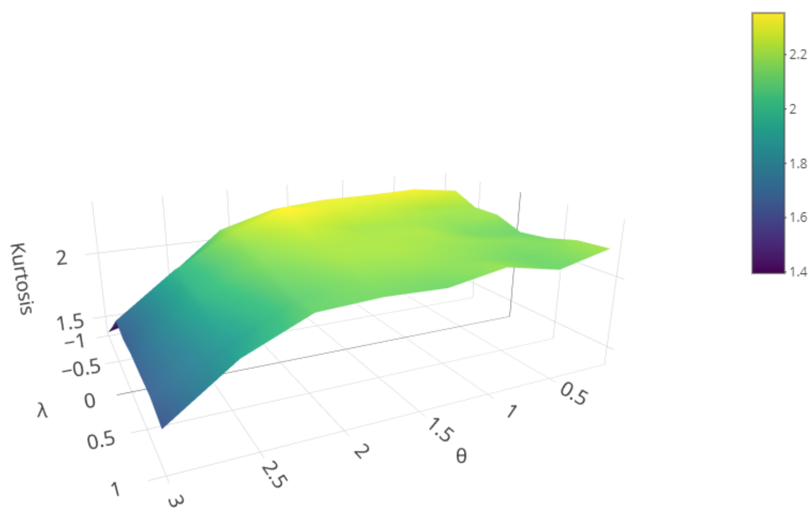


Figure 5 – Plot of kurtosis of TIXG distribution for different parameter choices.

By putting $p = 0.25, 0.5$ and 0.75 in the above Eq. (11), we will obtain the first quartile, median and third quartile of the distribution, respectively.

Skewness and kurtosis are the statistical measures which are used to calculate the degree of long tail and the degree of tail heaviness respectively. As there does not exist any classical moment for TIXG distribution, it may not measure the moment-based skewness and kurtosis of the distribution. However, on the basis of quantile function we can measure the skewness and kurtosis of the distribution with the help of Bowley measure of skewness (Bowley, 1920) and Moors measure of kurtosis (Moors, 1988).

The Bowley measure of skewness is given by

$$\mathscr{B} = \frac{Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

and the Moors kurtosis, is based on octiles is given as

$$\mathscr{M} = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}.$$

A numerical computation needs to be performed to calculate the value of skewness and kurtosis of the TIXG distribution. Plots of the skewness and kurtosis are displayed in Figures 4 and 5, respectively. It indicates that the shape of both skewness and kurtosis decreases when the parameter value increases.

3.5. *Stochastic ordering*

Stochastic ordering of positive continuous random variables is an important tool to study the structural properties of complicated stochastic systems. There are several forms of stochastic orderings that may be used to order random variables according to their characteristics. In this study we consider the hazard rate, the mean residual life and the likelihood ratio order for two independent TIXG random variables under a restricted parameter space. A random variable X is said to be smaller than a random variable Y with cdfs F_X and F_Y respectively, in the following conditions.

- Stochastic order ($X \leq_{\text{st}} Y$) if $F_X(x) \geq F_Y(x)$ for all x,
- Hazard rate order ($X \leq_{\text{hr}} Y$) if $h_X(x) \geq h_Y(x)$ for all x,
- Mean residual life order ($X \leq_{\text{mrl}} Y$) if $m_X(x) \leq m_Y(x)$ for all x,
- Likelihood ratio order ($X \leq_{\text{lr}} Y$) if $\frac{f_X(x)}{fY(x)}$ decreases in x.

The following implications based on this property are illustrated by Shaked and Shanthikumar (1994), which are well known for establishing stochastic ordering of distributions.

$$X \leq_{\text{lr}} Y \implies X \leq_{\text{hr}} Y \implies X \leq_{\text{mrl}} Y$$

and, hence,

$$X \leq_{\text{hr}} \implies X \leq_{\text{st}} Y.$$

The following theorem shows that the TIXG distribution is ordered with reference to "likelihood ratio" ordering.

THEOREM 1. *Let* $X \sim \text{TIXG}(\theta_1, \lambda_1)$ *and* $Y \sim \text{TIXG}(\theta_2, \lambda_2)$*. If* $\theta_1 = \theta_2 = \theta$ *and* $\lambda_1 > \lambda_2$*, then* $X \leq_{lr} Y$ *and hence it implies other orderings.*

PROOF.

$$\phi(x) = \frac{f_X(x)}{f_Y(x)}$$

$$= \frac{\theta_1^2(1+\theta_2)}{\theta_2^2(1+\theta_1)}\left(\frac{2x^2+\theta_1}{2x^2+\theta_2}\right)e^{\left(-\frac{\theta_1}{x}+\frac{\theta_2}{x}\right)}\left[\frac{1+\lambda_1-2\lambda_1\left\{\left(1+\frac{\theta_1 x+\frac{\theta_1^2}{2}}{(1+\theta_1)x^2}\right)e^{-\frac{\theta_1}{x}}\right\}}{1+\lambda_2-2\lambda_2\left\{\left(1+\frac{\theta_2 x+\frac{\theta_2^2}{2}}{(1+\theta_2)x^2}\right)e^{-\frac{\theta_2}{x}}\right\}}\right]. \tag{12}$$

