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OSCILLATING SERVICE SYSTEM BETWEEN CONVENTIONAL AND RETRIAL QUEUES FOR IMPATIENT CUSTOMERS WITH SWITCH-OFF AND CLOSE-DOWN PERIODS

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SUMMARY

In this article, we analyze a single server dynamic service system between conventional and retrial queueing modes with impatient customers and switch-off and close-down periods of the server. In such a system, analytical expressions for the steady-state joint probabilities of the status of the server and the orbit size are derived in terms of hypergeometric functions. The factorial moments of the orbit size are also determined. Several interesting and key performance measures have been obtained. Moreover, the regenerative cycle length of the system and its related characteristics are discussed. Finally, extensive numerical results are presented graphically to illustrate the effects of the system parameters on the vital performance measures.

Keywords: Retrial queue; Reneging; Switch-off time; Close-down time; Regenerative analysis; Hypergeometric functions.

1. INTRODUCTION

Queueing systems play a prominent role in the dimensioning and the performance analysis of a wide range of computer networks and communication systems. In fact, new

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results in queueing models have often been inspired by new technological advances in wireless mobile networks and Wi-Fi internet systems. The classical/conventional queueing model is the single-server service station where the server works at a constant speed. In such a model, customers arrive at service station according to a Poisson process and the service times of customers are independent and exponentially distributed random variables. An arriving customer that finds an idle server, it immediately accesses the server for the service. On the other hand, in the conventional queueing systems, it is always assumed that when an arriving customer sees the busy server, either it joins the infinite capacity waiting line to receive its service leading to unbounded queue size or leaves the system forever without service, known as loss system (see Takagi, 1991; Medhi, 2002). To alleviate the situations of either loss of the customers or overloading the service area in the conventional queues, a new class of queueing system, often called retrial queueing system, has been proposed as an alternative solution among others.

In the retrial queueing system, the buffer does not exist and hence an arriving customer or request, not able to get service immediately, leaves the service zone and enters a virtual place, called as orbit/retrial group. While waiting in the orbit, each orbiting customer tries again independently to capture a free server in a random order and at random time intervals by competing with other customers present in the orbit and new primary arrivals. Analysis of retrial queues is more difficult than the investigation of the corresponding system with buffers due to the space inhomogeneous of the number of customers in the system and the necessity of monitoring the status of the server.

Nowadays, retrial queues are widely used for many real application systems such as telephone switching systems, digital mobile communication networks, random access protocols in wireless networks, automatically repeated request call centers, wavelength-routed optical networks, cloud computing systems, just to name a few. For instance, Choi *et al.* (1992) have studied the stability of the CSMA / CD (Carrier Sense Multiple Access with Collision Detection) protocol, by a retrial queue. Later, Kumar *et al.* (2010) have investigated the retrial queues with collision of packets in wireless LAN (Local Area Network) with random access protocols and obtained vital performance measures of the system. Avrachenkov and Yechiali (2008) have used a retrial queueing network to model TCP (Transmission Control Protocol) traffic and derived related performance measures of interest. Tran-Gia and Mandjes (1997) and Economou and Herrero (2009) have proposed the retrial queueing systems for cellular communication networks and provided key performance measures such as waiting time distribution, idle time of guard channels, etc. Aguir *et al.* (2008, 2004) have analyzed the impact of retrial phenomenon of calls in the call centers and obtained the mean number of calls in the orbit and the blocking probability of calls by employing the retrial queueing system. Retrial queues have also been used for performance analysis of optical communication networks by Abidini *et al.* (2017). Artalejo and Phung-Duc (2012) have also performed a detailed study of a single server call center retrial queue with two-way communication by adopting the generating function technique. Recently, Kumar *et al.* (2021) have examined the multi-processor two-stage tandem call center retrial queues with non-reliable processors and derived some performance measures such as waiting time distribution of calls, the busy period of the call center, availability of the processors, etc. For excellent and comprehensive overviews of several results and the bibliographical information about retrial queues, we refer the reader to the outstanding survey papers by Falin (1990), Kulkarni and Liang (1997), Choi and Chang (1999), Artalejo (2010) and Kim and Kim (2016). Moreover, the detailed review of the analytical results and algorithmic approach on these topics can be found in the monographs by Falin and Templeton (1997) and Artalejo and Gomez-Corral (2008) and the references therein.

Impatience of customers is another vital phenomenon which is commonly encountered in several service systems. Impatient customers may leave (renege from) the system before receiving service, if either a long wait already experienced in the queue or a long wait anticipated by a customer upon arrival. As a result, customers' impatience or reneging leads to loss in revenues and customers' goodwill to the service provider. In fact, customers' impatience will affect the system performance measures. In recent years, queueing systems with customers' impatience have drawn significant attention in the designing and the performance evaluation of the complex computer networks and the modern telecommunication systems. For instance, (i) the call center in which customers' hang up occurs due to impatience before they are served; (ii) in the real-time data transmission network system, when the data packet is received after a hard deadline, it turns out to be useless, and (iii) the telecommunication system where the subscribers give up owing to impatience before the required connection is established. For significant aspects of queues with impatient customers and their applications, see the research articles by Brandt and Brandt (2004), Altman and Yechiali (2006), Economou and Kapodistria (2010), Phung-Duc (2014), Yue *et al.* (2016) and Peng and Wu (2020).

Enhancing energy efficiency in ICT (Information and Communication Technology) service systems such as data transmission centers, cloud computing systems, call centers, etc., is one of the challenging issues because the server in operative state consumes a huge amount of energy. The simple way to reduce energy consumption is dynamic control of switch on/off server according to the traffic load of the system. However, this approach causes the additional energy and time loss due to switching off/on of the server. Moreover, an off server needs some time in order to be active during which the server consumes energy but cannot process a job. Hence, there is a delay in processing of jobs when they arrive during the switch-off state. Thus, there exists a trade-off between energy consumption and delay performance. Therefore, it is highly important to know under what conditions it will be advantageous to put the server in switch-off state or leave it in the operative state.

Taking into consideration of the above facts, ICT service systems are redesigned by adding one more new feature apart from the server's switch-on and switch-off states. Specifically, when there is no job in the system to be served, the server is not immediately switched-off but stays idle for some random time duration, the so-called close-down period. During the close-down time period, if a job arrives, the server immediately returns to the switch-on state and begins service for that job, thereby the system busy period starts again. On the other hand, if no job arrives until the end of close-down period, the server enters into switch-off state for some random time duration. The systems dealing with the concept of close-down and switch-off periods along with their applications have been investigated in various communication network systems by Hassan and Atiquzzaman (1997), Niu *et al.* (2003), Sakai *et al.* (1998), Mitrani (2013), Kumar *et al.* (2015), Phung-Duc (2017), Daraseliya and Sopin (2017) and Chang *et al.* (2019).

Another important aspect in the study of the queueing system is the oscillation of the server. In such oscillating service queue, depending on the evaluation of number of customers in the system, the server either provides the service at two different service rates or switches between two different queueing modes. The oscillating queueing analysis is often considered to be an effective instrument in modelling and performance analysis of communication networks and service industries. For related research works pertaining to the oscillating queueing systems and their applications, the readers are referred to research papers by Choi and Choi (1996), Boxma and Kurkova (2001), Chydzinski (2002), Mitrani (2013) and Banik (2015).

Several research articles have been appeared separately on retrial queueing systems, impatience of customers, oscillating service systems and switch-off and close-down periods of the server. However, the investigation of retrial queueing system taking together with the aforementioned features seems to be quite complicated and has not been studied in detail up to now. This motivates us to analyze a new class of oscillating retrial queueing system with impatient customers, where the server needs the switch-off and close-down periods.

Thus, here we study the oscillation of a single server between conventional queue and retrial queue with impatient customers in which the close-down and switch-off periods of the server are considered. Further, our oscillating queueing system is equipped with the mechanism of search of customers from the orbit in the following manner: Whenever the number of customers in the orbit is less than or equal to the threshold value N at a service completion epoch, the server immediately fetches the next customer directly from the orbit, if any, for service with probability one. The time taken for search is negligible. Hence, the system is operated in the classical queueing mode. While the orbit size reaches $N+1$, no more search is made for customers and the system switches to retial queueing mode and continue to operate under this mode until the size of the orbit comes down to threshold value N at a service completion epoch. Thus, in the retrial queueing mode, the customers from the orbit have to make retrials on their own to occupy the free server for service. In addition, it is assumed that customers waiting in the orbit may become impatient due to delay and decide to renege from the system without getting service. The event of customers' impatience occurs only when the size of the orbiting customer is above the threshold value level N and at the same time the server is being busy in the retrial queueing mode.

The motivation of this article is twofold. The first one is to study a versatile dynamic operating of a single server retrial queueing system with orbit search mechanism. This provides insight of the link between the conventional queue with close-down and switch-off periods of the server and the retrial queue with impatience of customers. The notion of searching for customers in the orbit and the threshold value N would reduce the idle time of the server resulting in improvement of the quality of service. More specifically, for a large threshold value N, the retrial queue resembles to a conventional queue, whereas for small threshold value N, the idle time of server between two consecutive services will be rather lengthy if the system is in the retrial queueing mode. To alleviate such long idle times, the server fetches the next customer right away from the orbit as long as the number of customers in the orbit is below the pre-specified value N. Furthermore, when the orbit size increases, retrial intensity also correspondingly increases and hence the customers from the orbit generate the retrial flow quite often in order to access service. As a result, the frequent retrial attempts reduce the idle time of the server between services and hence the customers' impatience while waiting in the orbit is controlled and thus the loss of customers is reduced.

The second objective is to derive the analytical expressions for the steady-state joint probabilities of the status of server and the number of customers in the orbit by employing the recursive scheme along with various performance descriptors of the system. Further, we also show usefulness of a regenerative cycle approach, which is only concerned with a single busy cycle, and uses a well known regenerative cycle formula to study general structures of the average measures of the system such as the expected lengths of server's switch-off and close-down periods, the mean of the server being busy in both conventional and retrial queueing systems and the average busy period of the system.

This article is organized in the following manner. In Section 2, the mathematical model is described briefly. The compact expressions for the steady-state joint probabilities of the status of server and the number of customers in the orbit are determined in terms of generalized hypergeometric functions in Section 3. Section 4 deals with the factorial moments for the system under discussion. In Section 5, some key performance measures of the system are obtained. Section 6 discusses several important characteristics in terms of the cycle period of the system. Numerical examples are provided to illustrate the effects of system parameters on the vital performance measures in Section 7. In the final Section, we make some conclusions and remarks.

2. Model Description

We consider the oscillation of a single server between conventional/classical queueing mode and retrial queueing mode with unlimited orbit capacity. In this system, the server resides in one of three phases, namely operative period, close-down period and switch-off period. During the operative period, primary customers arrive at the service area according to a Poisson process with rate λ. The service times for all customers are independent and exponentially distributed with mean $\frac{1}{\nu}$. An arriving primary customer immediately begins its service, if idle server is available at the arrival epoch. Otherwise, it leaves the service area and joins the orbit. It is assumed that whenever the number of customers in the orbit is less than or equal to the threshold value N at a service completion instant, the server immediately fetches the next customer directly from the orbit, if any, for service with probability one and the time for this processing is negligible. It means that the system is operated in a conventional queueing mode. As soon as the orbit size becomes greater than or equal to $N+1$, i.e., once exceeds the threshold value N, the system transits to retrial queueing mode and continues to operate under this mode until the size of the orbit comes down to threshold value N at a service completion epoch. In the retrial queueing mode, the customers in the orbit conduct retrials to access the server under the classical retrial policy until occupy the server. That is, the inter-retrial time intervals are assumed to be exponentially distributed with intensity $\mu_j = j\mu$, where j (with $j \geq N+1$) is the number of customers in the orbit/retrial group and μ (with $\mu > 0$) is the retrial rate. Consequently, there is a competition between primary and orbital customers for getting into the server for the next service. Besides, customers waiting in the orbit may become impatient, so that they decide to leave the system forever without getting the service. The events of customers' impatience occur only when the size of the orbiting customers is above the level N and at the same time the server is being busy in the retrial queueing mode. More specifically, while the number of customers in the orbit is at least $N+1$ and the server is also busy in serving a customer, each waiting customer in the orbit activates an independent "impatience timer", τ, which is restricted by a random time having exponential distribution with mean $\frac{1}{\xi}$. If a customer's retrial attempt has not been completed successfully before the impatience timer expires, the customer abandons the system without getting service and never returns. Whereas, if the orbit size $j \leq N$, the waiting customers in the orbit do not leave but stay in the system until their services are completed, behaving like the conventional/classical queueing system. Thus, our dynamic service system deals with a back and forth movement between conventional queue and retrial queue modes in the operative period.