Taking logarithm on both side of Eq. (12) and differentiating w.r.t. x by assuming $\theta_1 = \theta_2 = \theta$ we get,

$$\frac{\phi'(x)}{\phi(x)} = \frac{-2\lambda_1 e^{-\frac{\theta}{x}}\left(\frac{2\theta^2x^2+\theta^3}{2(1+\theta)x^4}\right)}{1+\lambda_1-2\lambda_1\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}} - \frac{-2\lambda_2 e^{-\frac{\theta}{x}}\left(\frac{2\theta^2x^2+\theta^3}{2(1+\theta)x^4}\right)}{1+\lambda_2-2\lambda_2\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}}$$

$$\phi'(x) = \phi(x)(\lambda_2-\lambda_1)$$

$$\frac{e^{-\frac{\theta}{x}}\left(2\theta^2x^2+\theta^3\right)}{x^4(1+\theta)\left\{1+\lambda_1-2\lambda_1\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\left\{1+\lambda_2-2\lambda_2\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}}$$

$\Rightarrow \phi'(x) < 0$ if $\lambda_1 > \lambda_2$. Hence, $\phi(x)$ is decreasing in x if $\lambda_1 > \lambda_2$ and $\theta_1 = \theta_2 = \theta$, which implies that $X \leq_{\text{lr}} Y$ and hence, $X \leq_{\text{hr}} Y, X \leq_{\text{mrl}} Y$ and $X \leq_{\text{st}} Y$. □

3.6. *Ageing intensity function*

Jiang *et al.* (2003) proposed a qualitative measure of a lifetime random variable X, termed as Ageing intensity (AI) function for quantifying the ageing property of a unit which is either be a component or a living being. The AI function for a random variable X, denoted by $L_X(t)$, is defined as the ratio of the instantaneous failure rate to the failure rate average (Nanda *et al.*, 2007), i.e.

$$L_X(t) = \frac{h(t)}{G(t)}, \text{ for any } t > 0,$$

where, $G(t) = \frac{1}{t}\int_0^t h(u)\mathrm{d}u$ is the failure rate average and $h(t)$ is hazard rate function. Then, finally AI can be written as

$$L_X(t) = \frac{-tf(t)}{S(t)\log S(t)}.$$

Now, by using the pdf and survival function of TIXG(θ, λ) distribution, the expression

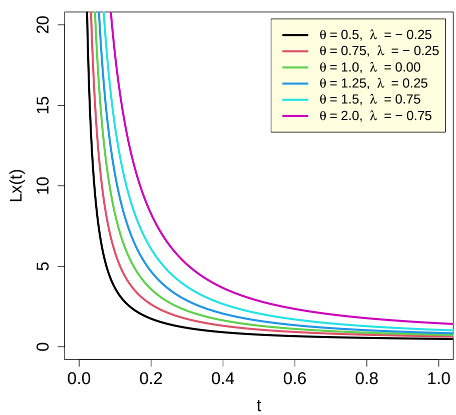


Figure 6 – The ageing intensity plot of TIXG distribution for different parameter choices.

for $L_X(t)$ becomes

$$L_X(t) = \frac{-\frac{\theta^2}{(1+\theta)t}\left(1+\frac{\theta}{2t^2}\right)e^{-\frac{\theta}{t}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta t+\frac{\theta^2}{t}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}\right]}{\left[\lambda\left\{\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}^2-(1+\lambda)\left\{\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}+1\right] \log\left[\lambda\left\{\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}^2-(1+\lambda)\left\{\left(1+\frac{\theta t+\frac{\theta^2}{2}}{(1+\theta)t^2}\right)e^{-\frac{\theta}{t}}\right\}+1\right]}.$$

The larger the value of AI function, stronger is the tendency of ageing for the associated random variable. Also, the failure rate function determines the AI function in an unique way, but the converse is not true (Sen *et al.*, 2018). For any lifetime distribution, if $L_X(t) = 1$, then the associated failure rate is constant; if $L_X(t) < 1$, then the associated failure rate is decreasing and if $L_X(t) > 1$, then the associated failure rate is increasing. As shown in Figure 6, the ageing intensity function varies with the choice of different values of θ and λ. It is observed that AI also shows non-increasing pattern with increment in time for different parameter choices.

3.7. Order statistics

Suppose $X_{(1)},X_{(2)},\ldots,X_{(n)}$ denotes the order statistics of a random sample $X_1,X_2,\ldots,X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the pdf of $X_{(j)}$ is expressed as follows

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j},\ j=1,2,\ldots,n. \tag{13}$$

Now, by using the pdf in Eq. (6) and cdf in Eq. (5) into Eq. (13), density of the j^{th} order statistic of the TIXG distribution is easily obtained as

$$\begin{aligned}f_{X_{(j)}}(x)=&\frac{n!}{(j-1)!(n-j)!}\left[(1+\lambda)\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}-\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}^2\right]^{j-1}\\&\left[\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}^2-\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\left(1+\lambda\right)+1\right]^{n-j}\\&\frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\right].\end{aligned} \tag{14}$$

The density function of the smallest and largest order statistic $X_{(1)}$ and $X_{(n)}$ are derived by putting $j=1$ and $j=n$ in the above Eq. (14), respectively. Therefore, the pdf of the smallest order statistic $X_{(1)}$ is given by

$$\begin{aligned}f_{X_{(1)}}(x)=&n\left[\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}^2-\left(1+\lambda\right)\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}+1\right]^{n-1}\\&\frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\right],\end{aligned}$$

and the expression for the pdf of the largest order statistic $X_{(n)}$ is given by

$$\begin{aligned}f_{X_{(n)}}(x)=&n\left[(1+\lambda)\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}-\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}^2\right]^{n-1}\\&\frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\right].\end{aligned}$$

The corresponding j^{th} order cdf $F_{X_{(j)}}(x)$ is given by the following expression

$$F_{X_{(j)}}(x)=\sum_{i=j}^{n}\binom{n}{i}F_X^i(x)[1-F_X(x)]^{(n-i)} \tag{15}$$

and using $F_X(x)$ from Eq. (5), the above Eq. (15) can be written as

$$F_{X_{(j)}}(x)=\sum_{i=j}^{n}\binom{n}{i}\left[(1+\lambda)\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}-\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}^2\right]^i$$
$$\left[\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}^2-\left(1+\lambda\right)\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}+1\right]^{n-i}.$$

4. Methods of estimation of TIXG distribution

In this Section, we describe five different classical estimation methods to estimate the parameters θ and λ of the TIXG distribution. All the estimation approaches are carried out when both θ and λ are unknown.

4.1. *Maximum likelihood estimation method*

The maximum likelihood approach is frequently utilized among the available statistical estimation methods in literature because of its desirable properties such as consistency, asymptotic efficiency, and invariance property (Casella and Berger, 2002). Let $X_1, X_2, \ldots, X_n$ be the random sample drawn from the TIXG(θ,λ) distribution with the pdf in Eq.(6), then the likelihood function is expressed as

$$\prod_{i=1}^{n} f(x_i;\theta,\lambda)=\prod_{i=1}^{n}\left[\frac{\theta^2}{(1+\theta)x^2}\left(1+\frac{\theta}{2x^2}\right)e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda\left\{\left(1+\frac{\theta x+\frac{\theta^2}{2}}{(1+\theta)x^2}\right)e^{-\frac{\theta}{x}}\right\}\right]\right].$$

Taking logarithm of the likelihood function, the expression of the log likelihood function can be written as

$$\begin{aligned}\ln L(\theta,\lambda)=&2n\ln\theta-n\ln(1+\theta)-2\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}\ln\left(1+\frac{\theta}{2x_i^2}\right)\\&-\theta\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\ln\left(1+\lambda-2\lambda\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right).\end{aligned}\tag{16}$$

Now, differentiating the logarithmic likelihood function given by Eq. (16) with respect to the parameters θ and λ and equating them to zero, we have the following likelihood expressions

$$\begin{aligned}\frac{\partial L}{\partial\theta}=&\frac{2n}{\theta}-\frac{n}{1+\theta}+\sum_{i=1}^{n}\left(\frac{1}{\theta+2x_i^2}\right)-\sum_{i=1}^{n}\frac{1}{x_i}-2\lambda\\&\sum_{i=1}^{n}\frac{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\left(-\frac{1}{x_i}\right)+e^{-\frac{\theta}{x_i}}\left(\frac{1}{x_i^2}\right)\left(\frac{x_i+\theta+\frac{\theta^2}{2}}{(1+\theta)^2}\right)}{1+\lambda-2\lambda\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}}=0,\end{aligned}\tag{17}$$

and

$$\frac{\partial L}{\partial \lambda}=\sum_{i=1}^{n}\left[\frac{1-2\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}}{1+\lambda-2\lambda\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}}\right]=0. \tag{18}$$

The MLE of the parameters θ and λ will be obtained by solving the above nonlinear system of equations given by Eq. (17) and Eq. (18), respectively. These nonlinear equations are difficult to solve analytically. However, to obtain the estimates $\hat{\theta}_{\text{MLE}}$ and $\hat{\lambda}_{\text{MLE}}$, some numerical methods such as the Newton-Raphson algorithm is recommended.

4.2. Maximum product spacings method

For the estimation of unknown parameters of continuous univariate distributions, the maximum product of spacings (MPS) technique introduced by Cheng and Amin (1979, 1983), provides a strong alternative to the MLE. They explored the consistency of MPS estimators in details, concluding that MPS estimators are at least asymptotically as efficient as MLE estimators upon exist. Ranneby (1984) also separately developed the same procedure as an approximation to the Kullback-Leibler measure of information.

Let $x_1, x_2, \ldots, x_n$ be the n ordered samples drawn from the TIXG(θ,λ) distribution with the cdf given in Equation (5). The expression for uniform spacings based on two consecutive cdfs is defined as follows

$$D_i(\theta,\lambda)=F(x_i|\theta,\lambda)-F(x_{i-1}|\theta,\lambda), \quad i=1,2,\ldots,n+1,$$

where, $F(x_0|\theta,\lambda)=0$ and $F(x_{n+1}|\theta,\lambda)=1$ such that $\sum_{i=1}^{n+1} D_i(\theta,\lambda)=1$.