Apart from oscillation of the server between the two queueing modes with impatience of customers, the server, upon completion of the services of all customers and the system is empty, turns to close-down period for a random duration. The close-down time periods are exponentially distributed random variables with mean $\frac{1}{\eta}$. During the close-down period, if a new primary customer arrives, the close-down period can be interrupted and the server immediately returns to the operative period and starts serving for that customer, thereby the system busy period commences again. On the other hand, if the close-down period expires before the arrival of a new primary customer, the server immediately goes to switch-off mode for maintenance whose time duration is exponentially distributed with mean $\frac{1}{\beta}$. As the server is under the maintenance during the switch-off period, the processing of service is suspended. Hence the new primary arrivals are not allowed to join the system and loss forever. After completion of the switch-off period, the server returns to the operative period and stays in an idle state. The server is now ready to provide service if the primary customer arrives during the idle state of the conventional queueing mode, thus a new busy period of the system begins. For mathematical tractability and to provide exact analytical results of the system, all the random variables defined above are assumed to be mutually independent.

From the above assumptions, we can model our dynamic operating of the single server retrial queueing system with impatient customers, where the server involves the close-down and the server switch-off periods as a continuous time Markov chain (CTMC).

Let $N_q(t)$ denote the number of customers in the retrial group/orbit at time t and $J(t)$ describe the status of the server at time t defined as:

$$J(t)=\begin{cases}0 & \text{if the server is idle,}\\ 1 & \text{if the server is busy,}\\ S & \text{if the server is in switch-off state,}\\ C & \text{if the server is in close-down state.}\end{cases}$$

Thus, under these settings, the system states at time t can be described by the bivariate process $\{X(t);\ t \geq 0\} = \{(J(t), N_q(t));\ t \geq 0\}$ taking the values on the state space $\Omega = \{0,1,S,C\} \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0,1,2,\ldots\}$. Consequently, the bivariate process $\{X(t);\ t \geq 0\}$ is an irreducible and aperiodic CTMC. In what follows, $\pi(i,j,t) = P(J(t) = i,\ N_q(t) = j)$, $i \in \{0,1,S,C\}$, $j \in \mathbb{Z}_+$, is the joint probability distribution of the status of the server and the number of customers in the orbit at time t. By our hypothesis, whenever orbit size $j \leq N$, at a service completion epoch, evidently, $\pi(0,j,t) = 0$ for $j = 1,2,3,\ldots,N$, since on service completion the server immediately fetches a customer from the orbit with probability one and the time for this procedure is negligible. Thus, the infinitesimal generator matrix, $Q = \left(q_{(i,j)(m,n)}\right)$, of the CTMC $\{X(t);\ t \geq 0\}$ is as follows:

$$q_{(i,0)(m,n)}=\begin{cases}-\beta & \text{if } (m,n)=(S,0),\ i=S,\\ \beta & \text{if } (m,n)=(0,0),\ i=S,\\ -(\lambda+\eta) & \text{if } (m,n)=(C,0),\ i=C,\\ \lambda & \text{if } (m,n)=(1,0),\ i=C,\\ \eta & \text{if } (m,n)=(S,0),\ i=C,\\ 0 & \text{otherwise;}\end{cases}$$

$$q_{(0,j)(m,n)}=\begin{cases}-(\lambda+j\mu) & \text{if } (m,n)=(0,j),\ j=0,\ j\geq N+1,\\ \lambda & \text{if } (m,n)=(1,j),\ j=0,\ j\geq N+1,\\ j\mu & \text{if } (m,n)=(1,j-1),\ j\geq N+1,\\ 0 & \text{otherwise;}\end{cases}$$

$$q_{(1,j)(m,n)} = \begin{cases} -(\lambda+\nu) & \text{if } (m,n)=(1,j),\ 0\le j\le N,\\ -(\lambda+\nu+j\xi) & \text{if } (m,n)=(1,j),\ j\ge N+1,\\ \lambda & \text{if } (m,n)=(1,j+1),\ j\ge 0,\\ \nu & \text{if } (m,n)=(C,0),\ j=0,\\ \nu & \text{if } (m,n)=(1,j-1),\ 1\le j\le N,\\ \nu & \text{if } (m,n)=(0,j),\ j\ge N+1,\\ j\xi & \text{if } (m,n)=(1,j-1),\ j\ge N+1,\\ 0 & \text{otherwise.} \end{cases}$$

These transition rates among states are depicted in Fig. 1.

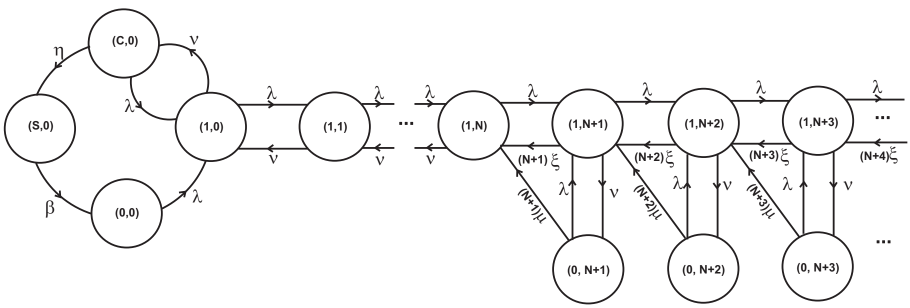

Figure 1 – Dynamically controlled oscillating queueing system.

It is easily seen that the matrix $Q = (q_{(i,j)(m,n)})$ is irreducible and conservative for the system under discussion. Additionally, it is assumed that the process is standard in the sense that

$$\lim_{t\to 0} P\{X(t)=(m,n)|X(0)=(i,j)\} = \delta_{(i,j)(m,n)}\ \forall\,(i,j),(m,n)\in\Omega,$$

where δ_{ab} represents the Kronecker's delta function, i.e., $\delta_{ab} = 1,$ if $a = b$ and $\delta_{ab}=0$, if $a\neq b$.

It is not hard to show formally that the system is always stable due to the reneging of customers from the orbit when the server is busy. In other words, owing to the impatience of customers while the server is busy, the overall reneging rate $\xi>0$ increases with the orbit size j which prevents explosion of the system size. Hence, the system is ergodic and stationary probabilities of the system exist as long as reneging rate $\xi>0$ even if $\lambda>\nu$ (see Rao, 1967). However, if the reneging rate $\xi=0$, i.e., the orbital customers are patient (persistent), we verified formally, though it is not being presented here, that the condition for ergodicity turns out be $\lambda<\nu$ (see Falin, 1986).

Based on the above observations, we conclude that the system under investigation is regular and ergodic. To this end, we now define the limiting joint probabilities of the process $\{(J(t), N_q(t)); t \geq 0\}$ as

$$\pi(i,j) = \lim_{t\to\infty} P\{J(t) = i,\ N_q(t) = j\},\ i \in \{0,1,S,C\},\ j \in \mathbb{Z}_+,$$

which always exist and are positive.

In what follows, we study the steady-state joint probabilities, $\pi(i,j)$ for $i \in \{0,1,S,C\}$ and $j \in \mathbb{Z}_+$, of the status of the server and the number of customers in the orbit for regular and ergodic CTMC $\{X(t);\ t \geq 0\}$.

3. Analysis of Steady-State Distribution

In the following Theorem, the compact analytical expressions for the steady-state joint probabilities $\pi(i,j)$ for $i \in \{0,1,S,C\}$ and $j \in \mathbb{Z}_+$, are derived in terms of generalized hypergeometric functions by solving the set of difference equations.

THEOREM 1. *For $\lambda > 0$, $\beta > 0$, $\eta > 0$, $\nu > 0$, $\mu > 0$ and $\xi > 0$, the steady-state joint probabilities, $\pi(i,j)$, of the status of the server and the number of customers in the orbit are determined as:*

$$\pi(S,0) = \begin{cases} \dfrac{1}{\beta}\Bigg\{\dfrac{1}{\beta} + \dfrac{1}{\eta} + \dfrac{1}{\lambda} + \left(\dfrac{1}{\eta} + \dfrac{1}{\lambda}\right)\rho\Bigg[\dfrac{1-\rho^{N+1}}{1-\rho} + \dfrac{\left(\frac{\lambda}{\xi}\right)\left(\frac{\lambda+\nu}{\mu} + N + 1\right)\rho^N}{(N+1)\left(\frac{\nu}{\xi} + \frac{\lambda}{\mu} + N + 1\right)} \\ \quad \times F\left(1, \dfrac{\lambda}{\mu} + N + 1, \dfrac{\lambda+\nu}{\mu} + N + 2; N+2, \dfrac{\lambda+\nu}{\mu} + N + 1, \dfrac{\nu}{\xi} + \dfrac{\lambda}{\mu} + N + 2; \dfrac{\lambda}{\xi}\right)\Bigg]\Bigg\}^{-1} \\ \hfill \text{if } \rho \neq 1, \\ \dfrac{1}{\beta}\Bigg\{\dfrac{1}{\beta} + \dfrac{1}{\eta} + \dfrac{1}{\lambda} + \left(\dfrac{1}{\eta} + \dfrac{1}{\lambda}\right)\Bigg[(N+1) + \dfrac{\left(\frac{\lambda}{\xi}\right)\left(\frac{2\lambda}{\mu} + N + 1\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi} + \frac{1}{\mu}\right) + N + 1\right)} \\ \quad \times F\left(1, \dfrac{\lambda}{\mu} + N + 1, \dfrac{2\lambda}{\mu} + N + 2; N+2, \dfrac{2\lambda}{\mu} + N + 1, \lambda\left(\dfrac{1}{\xi} + \dfrac{1}{\mu}\right) + N + 2; \dfrac{\lambda}{\xi}\right)\Bigg]\Bigg\}^{-1} \\ \hfill \text{if } \rho = 1; \end{cases} \quad (1)$$

$$
\pi(0,0)=\begin{cases}
\frac{1}{\lambda}\left\{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\rho\left[\frac{1-\rho^{N+1}}{1-\rho}+\frac{\left(\frac{\lambda}{\xi}\right)\left(\frac{\lambda+\nu}{\mu}+N+1\right)\rho^{N}}{(N+1)\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)}\right.\right. \\
\left.\left.\times F\left(1,\frac{\lambda}{\mu}+N+1,\frac{\lambda+\nu}{\mu}+N+2;N+2,\ \frac{\lambda+\nu}{\mu}+N+1,\ \frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2;\frac{\lambda}{\xi}\right)\right]\right\}^{-1} \\
\hfill if\,\rho\neq 1, \\
\frac{1}{\lambda}\left\{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\left[(N+1)+\frac{\left(\frac{\lambda}{\xi}\right)\left(\frac{2\lambda}{\mu}+N+1\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)}\right.\right. \\
\left.\left.\times F\left(1,\frac{\lambda}{\mu}+N+1,\frac{2\lambda}{\mu}+N+2;N+2,\ \frac{2\lambda}{\mu}+N+1,\ \lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2;\frac{\lambda}{\xi}\right)\right]\right\}^{-1} \\
\hfill if\,\rho=1;
\end{cases}
\tag{2}
$$