The maximum product of spacings estimators, denoted as $\hat{\theta}_{\text{MPS}}$ and $\hat{\lambda}_{\text{MPS}}$, are obtained by maximizing the geometric mean of the spacings,

$$G(\theta,\lambda)=\left[\prod_{1=1}^{n+1} D_i(\theta,\lambda)\right]^{\frac{1}{n+1}},$$

with respect to θ and λ or equivalently by maximizing the logarithm of the geometric mean of sample spacings

$$P(\theta,\lambda)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(\theta,\lambda). \tag{19}$$

After some simplifications, Eq. (19) reduces to the following expression

$$P=\frac{1}{n+1}\left[\ln\left[(1+\lambda)\left\{\left(1+\frac{\theta x_1+\frac{\theta^2}{2}}{(1+\theta)x_1^2}\right)e^{-\frac{\theta}{x_1}}\right\}-\lambda\left\{\left(1+\frac{\theta x_1+\frac{\theta^2}{2}}{(1+\theta)x_1^2}\right)e^{-\frac{\theta}{x_1}}\right\}^2\right]\right.$$

$$+\sum_{i=2}^{n}\ln\left[(1+\lambda)\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}-\lambda\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2-\right.$$
$$\left.(1+\lambda)\left\{\left(1+\frac{\theta x_{i-1}+\frac{\theta^2}{2}}{(1+\theta)x_{i-1}^2}\right)e^{-\frac{\theta}{x_{i-1}}}\right\}-\lambda\left\{\left(1+\frac{\theta x_{i-1}+\frac{\theta^2}{2}}{(1+\theta)x_{i-1}^2}\right)e^{-\frac{\theta}{x_{i-1}}}\right\}^2\right]+\ln$$
$$\left[\lambda\left\{\left(1+\frac{\theta x_n+\frac{\theta^2}{2}}{(1+\theta)x_n^2}\right)e^{-\frac{\theta}{x_n}}\right\}^2-(1+\lambda)\left\{\left(1+\frac{\theta x_n+\frac{\theta^2}{2}}{(1+\theta)x_n^2}\right)e^{-\frac{\theta}{x_n}}\right\}+1\right]\right]. \tag{20}$$

Therefore, the MPS estimate of the unknown parameters are obtained by setting the partial derivatives of Eq. (20) with respect to θ and λ. By solving the resulting expression after equating them to zero, it may be observed that the above non-linear equation is not in closed form solution. So, we have used some iterative method like Newton-Raphson to solve them numerically.

4.3. Methods of ordinary and weighted least square estimation

Swain *et al.* (1988) introduced least square (LS) and weighted least square (WLS) estimations for estimating the parameters of Beta distributions. Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the ordered sample of size n having a distribution function $F(X_{(i)})$. According to the least square technique, the estimation of unknown parameters θ and λ of TIXG distribution can be obtained by minimizing

$$S=\sum_{i=1}^{n}\left[F(x_i)-\frac{i}{n+1}\right]^2,$$

with respect to parameters θ and λ, respectively. Using the cdf from Eq. (5), we have

$$S=\sum_{i=1}^{n}\left[(1+\lambda)\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}-\lambda\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2-\frac{i}{n+1}\right]^2. \tag{21}$$

We differentiate the above Eq. (21) with respect to θ and λ and, then, equate them to zero to obtain the least square estimates, respectively.

$$\frac{\partial S}{\partial\theta}=\sum_{i=1}^{n}\left[(1+\lambda)\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}-\lambda\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2-\frac{i}{n+1}\right]$$
$$(1-\lambda)\left\{\frac{e^{-\frac{\theta}{x_i}}}{x_i^2}\frac{x_i+\theta+\frac{\theta^2}{2}}{(1+\theta)^2}-\frac{e^{-\frac{\theta}{x_i}}}{x_i}\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)\right\}=0 \tag{22}$$

and

$$\frac{\partial S}{\partial\lambda}=\sum_{i=1}^{n}\left[(1+\lambda)\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}-\lambda\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2-\frac{i}{n+1}\right]$$

$$\left[\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}-\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2\right]=0. \tag{23}$$

Also, the WLS estimates of the unknown parameters can be obtained by minimizing

$$W=\sum_{i=1}^{n}w_i\left[F(x_i)-\frac{i}{n+1}\right]^2,$$

with respect to θ and λ. The weights w_i are equal to $\frac{1}{V(x_{(i)})}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$. Thus, for the TIXG distribution weighted least square estimators, say $\hat{\theta}_{\mathrm{WLS}}$ and $\hat{\lambda}_{\mathrm{WLS}}$ can be obtained by minimizing

$$\begin{aligned}W=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\Bigg[&(1+\lambda)\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}\\&-\lambda\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2-\frac{i}{n+1}\Bigg]^2,\end{aligned} \tag{24}$$

with respect to θ and λ, respectively. So, we have the following non-linear equations

$$\begin{aligned}\frac{\partial W}{\partial\theta}=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\Bigg[&(1+\lambda)\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}-\lambda\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2\\&-\frac{i}{n+1}\Bigg](1-\lambda)\left\{\frac{e^{-\frac{\theta}{x_i}}}{x_i^2}\frac{x_i+\theta+\frac{\theta^2}{2}}{(1+\theta)^2}-\frac{e^{-\frac{\theta}{x_i}}}{x_i}\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)\right\}=0\end{aligned} \tag{25}$$

and

$$\begin{aligned}\frac{\partial W}{\partial\lambda}=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\Bigg[&(1+\lambda)\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}-\lambda\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2\\&-\frac{i}{n+1}\Bigg]\left[\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}-\left\{\left(1+\frac{\theta x_i+\frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{-\frac{\theta}{x_i}}\right\}^2\right]=0.\end{aligned} \tag{26}$$

As the closed form solution of the estimates for unknown parameters is difficult to calculate, so we derive the LS and WLS estimators by solving the above non-linear Equations (22)-(23) and Equations (25)-(26), respectively, with the help of some iterative method such as Newton-Raphson approach.