$$
\pi(C,0)=\begin{cases}
\frac{1}{\eta}\left\{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\rho\left[\frac{1-\rho^{N+1}}{1-\rho}+\frac{\left(\frac{\lambda}{\xi}\right)\left(\frac{\lambda+\nu}{\mu}+N+1\right)\rho^{N}}{(N+1)\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)}\right.\right. \\
\left.\left.\times F\left(1,\frac{\lambda}{\mu}+N+1,\frac{\lambda+\nu}{\mu}+N+2;N+2,\ \frac{\lambda+\nu}{\mu}+N+1,\ \frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2;\frac{\lambda}{\xi}\right)\right]\right\}^{-1} \\
\hfill if\,\rho\neq 1, \\
\frac{1}{\eta}\left\{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\left[(N+1)+\frac{\left(\frac{\lambda}{\xi}\right)\left(\frac{2\lambda}{\mu}+N+1\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)}\right.\right. \\
\left.\left.\times F\left(1,\frac{\lambda}{\mu}+N+1,\frac{2\lambda}{\mu}+N+2;N+2,\ \frac{2\lambda}{\mu}+N+1,\ \lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2;\frac{\lambda}{\xi}\right)\right]\right\}^{-1} \\
\hfill if\,\rho=1;
\end{cases}
\tag{3}
$$

$$\pi(1,j)=\begin{cases}\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\rho^{j}\ \pi(S,0) & \\ \qquad\qquad if\, 0\le j\le N \ and\ \rho\neq 1, \\ \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\pi(S,0) & \\ \qquad\qquad if\, 0\le j\le N \ and\ \rho=1, \\ \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\left(\frac{\lambda}{\mu}+N+1\right)_{j-N}\left(\frac{\lambda}{\xi}\right)^{j-N}\rho^{N}}{(N+1)_{j-N}\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)_{j-N}}\pi(S,0) & \\ \qquad\qquad if\, j\ge N+1 \ and\ \rho\neq 1, \\ \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\left(\frac{\lambda}{\mu}+N+1\right)_{j-N}\left(\frac{\lambda}{\xi}\right)^{j-N}}{(N+1)_{j-N}\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)_{j-N}}\pi(S,0) & \\ \qquad\qquad if\, j\ge N+1 \ and\ \rho=1; \end{cases} \tag{4}$$

$$\pi(0,j)=\begin{cases}\dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\mu}+N+1\right)_{j-(N+1)}\left(\frac{\lambda}{\xi}\right)^{j-N}\rho^{N}}{(N+1)_{j-N}\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)_{j-N}}\pi(S,0) & \\ \qquad\qquad if\, j\ge N+1,\ and\ \rho\neq 1, \\ \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\mu}+N+1\right)_{j-(N+1)}\left(\frac{\lambda}{\xi}\right)^{j-N}}{(N+1)_{j-N}\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)_{j-N}}\pi(S,0) & \\ \qquad\qquad if\, j\ge N+1,\ and\ \rho=1; \end{cases} \tag{5}$$

where $\rho=\frac{\lambda}{\nu}$ *and* F *represents the generalized hypergeometric series (Gradshteyn and Ryzhik, 2000) defined as*

$$F\left(\ \alpha_1,\alpha_2,\ldots,\alpha_p;\ \beta_1,\beta_2,\ldots,\beta_q;\quad z\ \right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n}\frac{z^n}{n!},\ |\,z\,|<\infty,$$

in which no denominator parameter β_j *is allowed to be* 0 *or a negative integer and* $(x)_n$ *is* *the Pochhammer symbol defined by*

$$(x)_n = \begin{cases} 1 & \textit{for } n = 0, \\ x(x+1)(x+2)\ldots(x+n-1) & \textit{for } n \geq 1. \end{cases}$$

PROOF. From the state transition diagram of Figure 1, we get the set of balance equations as follows:

$$\beta\,\pi(S,0) = \eta\,\pi(C,0), \tag{6}$$
$$(\lambda+\eta)\,\pi(C,0) = \nu\,\pi(1,0), \tag{7}$$
$$\lambda\,\pi(0,0) = \beta\,\pi(S,0), \tag{8}$$
$$(\lambda+j\mu)\,\pi(0,j) = \nu\,\pi(1,j), \qquad j \geq N+1, \tag{9}$$
$$(\lambda+\nu)\,\pi(1,0) = \lambda\,\pi(0,0) + \nu\,\pi(1,1) + \lambda\,\pi(C,0), \tag{10}$$

$$(\lambda+\nu)\,\pi(1,j) = \lambda\,\pi(1,j-1) + \nu\,\pi(1,j+1),\ 1 \leq j \leq N-1, \tag{11}$$

$$\begin{aligned}(\lambda+\nu)\,\pi(1,N) = {} & \lambda\,\pi(1,N-1) + (N+1)\mu\,\pi(0,N+1) \\ & + (N+1)\,\xi\,\pi(1,N+1),\end{aligned} \tag{12}$$
$$\begin{aligned}(\lambda+\nu+j\xi)\,\pi(1,j) = {} & \lambda\,\pi(1,j-1) + (j+1)\,\mu\,\pi(0,j+1) \\ & + \lambda\,\pi(0,j) + (j+1)\xi\,\pi(1,j+1),\ j \geq N+1,\end{aligned} \tag{13}$$

and the normalizing condition

$$\pi(S,0) + \pi(C,0) + \pi(0,0) + \sum_{j=N+1}^{\infty} \pi(0,j) + \sum_{j=0}^{\infty} \pi(1,j) = 1. \tag{14}$$

From Eq. (6) and Eq. (8), we get

$$\pi(C,0) = \frac{\beta}{\eta}\,\pi(S,0) \tag{15}$$

and

$$\pi(0,0) = \frac{\beta}{\lambda}\,\pi(S,0). \tag{16}$$

Making use of Eq. (15) in Eq. (7) yields

$$\pi(1,0) = \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\pi(S,0). \tag{17}$$

By virtue of Equations (15)-(17), the joint probabilities $\pi(1,j)$, $j=1,2,3,\ldots,N$, can be computed recursively from Eq. (10) and Eq. (11) as

$$\pi(1,j) \quad = \quad \begin{cases} \left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\nu}\right)\rho^{j}\,\pi(S,0), & \text{if } 0\leq j\leq N,\ \rho\neq 1,\\ \left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\lambda}\right)\pi(S,0), & \text{if } 0\leq j\leq N,\ \rho=1. \end{cases}$$

Thus, the steady-state joint probabilities $\pi(1,j)$, $0\leq j\leq N$, given in Eq. (4), have been determined.

Next, from Eq. (9), we get the relation

$$\pi(1,j)=\frac{1}{\nu}(\lambda+j\mu)\,\pi(0,j),\ \ j\geq N+1. \tag{18}$$

Plugging Eq. (18) into Eq. (13), after a little algebra, results in

$$\begin{aligned}(j+1)[\mu\nu+\lambda\xi+(j+1)\mu\xi]\,\pi(0,j+1)-\lambda[\lambda+j\mu]\,\pi(0,j)\\ =(j+2)[\mu\nu+\lambda\xi+(j+2)\mu\xi]\,\pi(0,j+2)\\ -\lambda[\lambda+(j+1)\mu]\,\pi(0,j+1),\quad j\geq N+1.\end{aligned} \tag{19}$$

The above can be expressed as

$$x_{j+1}\,\pi(0,j+1)-y_j\,\pi(0,j)=x_{j+2}\,\pi(0,j+2)-y_{j+1}\,\pi(0,j+1),\quad j\geq N+1, \tag{20}$$

where

$$x_j=j[\mu\nu+\lambda\xi+j\mu\xi]\quad\text{and}\quad y_j=\lambda[\lambda+j\mu]\ \text{for}\ j\geq N+1.$$

Thus, from Eq. (20), we conclude that for $j\geq N+1$,

$$x_{j+1}\,\pi(0,j+1)-y_j\,\pi(0,j)=K, \tag{21}$$

where K is an arbitrary constant.
Now, substituting Eq. (9) into Eq. (21), after simplification, we obtain

$$j[\mu\nu+\lambda\xi+j\mu\xi]\,\pi(0,j)-\lambda\nu\,\pi(1,j-1)=K. \tag{22}$$

Replacing j by $N+1$ in Eq. (22) and using Eq. (12) in the resulting expression, we get

$$\nu^2\,\pi(1,N)-\lambda\nu\,\pi(1,N-1)=K. \tag{23}$$

By inserting Eq. (4) into Eq. (23), we determine the unknown constant $K=0$. Therefore, for $K=0$, Eq. (21) implies that

$$\pi(0,j+1)=\frac{\lambda[\lambda+j\mu]}{(j+1)[\mu\nu+\lambda\xi+(j+1)\mu\xi]}\pi(0,j),\quad j\geq N+1. \tag{24}$$

By making use of Equations (4) and (18) in Eq. (12) leads to

$$\pi(0,N+1)=\begin{cases}\dfrac{\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\xi}\right)\rho^N}{(N+1)\left[\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right]}\pi(S,0), & \text{if } \rho\neq 1,\\[2ex] \dfrac{\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\xi}\right)}{(N+1)\left[\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+1\right]}\pi(S,0), & \text{if } \rho=1.\end{cases} \tag{25}$$

Usage of the above expression in Eq. (24) recursively, we arrive to the required result of Eq. (5).

We now use Equations (25) and (5) in Eq. (18) recursively, the expressions for $\pi(1,j)$, $j\geq N+1$, given in Eq. (4), are obtained. Subsequently, the unknown joint steady-state probability $\pi(S,0)$ in Eq. (1) is determined by utilizing the normalization condition in Eq. (14).

Finally, with the help of Equations (15), (16) and (1), the required steady-state joint probabilities $\pi(0,0)$ and $\pi(C,0)$ are obtained by Equations (2) and (3), respectively. Hence the proof is completed. □