4.4. *Cramér-Von-Mises method*

The Cramér-Von-Mises (CVM) is a type of minimum distance estimator, also popularly known as maximum goodness of-fit estimator. It is based on the difference between the estimate of the cumulative distribution function and the empirical distribution function (D'Agostino and Stephens, 1986; Luceño, 2006). Macdonald (1971) justified the use of Cramèr-Von-Mises type minimal distance estimators by demonstrating empirical evidence that their bias is smaller than that of other minimum distance estimators.

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the ordered sample from the TIXG distribution. Therefore, the Cramèr-Von-Mises estimators $\hat{\theta}_{\text{CVM}}$ and $\hat{\lambda}_{\text{CVM}}$ are obtained by minimizing the function $C(\theta,\lambda)$ with respect to θ and λ respectively where,

$$C(\theta,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_{i:n}|\theta,\lambda) - \frac{2i-1}{2n}\right)^2.$$

By differentiating the above expression w.r.t. θ and λ, we have the following non-linear equations,

$$\frac{\partial C}{\partial \theta} = \sum_{i=1}^{n}\Bigg[(1+\lambda)\left\{\left(1+\frac{\theta x_i + \frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{\left(-\frac{\theta}{x_i}\right)}\right\} - \lambda\left\{\left(1+\frac{\theta x_i + \frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{\left(-\frac{\theta}{x_i}\right)}\right\}^2 - \frac{2i-1}{2n}\Bigg](1-\lambda)\left\{\frac{e^{-\frac{\theta}{x_i}}}{x_i^2}\frac{x_i+\theta+\frac{\theta^2}{2}}{(1+\theta)^2} - \frac{e^{-\frac{\theta}{x_i}}}{x_i}\left(1+\frac{\theta x_i + \frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)\right\} = 0 \qquad (27)$$

and

$$\frac{\partial C}{\partial \lambda} = \sum_{i=1}^{n}\Bigg[(1+\lambda)\left\{\left(1+\frac{\theta x_i + \frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{\left(-\frac{\theta}{x_i}\right)}\right\} - \lambda\left\{\left(1+\frac{\theta x_i + \frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{\left(-\frac{\theta}{x_i}\right)}\right\}^2 - \frac{2i-1}{2n}\Bigg]\Bigg[\left(1+\frac{\theta x_i + \frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{\left(-\frac{\theta}{x_i}\right)} - \left\{\left(1+\frac{\theta x_i + \frac{\theta^2}{2}}{(1+\theta)x_i^2}\right)e^{\left(-\frac{\theta}{x_i}\right)}\right\}^2\Bigg] = 0. \qquad (28)$$

To obtain the CVM estimators from the above non-linear Equations (27) and (28), Newton-Raphson approach is recommended.

5. Simulation study and discussion

In this Section, Monte Carlo simulation study is performed in order to compare the behaviour of the different estimators of the TIXG distribution. The performance of the estimators obtained in the previous Section are evaluated by using maximum likelihood, maximum product spacings, least square, weighted least square and Cramér-Von-Mises estimation methods based on their biases and mean square errors (MSEs).

$$\text{Bias} = \frac{1}{K}\sum_{i=1}^{K}\left(\hat{\theta}_i - \theta\right) \quad \text{and} \quad \text{MSE} = \frac{1}{K}\sum_{i=1}^{K}\left(\hat{\theta}_i - \theta\right)^2.$$

TABLE 1
Estimates, Bias and MSE of the parameters of TIXG distribution when $\lambda = -0.25$.