COROLLARY 2. *If $\mu\to\infty$, then the stationary joint probabilities for the conventional queueing system with occurrence of customers' impatience phenomenon above the threshold value N subject to switch-off and close-down periods can be deduced from Equations (1)-(5) of Theorem 1 as*

$$\pi(S,0)=\begin{cases}\dfrac{\dfrac{1}{\beta}}{\left\{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho\left[\dfrac{1-\rho^N}{1-\rho}+\rho^N F\left(1;\dfrac{\nu}{\xi}+N+1;\dfrac{\lambda}{\xi}\right)\right]\right\}} & \textit{if } \rho\neq 1,\\[2ex] \dfrac{\dfrac{1}{\beta}}{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\left[N+F\left(1;\dfrac{\lambda}{\xi}+N+1;\dfrac{\lambda}{\xi}\right)\right]} & \textit{if } \rho=1;\end{cases} \tag{26}$$

$$\pi(0,0) = \begin{cases} \dfrac{\frac{1}{\lambda}}{\left\{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\rho\left[\frac{1-\rho^N}{1-\rho}+\rho^N F\left(1;\frac{\nu}{\xi}+N+1;\frac{\lambda}{\xi}\right)\right]\right\}} & if\, \rho \neq 1, \\ \dfrac{\frac{1}{\lambda}}{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\left[N+F\left(1;\frac{\lambda}{\xi}+N+1;\frac{\lambda}{\xi}\right)\right]} & if\, \rho = 1; \end{cases} \tag{27}$$

$$\pi(C,0) = \begin{cases} \dfrac{\frac{1}{\eta}}{\left\{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\rho\left[\frac{1-\rho^N}{1-\rho}+\rho^N F\left(1;\frac{\nu}{\xi}+N+1;\frac{\lambda}{\xi}\right)\right]\right\}} & if\, \rho \neq 1, \\ \dfrac{\frac{1}{\eta}}{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\left[N+F\left(1;\frac{\lambda}{\xi}+N+1;\frac{\lambda}{\xi}\right)\right]} & if\, \rho = 1; \end{cases} \tag{28}$$

$$\pi(1,j) = \begin{cases} \dfrac{\left(\frac{1}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\rho^j}{\left\{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\rho\left[\frac{1-\rho^N}{1-\rho}+\rho^N F\left(1;\frac{\nu}{\xi}+N+1;\frac{\lambda}{\xi}\right)\right]\right\}} & if\, 0 \leq j \leq N,\ \rho \neq 1, \\ \dfrac{\left(\frac{1}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)}{\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\left[N+F\left(1;\frac{\lambda}{\xi}+N+1;\frac{\lambda}{\xi}\right)\right]} & if\, 0 \leq j \leq N,\ \rho = 1; \end{cases} \tag{29}$$

$$\pi(1,j) = \begin{cases} \dfrac{\left(\dfrac{1}{\eta}\right)\left(\dfrac{\lambda+\eta}{\nu}\right) \dfrac{1}{\left(\dfrac{\nu}{\xi}+N+1\right)_{j-N}} \left(\dfrac{\lambda}{\xi}\right)^{j-N} \rho^N}{\left\{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho\left[\dfrac{1-\rho^N}{1-\rho}+\rho^N F\left(1;\dfrac{\nu}{\xi}+N+1;\dfrac{\lambda}{\xi}\right)\right]\right\}} & \text{if } j \geq N+1,\ \rho \neq 1, \\ \dfrac{\left(\dfrac{1}{\eta}\right)\left(\dfrac{\lambda+\eta}{\lambda}\right) \dfrac{1}{\left(\dfrac{\lambda}{\xi}+N+1\right)_{j-N}} \left(\dfrac{\lambda}{\xi}\right)^{j-N}}{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\left[N+F\left(1;\dfrac{\lambda}{\xi}+N+1;\dfrac{\lambda}{\xi}\right)\right]} & \text{if } j \geq N+1,\ \rho = 1. \end{cases} \tag{30}$$

PROOF. The proof follows by some mathematical manipulation and thus we omitted the details. □

COROLLARY 3. *If $N = 0$, the results reported in Theorem 1 turn out be the joint steady-state probabilities of the classical retrial queue with impatient customers subject to switch-off and close-down periods as*

$$\pi(S,0) = \begin{cases} \dfrac{1}{\beta}\left\{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho\left[1+\dfrac{\left(\dfrac{\lambda}{\xi}\right)\left(\dfrac{\lambda+\nu}{\mu}+1\right)}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+1\right)}\right.\right. \\ \left.\left.\times F\left(1,\dfrac{\lambda}{\mu}+1,\dfrac{\lambda+\nu}{\mu}+2;2,\ \dfrac{\lambda+\nu}{\mu}+1,\ \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+2;\dfrac{\lambda}{\xi}\right)\right]\right\}^{-1} \\ \qquad \text{if } \rho \neq 1, \end{cases} \tag{31}$$

$$\pi(S,0) = \begin{cases} \dfrac{1}{\beta}\left\{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\left[1+\dfrac{\left(\dfrac{\lambda}{\xi}\right)\left(\dfrac{2\lambda}{\mu}+1\right)}{\left(\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+1\right)}\right.\right. \\ \left.\left.\times F\left(1,\dfrac{\lambda}{\mu}+1,\dfrac{2\lambda}{\mu}+2;2,\dfrac{2\lambda}{\mu}+1,\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+2;\dfrac{\lambda}{\xi}\right)\right]\right\}^{-1} \\ \qquad \text{if } \rho = 1, \end{cases}$$

$$\pi(0,0)=\begin{cases}\dfrac{1}{\lambda}\left\{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho\left[1+\dfrac{\left(\dfrac{\lambda}{\xi}\right)\left(\dfrac{\lambda+\nu}{\mu}+1\right)}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+1\right)}\right.\right.\\ \left.\left.\times F\left(1,\dfrac{\lambda}{\mu}+1,\dfrac{\lambda+\nu}{\mu}+2;2,\ \dfrac{\lambda+\nu}{\mu}+1,\ \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+2;\dfrac{\lambda}{\xi}\right)\right]\right\}^{-1} \\ \hfill if\ \rho\neq 1,\\ \dfrac{1}{\lambda}\left\{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\left[1+\dfrac{\left(\dfrac{\lambda}{\xi}\right)\left(\dfrac{2\lambda}{\mu}+1\right)}{\left(\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+1\right)}\right.\right.\\ \left.\left.\times F\left(1,\dfrac{\lambda}{\mu}+1,\dfrac{2\lambda}{\mu}+2;2,\dfrac{2\lambda}{\mu}+1,\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+2;\dfrac{\lambda}{\xi}\right)\right]\right\}^{-1} \\ \hfill if\ \rho=1,\end{cases} \tag{32}$$

$$\pi(C,0)=\begin{cases}\dfrac{1}{\eta}\left\{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho\left[1+\dfrac{\left(\dfrac{\lambda}{\xi}\right)\left(\dfrac{\lambda+\nu}{\mu}+1\right)}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+1\right)}\right.\right.\\ \left.\left.\times F\left(1,\dfrac{\lambda}{\mu}+1,\dfrac{\lambda+\nu}{\mu}+2;2,\ \dfrac{\lambda+\nu}{\mu}+1,\ \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+2;\dfrac{\lambda}{\xi}\right)\right]\right\}^{-1} \\ \hfill if\ \rho\neq 1,\\ \dfrac{1}{\eta}\left\{\dfrac{1}{\beta}+\dfrac{1}{\eta}+\dfrac{1}{\lambda}+\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\left[1+\dfrac{\left(\dfrac{\lambda}{\xi}\right)\left(\dfrac{2\lambda}{\mu}+1\right)}{\left(\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+1\right)}\right.\right.\\ \left.\left.\times F\left(1,\dfrac{\lambda}{\mu}+1,\dfrac{2\lambda}{\mu}+2;2,\dfrac{2\lambda}{\mu}+1,\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+2;\dfrac{\lambda}{\xi}\right)\right]\right\}^{-1} \\ \hfill if\ \rho=1,\end{cases} \tag{33}$$

$$\pi(1,j)=\begin{cases}\dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\left(\frac{\lambda}{\mu}+1\right)_j\left(\frac{\lambda}{\xi}\right)^j}{j!\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+1\right)_j}\pi(S,0), \; j\geq 0 \\ \qquad\qquad\qquad\qquad\qquad \textit{if}\, \rho\neq 1, \\ \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\left(\frac{\lambda}{\mu}+1\right)_j\left(\frac{\lambda}{\xi}\right)^j}{j!\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+1\right)_j}\pi(S,0), \; j\geq 0 \\ \qquad\qquad\qquad\qquad\qquad \textit{if}\, \rho\neq 1,\end{cases} \tag{34}$$

$$\pi(0,j)=\begin{cases}\dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\mu}+1\right)_{j-1}\left(\frac{\lambda}{\xi}\right)^j}{j!\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+1\right)_j}\pi(S,0), \; j\geq 1 \\ \qquad\qquad\qquad\qquad\qquad \textit{if}\, \rho\neq 1, \\ \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\mu}+1\right)_{j-1}\left(\frac{\lambda}{\xi}\right)^j}{j!\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+1\right)_j}\pi(S,0), \; j\geq 1 \\ \qquad\qquad\qquad\qquad\qquad \textit{if}\, \rho=1.\end{cases} \tag{35}$$

PROOF. It is straightforward procedure to derive the above results and thus we omitted the details for brevity. □

REMARK 4. *In the case of $\rho<1$, $N\to\infty$, the steady-state joint probability distribution of Theorem 1, boils down to the steady-state probability distribution studied in Theorem 4 of Kumar* et al. *(2015).*

REMARK 5. *As far as we know, the results presented in Corollaries 2 and 3 have not been investigated earlier in the literature.*

4. FACTORIAL MOMENTS

In the present Section, we determine the partial factorial moments for our dynamic oscillating service system in the steady-state regime.

Define the partial generating functions for joint probabilities $\pi(C,0)$, $\pi(0,0)$, $\pi(0,j)$, $j \geq N+1$ and $\pi(1,j)$, $j \geq 0$, as

$$P_0(z) = \pi(C,0) + \pi(0,0) + \sum_{j=N+1}^{\infty} \pi(0,j)\, z^j$$

and

$$P_1(z) = \sum_{j=0}^{\infty} \pi(1,j)\, z^j, \quad |z| \leq 1.$$

On account of Equations from (2) to (5), one can obtain $P_0(z)$ and $P_1(z)$ as

$$P_0(z) = \begin{cases} \left[\dfrac{\beta}{\eta} + \dfrac{\beta}{\lambda} + \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\xi}\right)\rho^N z^{N+1}}{(N+1)\left(\frac{\nu}{\xi} + \frac{\lambda}{\mu} + N + 1\right)} \right. \\ \quad \left. \times F\left(1, \dfrac{\lambda}{\mu} + N + 1; N+2, \dfrac{\nu}{\xi} + \dfrac{\lambda}{\mu} + N + 2; \dfrac{\lambda z}{\xi}\right)\right] \pi(S,0) & \text{if } \rho \neq 1, \\ \left[\dfrac{\beta}{\eta} + \dfrac{\beta}{\lambda} + \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\xi}\right) z^{N+1}}{(N+1)\left(\lambda\left(\frac{1}{\xi} + \frac{1}{\mu}\right) + N + 1\right)} \right. \\ \quad \left. \times F\left(1, \dfrac{\lambda}{\mu} + N + 1; N+2, \lambda\left(\dfrac{1}{\xi} + \dfrac{1}{\mu}\right) + N + 2; \dfrac{\lambda z}{\xi}\right)\right] \pi(S,0) & \text{if } \rho = 1 \end{cases} \tag{36}$$

and

$$P_1(z) = \begin{cases} \left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\nu}\right)\left[\dfrac{1-(\rho z)^{N+1}}{1-\rho z} + \dfrac{\left(\frac{\lambda}{\mu} + N + 1\right)\left(\frac{\lambda}{\xi}\right)\rho^N z^{N+1}}{(N+1)\left(\frac{\nu}{\xi} + \frac{\lambda}{\mu} + N + 1\right)} \right. \\ \quad \left. \times F\left(1, \dfrac{\lambda}{\mu} + N + 2; N+2, \dfrac{\nu}{\xi} + \dfrac{\lambda}{\mu} + N + 2; \dfrac{\lambda z}{\xi}\right)\right] \pi(S,0) & \text{if } \rho \neq 1, \\ \left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\lambda}\right)\left[\dfrac{1-z^{N+1}}{1-z} + \dfrac{\left(\frac{\lambda}{\mu} + N + 1\right)\left(\frac{\lambda}{\xi}\right) z^{N+1}}{(N+1)\left(\lambda\left(\frac{1}{\xi} + \frac{1}{\mu}\right) + N + 1\right)} \right. \\ \quad \left. \times F\left(1, \dfrac{\lambda}{\mu} + N + 2; N+2, \lambda\left(\dfrac{1}{\xi} + \dfrac{1}{\mu}\right) + N + 2; \dfrac{\lambda z}{\xi}\right)\right] \pi(S,0) & \text{if } \rho = 1, \end{cases} \tag{37}$$

where $\pi(S,0)$ is as given in Eq. (1).

Our next aim is to compute explicit expressions for the partial factorial moments for the dynamic control oscillating service system with impatient customers.