θ	n	Methods	$\hat{\theta}$	$\lvert$Bias $\hat{\theta}\rvert$	MSE $\hat{\theta}$	$\hat{\lambda}$	$\lvert$Bias $\hat{\lambda}\rvert$	MSE $\hat{\lambda}$
0.5		MLE	0.521	0.021	0.007	- 0.160	0.090	0.139
		MPS	0.481	0.019	0.006	- 0.330	0.080	0.126
	75	LS	0.518	0.018	0.006	- 0.188	0.062	0.119
		WLS	0.517	0.017	0.006	- 0.184	0.066	0.125
		CVM	0.532	0.032	0.008	- 0.126	0.124	0.141
		MLE	0.507	0.007	0.004	- 0.217	0.033	0.098
		MPS	0.478	0.022	0.005	- 0.349	0.099	0.109
	150	LS	0.511	0.011	0.004	- 0.210	0.040	0.085
		WLS	0.508	0.008	0.004	- 0.216	0.034	0.088
		CVM	0.519	0.019	0.004	- 0.171	0.079	0.094
		MLE	0.501	0.001	0.003	- 0.250	0.002	0.076
		MPS	0.479	0.021	0.004	- 0.354	0.104	0.095
	300	LS	0.507	0.007	0.003	- 0.226	0.024	0.064
		WLS	0.506	0.006	0.003	- 0.231	0.019	0.064
		CVM	0.513	0.013	0.003	- 0.202	0.048	0.066
0.75		MLE	0.780	0.030	0.016	-0.168	0.082	0.133
		MPS	0.718	0.032	0.015	-0.341	0.091	0.128
	75	LS	0.773	0.023	0.015	-0.204	0.046	0.118
		WLS	0.773	0.023	0.015	-0.198	0.052	0.124
		CVM	0.794	0.044	0.018	-0.148	0.102	0.140
		MLE	0.758	0.008	0.010	-0.227	0.023	0.096
		MPS	0.711	0.039	0.012	-0.362	0.112	0.114
	150	LS	0.762	0.012	0.010	-0.225	0.025	0.084
		WLS	0.759	0.009	0.009	-0.229	0.021	0.088
		CVM	0.775	0.025	0.011	-0.190	0.060	0.093
		MLE	0.749	0.001	0.007	-0.258	0.008	0.074
		MPS	0.715	0.035	0.009	-0.361	0.111	0.097
	300	LS	0.757	0.007	0.007	-0.240	0.010	0.064
		WLS	0.755	0.005	0.006	-0.242	0.008	0.063
		CVM	0.765	0.015	0.007	-0.218	0.032	0.067
1.0		MLE	1.039	0.039	0.030	- 0.175	0.075	0.132
		MPS	0.953	0.047	0.029	- 0.347	0.097	0.130
	75	LS	1.028	0.028	0.029	- 0.215	0.035	0.121
		WLS	1.029	0.029	0.029	- 0.206	0.044	0.125
		CVM	1.057	0.057	0.035	- 0.159	0.091	0.142
		MLE	1.009	0.009	0.018	- 0.233	0.017	0.095
		MPS	0.946	0.054	0.022	- 0.365	0.115	0.114
	150	LS	1.012	0.012	0.019	- 0.237	0.013	0.088
		WLS	1.011	0.011	0.018	- 0.233	0.017	0.087
		CVM	1.031	0.031	0.021	- 0.1983	0.052	0.094
		MLE	0.997	0.003	0.013	-0.261	0.011	0.073
		MPS	0.949	0.051	0.018	-0.368	0.117	0.100
	300	LS	1.007	0.007	0.013	-0.247	0.003	0.066
		WLS	1.005	0.005	0.012	-0.247	0.003	0.064
		CVM	1.018	0.018	0.014	-0.226	0.024	0.068

TABLE 2
Estimates, Bias and MSE of the parameters of TIXG distribution when $\lambda = 0.25$.

θ	n	Methods	$\hat{\theta}$	$\|$Bias $\hat{\theta}\|$	MSE $\hat{\theta}$	$\hat{\lambda}$	$\|$Bias $\hat{\lambda}\|$	MSE $\hat{\lambda}$
0.5		MLE	0.498	0.002	0.004	0.224	0.026	0.129
		MPS	0.464	0.036	0.007	0.041	0.208	0.223
	75	LS	0.482	0.018	0.005	0.144	0.106	0.151
		WLS	0.487	0.013	0.004	0.171	0.079	0.140
		CVM	0.495	0.005	0.004	0.213	0.037	0.134
		MLE	0.496	0.004	0.002	0.226	0.024	0.079
		MPS	0.476	0.024	0.004	0.115	0.135	0.138
	150	LS	0.489	0.011	0.003	0.189	0.061	0.085
		WLS	0.493	0.007	0.002	0.209	0.041	0.074
		CVM	0.496	0.004	0.003	0.225	0.025	0.077
		MLE	0.499	0.001	0.001	0.239	0.011	0.039
		MPS	0.488	0.012	0.002	0.185	0.065	0.059
	300	LS	0.494	0.006	0.001	0.217	0.033	0.046
		WLS	0.497	0.003	0.001	0.230	0.020	0.037
		CVM	0.498	0.002	0.001	0.234	0.016	0.044
0.75		MLE	0.747	0.003	0.009	0.225	0.024	0.122
		MPS	0.695	0.055	0.015	0.047	0.202	0.218
	75	LS	0.723	0.027	0.012	0.147	0.103	0.147
		WLS	0.731	0.019	0.011	0.173	0.077	0.136
		CVM	0.743	0.007	0.011	0.214	0.036	0.129
		MLE	0.745	0.005	0.005	0.230	0.020	0.071
		MPS	0.715	0.035	0.009	0.126	0.124	0.125
	150	LS	0.734	0.016	0.006	0.194	0.056	0.079
		WLS	0.740	0.010	0.005	0.213	0.037	0.069
		CVM	0.744	0.006	0.006	0.228	0.022	0.072
		MLE	0.749	0.001	0.002	0.243	0.007	0.032
		MPS	0.734	0.016	0.003	0.193	0.057	0.049
	300	LS	0.742	0.008	0.003	0.219	0.031	0.043
		WLS	0.746	0.004	0.003	0.233	0.017	0.034
		CVM	0.747	0.003	0.003	0.236	0.014	0.040
1.0		MLE	0.997	0.003	0.017	0.225	0.025	0.121
		MPS	0.926	0.074	0.028	0.050	0.200	0.216
	75	LS	0.964	0.036	0.022	0.148	0.102	0.146
		WLS	0.975	0.025	0.019	0.176	0.074	0.132
		CVM	0.991	0.009	0.020	0.215	0.035	0.128
		MLE	0.995	0.005	0.009	0.232	0.018	0.067
		MPS	0.954	0.046	0.015	0.131	0.119	0.119
	150	LS	0.978	0.022	0.012	0.195	0.055	0.078
		WLS	0.987	0.013	0.010	0.216	0.034	0.065
		CVM	0.993	0.007	0.011	0.230	0.020	0.069
		MLE	0.999	0.001	0.004	0.244	0.006	0.031
		MPS	0.979	0.021	0.006	0.195	0.056	0.047
	300	LS	0.989	0.011	0.006	0.221	0.029	0.041
		WLS	0.995	0.005	0.005	0.235	0.015	0.032
		CVM	0.996	0.004	0.006	0.238	0.012	0.038