THEOREM 6. *Let M_k^i, for $i \in \{0,1\}$ and $k \geq 1$, be the partial k^{th} factorial moments of the number of customers in the orbit given that the status of the server is i, defined as*

$$M_k^i = \sum_{j=k}^{\infty} j(j-1)\dots(j-k+1)\,\pi(i,j).$$

The partial k^{th} factorial moments for $k \geq 1$ are:

- *for $\rho \neq 1$*

$$M_k^0 = \begin{cases} \left[\dfrac{N!}{(N-k+1)!}\dfrac{\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\xi}\right)\rho^N}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right)} \right. \\ \left. \times F\left(1,\dfrac{\lambda}{\mu}+N+1;N-k+2,\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+2;\dfrac{\lambda}{\xi}\right)\right]\pi(S,0) \\ \hfill \text{if } k \leq N, \\ \left[\dfrac{N!\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\mu}+N+1\right)_{k-(N+1)}\left(\dfrac{\lambda}{\xi}\right)^{k-N}\rho^N}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right)_{k-N}} \right. \\ \left. \times F\left(\dfrac{\lambda}{\mu}+k;\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+k+1;\dfrac{\lambda}{\xi}\right)\right]\pi(S,0) \\ \hfill \text{if } k \geq N+1; \end{cases} \tag{38}$$

- *for* $\rho = 1$

$$M_k^0 = \begin{cases} \left[\dfrac{N!}{(N-k+1)!} \dfrac{\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\xi}\right)}{\left(\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+1\right)} \right. \\ \left. \times F\left(1, \dfrac{\lambda}{\mu}+N+1; N-k+2, \lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+2; \dfrac{\lambda}{\xi}\right)\right] \pi(S,0) \\ \hfill if\, k \le N, \\ \left[\dfrac{N!\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\mu}+N+1\right)_{k-(N+1)}\left(\dfrac{\lambda}{\xi}\right)^{k-N}}{\left(\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+1\right)_{k-N}} \right. \\ \left. \times F\left(\dfrac{\lambda}{\mu}+k; \lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+k+1; \dfrac{\lambda}{\xi}\right)\right] \pi(S,0) \\ \hfill if\, k \ge N+1; \end{cases} \tag{39}$$

- *for* $\rho \neq 1$

$$M_k^1 = \begin{cases} \left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\nu}\right)\left[\displaystyle\sum_{m=k}^{N} \dfrac{m!\,\rho^m}{(m-k)!} + \dfrac{N!}{(N-k+1)!} \dfrac{\left(\dfrac{\lambda}{\mu}+N+1\right)\left(\dfrac{\lambda}{\xi}\right)\rho^N}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right)} \right. \\ \left. \times F\left(1, \dfrac{\lambda}{\mu}+N+2; N-k+2, \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+2; \dfrac{\lambda}{\xi}\right)\right] \pi(S,0) \\ \hfill if\, k \le N, \end{cases}$$

$$M_k^1 = \begin{cases} \left[N!\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\nu}\right) \dfrac{\left(\dfrac{\lambda}{\mu}+N+1\right)_{k-N}\left(\dfrac{\lambda}{\xi}\right)^{k-N}\rho^N}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right)_{k-N}} \right. \\ \left. \times F\left(\dfrac{\lambda}{\mu}+k+1; \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+k+1; \dfrac{\lambda}{\xi}\right)\right] \pi(S,0) \\ \hfill if\, k \ge N+1; \end{cases} \tag{40}$$

- *for* $\rho = 1$

$$M_k^1 = \begin{cases} \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\Bigg[\sum_{m=k}^{N}\frac{m!}{(m-k)!} + \frac{N!}{(N-k+1)!}\frac{\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)}{\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)} \\ F\left(1,\frac{\lambda}{\mu}+N+2;N-k+2,\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2;\frac{\lambda}{\xi}\right)\Bigg]\pi(S,0) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad if\, k \le N, \\ \Bigg[N!\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\frac{\left(\frac{\lambda}{\mu}+N+1\right)_{k-N}\left(\frac{\lambda}{\xi}\right)^{k-N}}{\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)_{k-N}} \\ \quad \times F\left(\frac{\lambda}{\mu}+k+1;\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+k+1;\frac{\lambda}{\xi}\right)\Bigg]\pi(S,0) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad if\, k \ge N+1. \end{cases} \tag{41}$$

Besides, the k^{th} *factorial moments,* M_k*, of the number of customers in the orbit are derived as*

$$M_k = M_k^0 + M_k^1, \quad k = 1,2,3,\ldots.$$

PROOF. The proof follows by taking successive derivatives of Equations (36) and (37) for $k-$times with respect to z at $z=1$ and after tedious algebraic calculations. □

Specifically, by the results of the Theorem 6, one can prove immediately that the first and second partial factorial moments for the system under consideration as

$$M_1^0 = P_0^{'}(1) = \begin{cases} \Bigg[\frac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\xi}\right)\rho^N}{\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)} \\ \times F\left(1,\frac{\lambda}{\mu}+N+1;N+1,\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2;\frac{\lambda}{\xi}\right)\Bigg]\pi(S,0), \qquad \text{if } \rho \ne 1, \end{cases}$$

$$M_1^0 = P_0^{'}(1) = \begin{cases} \left[\dfrac{\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\xi}\right)}{\left(\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+1\right)} \right. \\ \left. \times F\left(1, \dfrac{\lambda}{\mu}+N+1; N+1,\ \lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+2; \dfrac{\lambda}{\xi}\right)\right] \pi(S,0), \text{ if } \rho = 1, \end{cases} \tag{42}$$

$$M_1^1 = P_1^{'}(1) = \begin{cases} \left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\nu}\right)\left[\dfrac{\rho}{(1-\rho)^2}\left(1-(N+1)\rho^N + N\rho^{N+1}\right) + \dfrac{\left(\dfrac{\lambda}{\mu}+N+1\right)\left(\dfrac{\lambda}{\xi}\right)\rho^N}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right)}\right. \\ \left. \times F\left(1, \dfrac{\lambda}{\mu}+N+2; N+1, \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+2; \dfrac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho \neq 1, \\ \left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\lambda}\right)\left[\dfrac{N(N+1)}{2} + \dfrac{\left(\dfrac{\lambda}{\mu}+N+1\right)\left(\dfrac{\lambda}{\xi}\right)}{\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+1}\right. \\ \left. \times F\left(1, \dfrac{\lambda}{\mu}+N+2; N+1,\ \lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+2; \dfrac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho = 1, \end{cases} \tag{43}$$

$$M_2^0 = P_0^{''}(1) = \begin{cases} \left[\dfrac{N\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\xi}\right)\rho^N}{\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right)}\right. \\ \left. \times F\left(1, \dfrac{\lambda}{\mu}+N+1; N, \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+2; \dfrac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho \neq 1, \\ \left[\dfrac{N\left(\dfrac{\beta}{\eta}\right)\left(\dfrac{\lambda+\eta}{\mu}\right)\left(\dfrac{\lambda}{\xi}\right)}{\left(\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+1\right)}\right. \\ \left. \times F\left(1, \dfrac{\lambda}{\mu}+N+1; N, \lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+2; \dfrac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho = 1, \end{cases} \tag{44}$$

$$M_2^1 = P_1''(1) = \begin{cases} \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\left[\sum_{m=2}^{N} \frac{m!\rho^m}{(m-2)!} + \frac{N\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)\rho^N}{\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)} \right. \\ \left. \times F\left(1, \frac{\lambda}{\mu}+N+2; N, \frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2; \frac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho \neq 1, \\ \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\left[\sum_{m=2}^{N} \frac{m!}{(m-2)!} + \frac{N\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)}{\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1} \right. \\ \left. \times F\left(1, \frac{\lambda}{\mu}+N+2; N, \lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2; \frac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho = 1, \end{cases} \tag{45}$$

where $\pi(S,0)$ is as given in Eq. (1). In particular, the mean and variance of the number of customers in the orbit are vital performance measures for the system under study. These average measures can be obtained immediately from the above results.

5. Performance Measures

In this Section, we present some key performance measures which help us to make a detailed study about the oscillating queueing system under consideration.

1. P(Server is busy)= $P_1(1)$

$$= \begin{cases} \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\left[\frac{(1-\rho^{N+1})}{(1-\rho)} + \frac{\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)\rho^N}{(N+1)\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)} \right. \\ \left. \times F\left(1, \frac{\lambda}{\mu}+N+2; N+2, \frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2; \frac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho \neq 1, \\ \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\left[(N+1) + \frac{\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)} \right. \\ \left. \times F\left(1, \frac{\lambda}{\mu}+N+2; N+2, \lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2; \frac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho = 1. \end{cases}$$

2. P(Server is available) = 1−$\pi(S,0)$.

3. P(Primary customer enters retrial orbit upon arrival) $=\sum_{j=N}^{\infty}\pi(1,j)$

$$=\begin{cases}\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\rho^N\left[1+\frac{\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)}{(N+1)\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)}\right.\\ \left.\times F\left(1,\frac{\lambda}{\mu}+N+2;N+2,\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2;\frac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho\neq 1,\\ \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\left[1+\frac{\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)}\right.\\ \left.\times F\left(1,\frac{\lambda}{\mu}+N+2;N+2,\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2;\frac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho=1.\end{cases}$$

4. P(Primary customer captures the server upon arrival)$=P_0(1)$

$$=\pi(C,0)+\pi(0,0)+\sum_{j=N+1}^{\infty}\pi(0,j)$$

$$=\begin{cases}\left[\frac{\beta}{\eta}+\frac{\beta}{\lambda}+\frac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\xi}\right)\rho^N}{(N+1)\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)}\right.\\ \left.\times F\left(1,\frac{\lambda}{\mu}+N+1;N+2,\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2;\frac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho\neq 1,\\ \left[\frac{\beta}{\eta}+\frac{\beta}{\lambda}+\frac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\mu}\right)\left(\frac{\lambda}{\xi}\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)}\right.\\ \left.\times F\left(1,\frac{\lambda}{\mu}+N+1;N+2,\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2;\frac{\lambda}{\xi}\right)\right]\pi(S,0), & \text{if } \rho=1.\end{cases}$$

5. The mean number, $E(X_Q)$, of customers in the orbit is given by

$$E(X_Q)=M_1^0+M_1^1.$$

6. The mean number, $E(X_S)$, of customers in the system is determined as

$$E(X_S) = M_1^0 + M_1^1 + P_1(1).$$

7. The expected number, U, of customers served per unit of time is

$$U = \nu \sum_{j=0}^{\infty} \pi(1, j) = \nu P_1(1).$$

8. The effective arrival rate (i.e., the total arrival rate while the server is available) is given as

$$\lambda_{eff} = \lambda[1 - \pi(S, 0)].$$

9. The total sojourn time, W_S, of a customer in the system is measured from the moment of its primary arrival until departure, either by completion of service or as a result of abandonment. By Little's law, the expected sojourn time $E(W_S)$ in the system is obtained as

$$\begin{aligned} E(W_S) &= \frac{1}{\lambda_{eff}} \left[M_1^0 + M_1^1 + P_1(1) \right] \\ &= \frac{E(X_S)}{\lambda[1 - \pi(S, 0)]}. \end{aligned}$$