To generate pseudo-random numbers of sizes $n = 75, 150, 300$ from the TIXG distribution, we use inverse transformation method by solving $F(x) = u$ numerically with the help of `uniroot()` function in `R` (R Core Team, 2021, Version 3.6.1), where $u \sim \text{Unif}(0, 1)$. For this purpose, we set the parameter choices at $\theta = 0.5, 0.75, 1.0$ and the transmutation parameter $\lambda = -0.25, 0.25$. To get the simulated results, we repeat the process for $K = 1000$ times and calculate the average estimate, bias and MSE of the corresponding estimators. The results of simulation study are presented in Tables 1 and 2.

According to the simulation results, it is clearly observed that as we increase the sample sizes, the bias and MSE values of all the estimators decrease and it verifies the consistency of the estimation methods used in this study. As we increase the value of θ, the absolute value of bias also increases for the given values of λ and n considering all the methods of estimation. In the case of $\lambda = -0.25$, when the sample size $n \geq 150$, $\hat{\theta}_{\text{WLS}}$ performs best since it yields low MSE values for all the chosen values of θ. Also, the least square and weighted least square estimators $\hat{\lambda}_{\text{LS}}$, $\hat{\lambda}_{\text{WLS}}$ produce least MSE values in most of the considered cases. Additionally, while $\lambda = 0.25$, $\hat{\theta}_{\text{MLE}}$ and $\hat{\lambda}_{\text{MLE}}$ outperform the other estimators with the smaller MSE values for the considered values of θ.

6. Real data application

In this Section, we verify the potentiality of the proposed TIXG distribution by considering Vinyl chloride data obtained from clean upgradient monitoring wells. Vinyl chloride is a volatile organic compound and both anthropogenic and carcinogenic in nature. The data has been originally extracted from Bhaumik *et al.* (2009). Also Krishnamoorthy *et al.* (2008) and Bhaumik and Gibbons (2006) considered this data in constructing prediction and tolerance intervals for gamma random variables. A Summary of the datasets including mean, median, coefficient of quantile deviation, Bowley's measure of skewness, Moor's kurtosis measure etc. are provided in Table 3.

TABLE 3

Summary statistics for the Vinyl chloride data.

length n	Min $x_{(1)}$	Max $x_{(n)}$	Mean $\bar{x}$	Median	Coefficient of Quartile Deviation	Bowley's Skewness	Moor's Kurtosis
34	0.1	8.0	1.879	1.150	0.6639	0.3418	1.2215

We compare the fitting of the TIXG distribution with some other well known comparative models such as Transmuted Inverse Rayleigh (TIR) (Ahmad *et al.*, 2014), Inverse Xgamma (IXG) (Yadav *et al.*, 2019), Length Biased Xgamma (LBXG) (Sen *et al.*, 2017) and Exponentiated Inverse Rayleigh (EIR) (Rao *et al.*, 2019) distributions (Figure 7).

In Table 4 estimates of unknown parameters of TIXG distribution using different estimation methods namely maximum likelihood, maximum product spacings, least square, weighted least square and the Cramér-Von-Mises are provided along with some

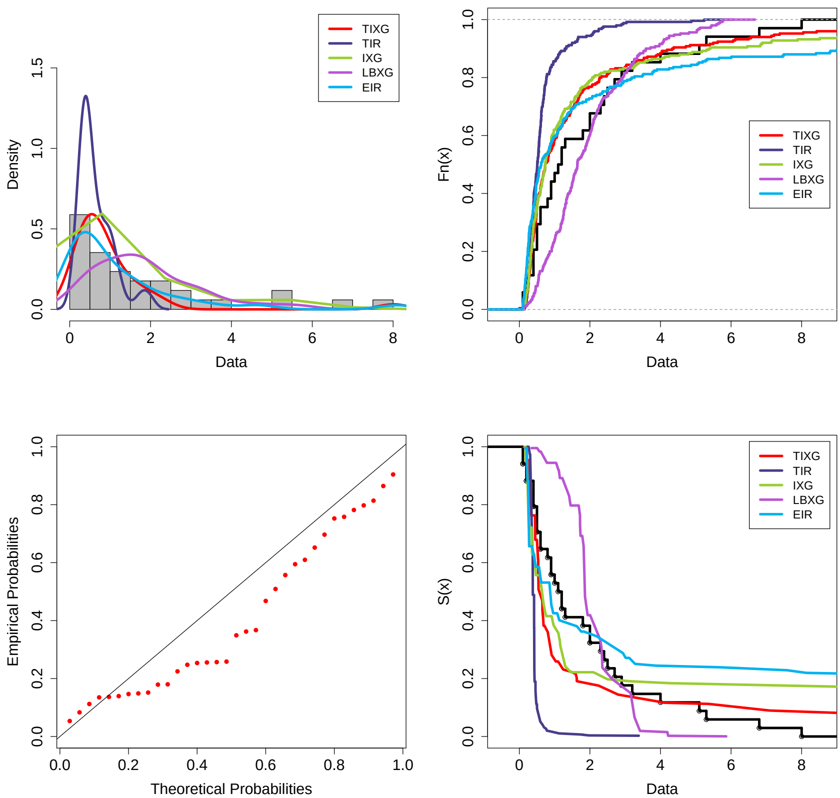


Figure 7 – Diagnostic plots of fitted TIXG distribution for Vinyl Chloride data. From upper-left to lower-right: histogram, empirical vs theoretical cdf, P-P plot, fitted survival function plot.

corresponding goodness-of-fit measures. From this Table it is clearly seen that all the estimation techniques worked well for the considered real dataset.