10. The rate, R_A, of reneging/abandonment due to impatience is defined by

$$R_A = \lambda_{eff} - \nu P_1'(1) = \xi \sum_{j=N+1}^{\infty} j \pi(1, j)$$

$$= \begin{cases} \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right) \frac{\left(\frac{\lambda}{\mu} + N + 1\right)\lambda\rho^N}{\left(\frac{\nu}{\xi} + \frac{\lambda}{\mu} + N + 1\right)} \\ \quad \times F\left(1, \frac{\lambda}{\mu} + N + 2; N + 1, \frac{\nu}{\xi} + \frac{\lambda}{\mu} + N + 2; \frac{\lambda}{\xi}\right) \pi(S, 0), & \text{if } \rho \neq 1, \\ \left(\frac{\beta}{\eta}\right) \frac{(\lambda+\eta)\left(\frac{\lambda}{\mu} + N + 1\right)}{\left(\lambda\left(\frac{1}{\xi} + \frac{1}{\mu}\right) + N + 1\right)} \\ \quad \times F\left(1, \frac{\lambda}{\mu} + N + 2; N + 1, \lambda\left(\frac{1}{\xi} + \frac{1}{\mu}\right) + N + 2; \frac{\lambda}{\xi}\right) \pi(S, 0), & \text{if } \rho = 1. \end{cases}$$

11. The proportion of customers served is calculated as

$$P_S = \frac{\nu P_1(1)}{\lambda_{eff}} = \frac{P_1(1)}{\rho[1-\pi(S,0)]}$$

$$= \begin{cases} \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)}{1-\pi(S,0)}\left[\dfrac{1-\rho^{N+1}}{1-\rho} + \dfrac{\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)\rho^N}{(N+1)\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)} \right. \\ \left. \times F\left(1, \frac{\lambda}{\mu}+N+2; N+2, \frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2; \frac{\lambda}{\xi}\right)\right] \pi(S,0), & \text{if } \rho \neq 1, \\ \dfrac{\left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)}{1-\pi(S,0)}\left[(N+1) + \dfrac{\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)} \right. \\ \left. \times F\left(1, \frac{\lambda}{\mu}+N+2; N+2, \lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2; \frac{\lambda}{\xi}\right)\right] \pi(S,0), & \text{if } \rho = 1. \end{cases}$$

12. Let $E(X_{CQ})$ be the expected number of customers in the conventional queueing system before it becomes the retrial queueing system mode, i.e., before reaching state $(1,N)$. This quantity is measured as

$$E(X_{CQ}) = \sum_{j=0}^{N-1}(j+1)\,\pi(1,j) = \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\pi(S,0)\sum_{j=0}^{N-1}(j+1)\,\rho^j,$$

where we have used the results given in Eq. (4). Hence, after using a little algebra and calculus, the above expression yields that

$$E(X_{CQ}) = \begin{cases} \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\nu}\right)\left[\dfrac{1+N\rho^{N+1}-N\rho^N-\rho^N}{(1-\rho)^2}\right]\pi(S,0), & \text{if } \rho \neq 1, \\ \left(\frac{\beta}{\eta}\right)\left(\frac{\lambda+\eta}{\lambda}\right)\dfrac{N(N+1)}{2}\,\pi(S,0), & \text{if } \rho = 1. \end{cases}$$

6. Regenerative Cycle Analysis of the System

We now investigate the regenerative cycle length and its related characteristics for our dynamic service system in which the server is subject to switch-off and close-down periods. Indeed, the regenerative approach is a versatile technique to compute the equilibrium distribution of the state probabilities for a wide variety of state-dependent arrival and service rates queueing systems in tractable forms (Tijms, 1994).

1. The regenerative cycle, T, of our system is defined as the length of time interval between two consecutive primary customer arrivals finding the server in operative idle state and there is no customer in the system, i.e., at state $(0,0)$. Hence, the mean length of the regenerative cycle of our oscillating service system is

$$E(T)=\frac{\frac{1}{\lambda}}{\pi(0,0)}$$
$$=\begin{cases} \frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\rho\left[\frac{1-\rho^{N+1}}{1-\rho}+\frac{\left(\frac{\lambda}{\xi}\right)\left(\frac{\lambda+\nu}{\mu}+N+1\right)\rho^{N}}{(N+1)\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)}\right. \\ \left.\times F\left(1,\frac{\lambda}{\mu}+N+1,\frac{\lambda+\nu}{\mu}+N+2;N+2,\frac{\lambda+\nu}{\mu}+N+1,\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+2;\frac{\lambda}{\xi}\right)\right], \\ \qquad \text{if } \rho\neq 1, \\ \frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}+\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\left[(N+1)+\frac{\left(\frac{\lambda}{\xi}\right)\left(\frac{2\lambda}{\mu}+N+1\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)}\right. \\ \left.\times F\left(1,\frac{\lambda}{\mu}+N+1,\frac{2\lambda}{\mu}+N+2;N+2,\frac{2\lambda}{\mu}+N+1,\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2;\frac{\lambda}{\xi}\right)\right], \\ \qquad \text{if } \rho=1. \end{cases}$$

2. The expected length, $E(T_{(S,0)})$, of server's switch-off period during the mean regenerative cycle is computed as

$$E[T_{(S,0)}]=E[T]\,\pi(S,0)=\frac{1}{\beta}.$$

3. The expected length, $E(T_{(C,0)})$, of server's close-down period during the mean regenerative cycle is given by

$$E[T_{(C,0)}]=E[T]\,\pi(C,0)=\frac{1}{\eta}.$$

4. The expected length, $E(T_{(0,0)})$, of the server's idle time in the conventional queueing mode while the system is empty during the mean regenerative cycle is calculated as

$$E[T_{(0,0)}]=E[T]\,\pi(0,0)=\frac{1}{\lambda}.$$

5. The expected length, $E(T_{IPNEO})$, of the server's idle time in the retrial queueing mode with nonempty orbit during the mean regenerative cycle is obtained by

$$E[T_{IPNEO}] = E[T] \sum_{j=N+1}^{\infty} \pi(0,j)$$

$$= \begin{cases} \dfrac{\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho^{N+1}\left(\dfrac{\lambda}{\mu}\right)\left(\dfrac{\nu}{\xi}\right)}{(N+1)\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right)} \times F\left(1, \dfrac{\lambda}{\mu}+N+1; N+2, \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+2, \dfrac{\lambda}{\xi}\right), & \text{if } \rho \neq 1, \\ \dfrac{\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\left(\dfrac{\lambda^2}{\mu\xi}\right)}{(N+1)\left(\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+1\right)} \\ \times F\left(1, \dfrac{\lambda}{\mu}+N+1; N+2, \lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+2, \dfrac{\lambda}{\xi}\right), & \text{if } \rho = 1, \end{cases}$$

6. The expected length, $E(T_{CQ})$, of the server being busy in the conventional queueing system before the system turns out to be the retrial queue during the mean regenerative cycle is determined as

$$E[T_{CQ}] = E[T] \sum_{j=0}^{N} \pi(1,j)$$

$$= \begin{cases} \left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho\left(\dfrac{1-\rho^{N+1}}{1-\rho}\right), & \text{if } \rho \neq 1, \\ \left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)(N+1), & \text{if } \rho = 1. \end{cases}$$

7. The expected length, $E(T_{BPNEO})$, of the server being busy in the retrial queueing mode with nonempty orbit during the mean regenerative cycle is derived as

$$E[T_{BPNEO}] = E[T] \sum_{j=N+1}^{\infty} \pi(1,j)$$

$$= \begin{cases} \dfrac{\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho^{N+2}\left(\dfrac{\lambda}{\mu}+N+1\right)\left(\dfrac{\nu}{\xi}\right)}{(N+1)\left(\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+1\right)} \\ \qquad \times F\left(1, \dfrac{\lambda}{\mu}+N+2; N+2, \dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+2, \dfrac{\lambda}{\xi}\right), & \text{if } \rho \neq 1, \end{cases}$$

$$E[T_{BPNEO}]=\begin{cases}\dfrac{\left(\frac{1}{\eta}+\frac{1}{\lambda}\right)\left(\frac{\lambda}{\mu}+N+1\right)\left(\frac{\lambda}{\xi}\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)}\\ \times F\left(1,\dfrac{\lambda}{\mu}+N+2;N+2,\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+2,\dfrac{\lambda}{\xi}\right), \quad \text{if } \rho=1.\end{cases}$$

8. Finally, the busy period, L, of the oscillating service system is defined as the period that starts at the epoch when an arriving primary customer finding the server is in the operative idle state and there is no customer in the system, i.e., in state (0,0) and ends at the epoch where the server leaves for the switch-off state $(S,0)$. Thus, the mean length, $E(L)$, of the system busy period of our dynamic service queueing system is obtained as

$$E(L)=E(T)-\left(\frac{1}{\beta}+\frac{1}{\eta}+\frac{1}{\lambda}\right)$$

$$=\begin{cases}\left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\rho\left[\dfrac{1-\rho^{N+1}}{1-\rho}+\dfrac{\left(\frac{\lambda}{\xi}\right)\left(\frac{\lambda+\nu}{\mu}+N+1\right)\rho^{N}}{(N+1)\left(\frac{\nu}{\xi}+\frac{\lambda}{\mu}+N+1\right)}\right.\\ \left.\times F\left(1,\dfrac{\lambda}{\mu}+N+1,\dfrac{\lambda+\nu}{\mu}+N+2;N+2,\dfrac{\lambda+\nu}{\mu}+N+1,\dfrac{\nu}{\xi}+\dfrac{\lambda}{\mu}+N+2;\dfrac{\lambda}{\xi}\right)\right], \quad \text{if } \rho\neq 1,\\ \left(\dfrac{1}{\eta}+\dfrac{1}{\lambda}\right)\left[(N+1)+\dfrac{\left(\frac{\lambda}{\xi}\right)\left(\frac{2\lambda}{\mu}+N+1\right)}{(N+1)\left(\lambda\left(\frac{1}{\xi}+\frac{1}{\mu}\right)+N+1\right)}\right.\\ \left.\times F\left(1,\dfrac{\lambda}{\mu}+N+1,\dfrac{2\lambda}{\mu}+N+2;N+2,\dfrac{2\lambda}{\mu}+N+1,\lambda\left(\dfrac{1}{\xi}+\dfrac{1}{\mu}\right)+N+2;\dfrac{\lambda}{\xi}\right)\right], \quad \text{if } \rho=1.\end{cases}$$

7. Numerical Illustrations

In this Section, we use some selected probability descriptors and average measures derived previously to bring out the qualitative and quantitative aspects of the proposed dynamic service system through numerical illustrations. To this end, numerical results are presented in the form of three dimensional graphs. In each graph, we have drawn three surfaces which correspond to the threshold value $N=4,8$ and 12.

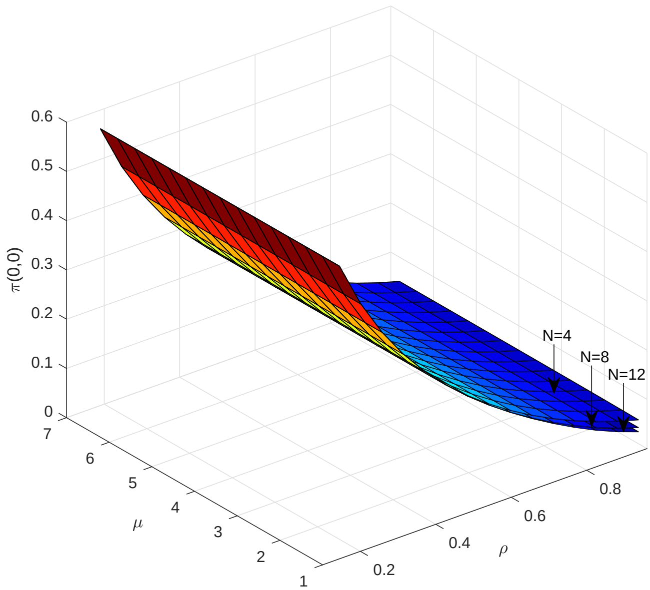

Figure 2 – $\pi(0,0)$ versus (ρ,μ) for $(\xi,\beta,\eta)=(4,5,6)$.

First, we look at the effects of the system parameters μ,ξ,β,η and ρ for the different threshold values of N on the probability descriptor $\pi(0,0)$ in Figures 2-4. We examine the sensitivity of $\pi(0,0)$ against the traffic intensity ρ and the retrial rate μ in Figure 2 by setting the other parameters as $(\xi,\beta,\eta)=(4,5,6)$. It is seen from Figure 2 that all three surfaces of $\pi(0,0)$ decrease sharply as convex functions of ρ, whereas they increase monotonically for increasing values of μ. This is due to the fact that higher the values of ρ cause the more congestion in the system and hence $\pi(0,0)$ is a decreasing function of ρ. But the larger μ results in a shorter mean retrial time $1/\mu$ and thus $\pi(0,0)$ is increasing with μ. For $(\mu,\beta,\eta)=(10,5,6)$, we display the trends of the surfaces of $\pi(0,0)$ versus ρ and ξ in Figure 3. As before, one can observe that as the values of ρ increase, all three surfaces of $\pi(0,0)$ appear to decrease drastically in convex manner. However, the surfaces of $\pi(0,0)$ increase at slower rate for increasing values of the impatient rate ξ. This phenomenon is intuitively true because the reneging of more customers from the orbit resulting in the reduction of congestion in the system and hence there is high chance of the system becomes empty. We now sketch three surfaces corresponding to $\pi(0,0)$ in Figure 4 as functions of μ and ξ by fixing $(\rho,\beta,\eta)=(0.778,5,6)$. As expected, all three surfaces show the increasing trend with increasing values of both μ and ξ. Moreover, it is worth mentioning that all three surfaces of the probability descriptor $\pi(0,0)$ appear to be decreasing functions of the threshold value N.

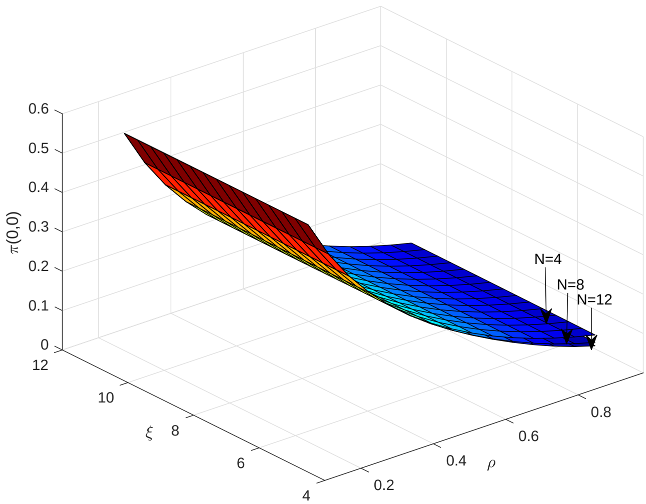

Figure 3 – $\pi(0,0)$ versus (ρ,ξ) for $(\mu,\beta,\eta) = (10,5,6)$.

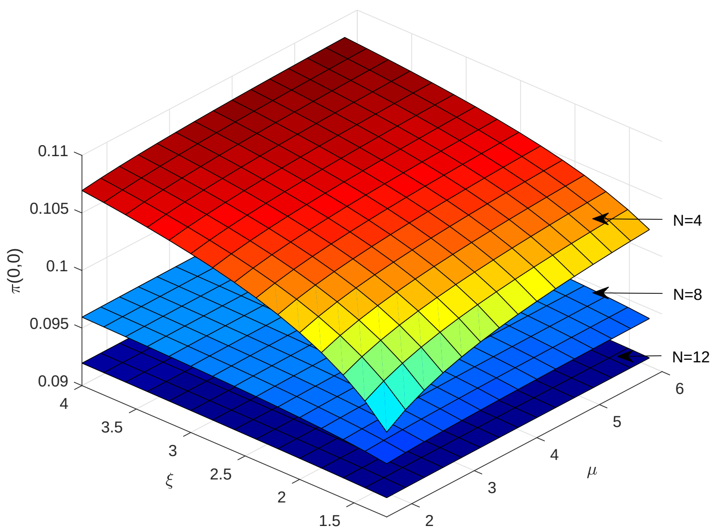

Figure 4 – $\pi(0,0)$ versus (μ,ξ) for $(\rho,\beta,\eta) = (0.778,5,6)$.

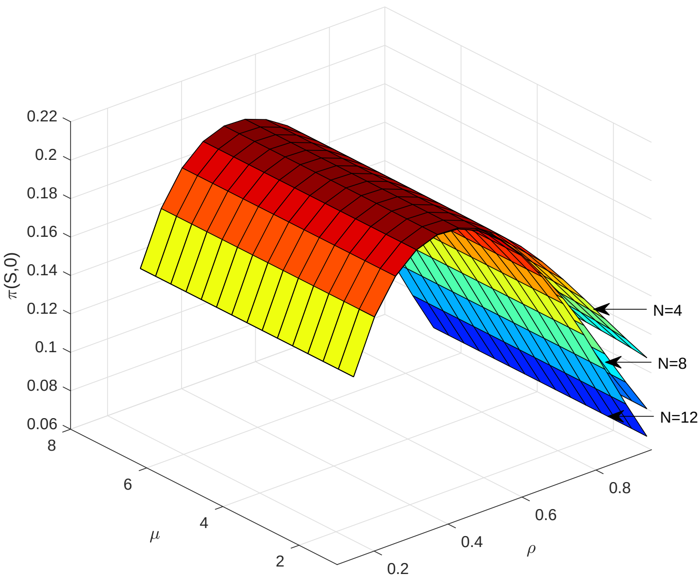

Figure 5 – $\pi(S,0)$ versus (ρ,μ) for $(\xi,\beta,\eta)=(4,5,6)$.

Next, we investigate the behaviour of the probability descriptor $\pi(S,0)$ by varying the values of the system parameters in Figures 5-7. We fix $(\xi,\beta,\eta)=(4,5,6)$ and plot three surfaces corresponding to $\pi(S,0)$ as functions of ρ and μ in Figure 5. As a result, while the values of retrial rate μ increase, all three surfaces of $\pi(S,0)$ increase gradually. This trend agrees with our intuitive expectation. On the other hand, one can see from Figure 5 that all three surfaces of $\pi(S,0)$ grow in the beginning for smaller values of ρ and then the trend is reversed for larger values of ρ.
In other words, all three surfaces of $\pi(S,0)$ are concave functions of ρ. The possible explanation of the above phenomenon is as follows. For smaller traffic intensity ρ, there is more chance of the server resides in the switch-off state, whereas the possibility of the server being in the switch-off state is very less for larger traffic intensity ρ. For chosen parametric values $(\mu,\beta,\eta)=(10,5,6)$, Figure 6 describes the behaviours of the surfaces of $\pi(S,0)$ against ρ and ξ. It is clearly seen that the trends of all three surfaces in Figure 6 are quite similar to those in Figure 5. Next, Figure 7 reveals the effects μ and ξ on the surfaces of $\pi(S,0)$ by setting $(\rho,\beta,\eta)=(0.778,5,6)$ for three different threshold values of N. One can see from Figure 7 that all three surfaces increase gradually with both μ and ξ, which is consistent with our intuition.

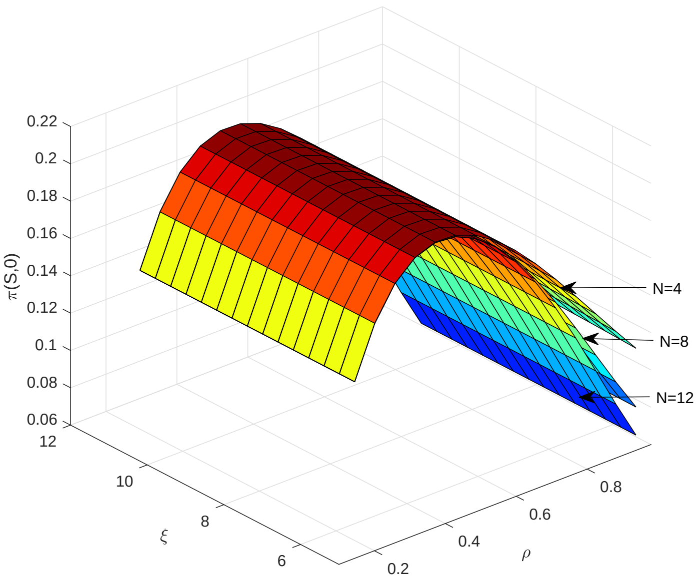

Figure 6 – $\pi(S,0)$ versus (ρ,ξ) for $(\mu,\beta,\eta)=(10,5,6)$.

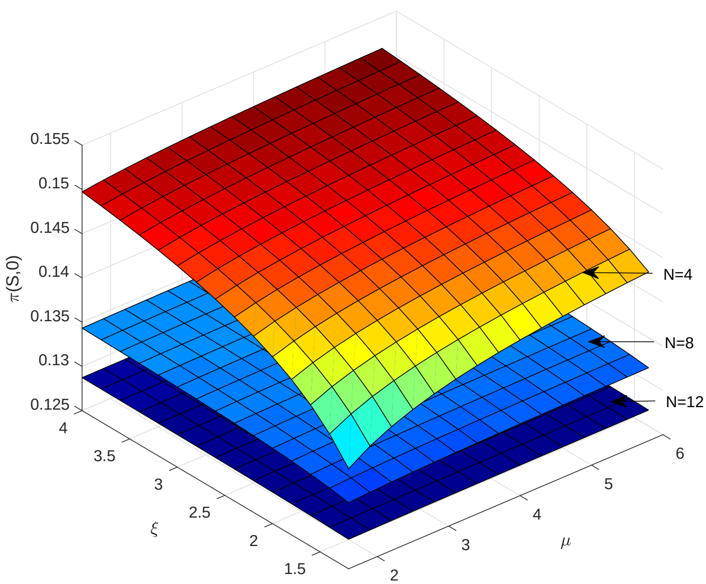

Figure 7 – $\pi(S,0)$ versus (μ,ξ) for $(\rho,\beta,\eta)=(0.778,5,6)$.

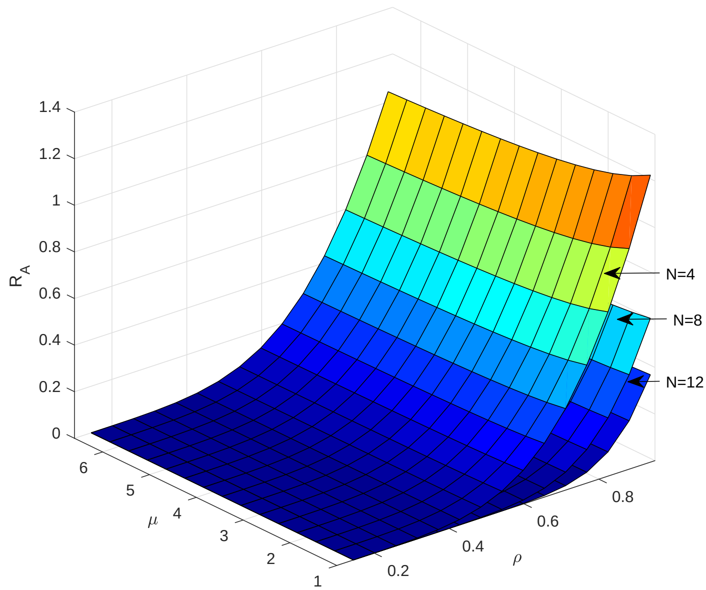

Figure 8 – R_A versus (ρ, μ) for $(\xi, \beta, \eta) = (4, 5, 6)$.

We also conducted the numerical experiments to provide some illustrations in studying the effects of parameters on the probability descriptor $\pi(C, 0)$. From our numerical experience, but not reported here, it is observed that the trends of the surfaces of $\pi(C, 0)$ versus $(\rho, \mu), (\rho, \xi)$ and (μ, ξ) behave very similar to the trends of the surfaces corresponding to $\pi(S, 0)$ as in Figures 5,6 and 7, respectively, where the other parameters have been kept fixed.