For the purpose of model comparison, we consider the CVM, the Anderson-Darling (AD) and the Kolmogorov-Smirnov (KS) test statistics. These statistics are frequently used to see how well a given cdf fits the empirical distribution of a dataset (Seal *et al.*, 2023). In addition, for more accuracy some goodness-of-fit measures including the Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC) and $-2\hat{l}$, where $\hat{l}$ is the maximized log-likelihood, have been chosen for comparing the superiority of the candidate model. Generally, the smaller value of these statistics indicates the better fit to the data. The required computational works are carried out with R (R

TABLE 4
Parameter estimates under different methods and the goodness-of-fit statistics for Vinyl chloride data.

Method	$\hat{\theta}$	$\hat{\lambda}$	-2logL	AIC	BIC	HQIC	CAIC
MLE	0.86	- 0.67	120.76	124.76	127.81	125.80	125.14
MPS	1.25	- 0.76	132.86	136.86	139.91	137.90	137.25
LS	1.04	- 0.61	122.87	126.87	129.92	127.91	127.26
WLS	0.96	- 0.72	121.91	125.91	128.96	126.95	126.29
CVM	1.00	- 0.70	122.57	126.57	129.62	127.61	126.95

Core Team, 2021, Version 3.6.1).

It is observed from the Table 5 that, TIXG distribution has the smallest value regarding the comparing criteria among all other competitive models. As a result, the suggested TIXG model is considered as the best model for the Vinyl chloride data.

TABLE 5
Goodness-of-fit measures of the TIXG model and the other competing models for Vinyl chloride data.

Model	MLE	-2logL	AIC	CAIC	HQIC	BIC	K-S	CVM	AD
TIXG	$\hat{\theta}$= 0.86	120.76	124.76	125.14	125.80	127.81	0.12	0.19	1.28
	$\hat{\lambda}$= - 0.67								
IXG	$\hat{\theta}$= 1.07	125.31	127.31	127.44	127.83	128.84	0.17	0.41	2.33
LBXG	$\hat{\theta}$= 1.68	128.86	130.86	130.99	131.38	132.39	0.25	0.64	4.85
EIR	$\hat{\sigma} = 0.29$	127.42	131.42	131.81	132.46	134.47	0.20	0.45	2.40
	$\hat{\alpha} = 0.20$								
TIR	$\hat{\theta}$= 0.10	170.79	174.79	175.18	175.83	177.84	0.40	2.28	17.92
	$\hat{\lambda}$= - 0.78								

In order to verify the shape of the failure rate function, a standard graphical approach called Total Time on Test (TTT) plot for the considered dataset is provided in Figure 8. According to Aarset (1987), if the shape of the TTT plot is straight diagonal, the hazard rate is constant. The failure rate function is increasing (decreasing) if the TTT plot is concave (convex), whereas the bathtub shaped hazard is obtained when first is convex and then concave. As from Figure 8, it has been seen that the TTT plot for the considered dataset implies the decreasing nature of hazard rate function. Therefore, the TIXG distribution provides a reasonable fit for modeling this dataset. The relative histogram with the fitted densities, the empirical cdf, the empirical survival function and P-P plots for the Vinyl chloride data are also displayed in Figure 7. These Figures substantially supported the results presented in Tables 4 and 5, respectively.

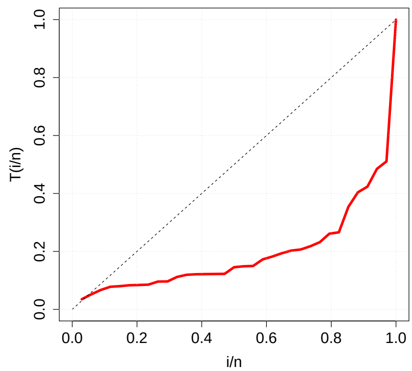


Figure 8 – Total Time on Test (TTT) plot.

7. Conclusion

In statistical research, the introduction of new lifetime distributions or modification to already available lifetime distributions has become a time-honored trend. In this article, a new two parameter TIXG distribution is introduced by using QRTM technique and the baseline Inverse Xgamma distribution. Some of the important statistical properties such as hazard rate function, survival function, moment generating function, characteristic function, moments, skewness, kurtosis, order statistics, ageing intensity function etc. are obtained. To estimate the unknown shape parameter θ and the transmutation parameter λ, we consider five different estimation methods such as maximum likelihood estimation, maximum product spacings, least square, weighted least square and Cramèr-Von-Mises estimation methods. Further, we compare those estimators by using Monte Carlo simulation for different sample sizes. The performance of the estimators is evaluated on the basis of the bias and the MSE. Simulation results indicate that all the estimators are asymptotically unbiased and consistent as the bias and MSE are gradually decreasing when the sample size increases. Further, a real data has been utilized to demonstrate the applicability of the newly obtained model and it has been found that the TIXG distribution provides better fit as compared to some of its competitive models. Therefore, it can be hoped that the proposed model might be taken as an alternative model to analyze several lifetime datasets.

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