We now intend to analyze the sensitivity of the rate, R_A, of reneging/abandonment of customers due to impatience in Figures 8-10 by varying the values of the system parameters μ, ξ, β, η and ρ and for three different values of threshold size N. We find from Figure 8 that all three surfaces of R_A are increasing convex functions of ρ and, in contrast, they decrease while values of μ increase for $(\xi, \beta, \eta) = (4, 5, 6)$. It is consistent with our intuition that more the customers in the system, greater the number of customers who may leave the system without getting the service due to reneging, leading to increase in values of R_A. On the other hand, when the retrial rate μ increases, the customers in the orbit will most likely get the service and hence this phenomenon controls the reneging of the customers from the system resulting in decreasing the values of R_A.

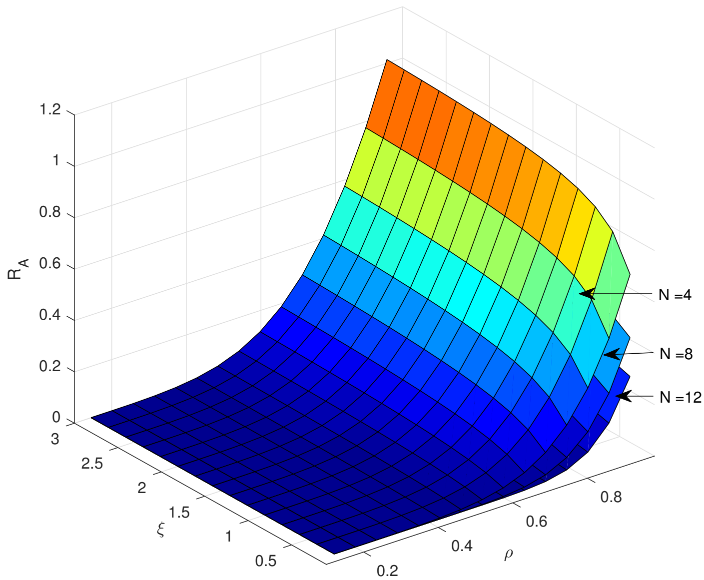

Figure 9 – R_A versus (ρ, ξ) for $(\mu, \beta, \eta) = (10, 5, 6)$.

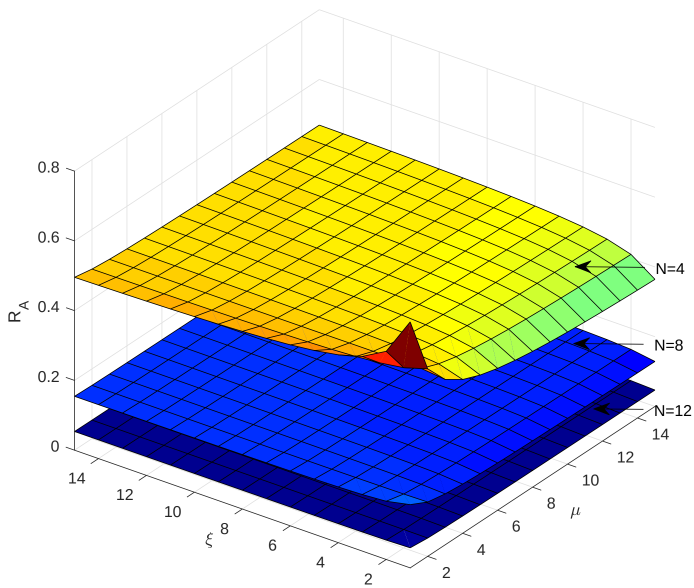

Figure 10 – R_A versus (μ, ξ) for $(\rho, \beta, \eta) = (0.778, 5, 6)$.

Next, Figure 9 describes the sensitivity of R_A with respect to ρ and ξ by taking the other parametric values $(\mu, \beta, \eta) = (10, 5, 6)$. As is to be expected, all three surfaces corresponding to R_A are always increasing convex functions of both ρ and ξ. However, it is noticed that the parameter ρ has a greater effect on all three surfaces of R_A than the impact of ξ. It is also seen that while the threshold values of N increase, the surfaces of R_A decrease for a fixed value of ρ. We now fix $(\rho, \beta, \eta) = (0.778, 5, 6)$ and present numerical examples for the impact of both μ and ξ on the descriptor R_A in Figure 10. By looking at Figure 10, we infer that all three surfaces of R_A decrease in slower manner for increasing values μ, whereas they decrease initially and then start increasing slightly for increasing values of ξ. Here too, it is seen that the surfaces are decreasing functions of N for fixed value of (μ, ξ).

Our next set of numerical illustrations deal with the behaviours of the expected total sojourn time, $E(W_S)$, of an arbitrary customer in the system for various values of the parameters. The results are presented in Figures 11-13. For the fixed parametric values $(\xi, \beta, \eta) = (4, 5, 6)$, the influences of both ρ and μ on $E(W_S)$ are reported in Figure 11. A quick examination of this figure reveals that all three surfaces of the measure $E(W_S)$ rapidly increase when ρ increases, whereas they decrease moderately for increasing values of μ. Intuitively, higher the values of the traffic intensity ρ, the more number of customers will join the system and they wait for their service. Obviously, the sojourn time, $E(W_S)$, of a newly arriving customer will increase in the system. But, increasing the values of the retrial rate μ will reduce the congestion of the system implying that the sojourn time, $E(W_S)$, of the customer decreases. It is also worth to note that all three surfaces increase with increasing values of N for fixed (ρ, μ). Figure 12 demonstrates the impacts of the parametric values of ρ and ξ on $E(W_S)$ by setting $(\mu, \beta, \eta) = (10, 5, 6)$. Referring to this figure, we see that the trends of surfaces of $E(W_S)$ in Figure 12 are quite similar to the behaviours of the surfaces of $E(W_S)$ as in Figure 11. Next, the influences of μ and ξ on $E(W_S)$ are depicted in Figure 13 by keeping the parametric values $(\rho, \beta, \eta) = (0.778, 5, 6)$. Evidently, all three surfaces of $E(W_S)$ gradually decrease for increasing values of both μ and ξ as the number of customers in the system decrease leading to decrease in the sojourn time. However, in Figure 13, the surfaces corresponding to $E(W_S)$ increase with respect to the threshold value N for each fixed value of (μ, ξ).

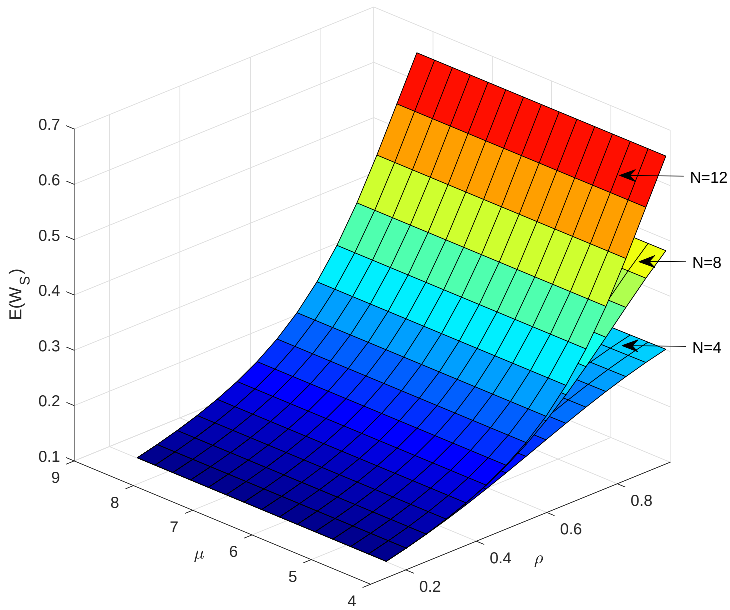

Figure 11 – $E(W_S)$ versus (ρ, μ) for $(\xi, \beta, \eta) = (4, 5, 6)$.

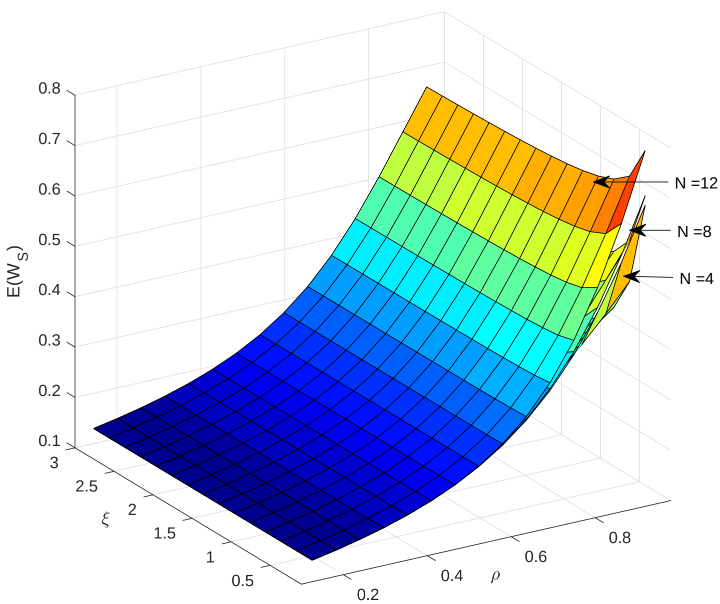

Figure 12 – $E(W_S)$ versus (ρ, ξ) for $(\mu, \beta, \eta) = (10, 5, 6)$.

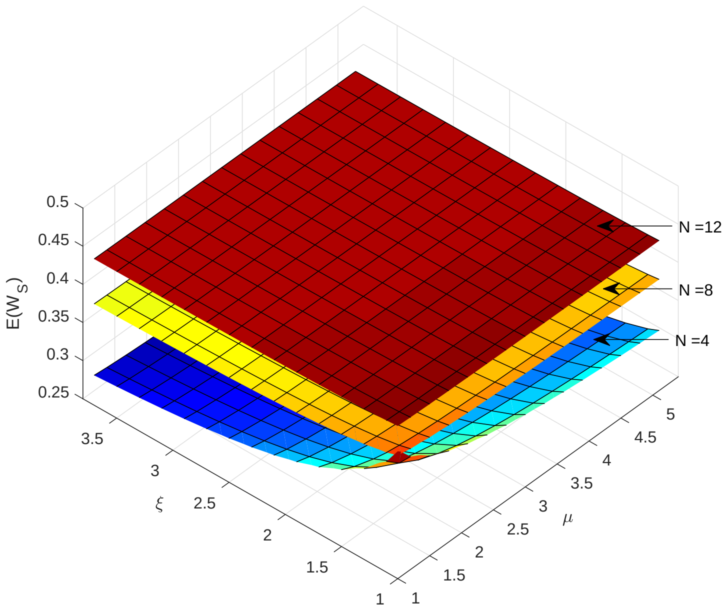

Figure 13 – $E(W_S)$ versus (μ, ξ) for $(\rho, \beta, \eta) = (0.778, 5, 6)$.

8. Conclusion

In this work, we have carried out an exhaustive steady-state analysis of a single server dynamic service system between conventional and retrial queueing modes with impatient customers. The possibility of the close-down and switch-off periods of the server has also been incorporated. For this oscillating service system, analytical expressions for the steady-state joint probabilities of the status of server and the orbit size are derived in terms of hypergeometric functions. We have investigated some interesting and important performance measures along with the partial factorial moments of the orbit size. Besides, the mean regenerative cycle length of the system and its related measures are studied by applying the theory of regenerative process. Finally, graphical illustrations of the selected performance measures have been provided to understand the system behaviour.

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